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Local Existence and Blow up for p-Laplacian Equation with Logarithmic Nonlinearity

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Abstract: This paper deals with a problem of a wave equation with p-Laplacian and logarithmic nonlinearity term. Under suitable conditions, we present the finite-time blow up of solutions for negative initial energy.

Keywords: Blow up, Existence, Logarithmic nonlinearity, p-Laplacian equation.

1 Introduction

In this work, we investigate the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \Delta u - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + |u_t|^{k-2}u_t = |u|^{p-2}u\ln|u|, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1}$$

where $p, k > 2$ are real numbers. $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$. The functions u_0, u_1 are given initial data and exponent p satisfies

$$\begin{cases} 2 < p < \infty, & \text{if } n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{cases} \tag{2}$$

In absence of p-Laplacian operator $div\left(|\nabla u|^{p-2}\nabla u\right)$, (2) becomes a wave equation with logarithmic source term

$$u_{tt} - \Delta u + h(u_t) = |u|^{p-2}u\ln|u|. \tag{3}$$

Logarithmic nonlinearity term appears frequently in partial differential equations due to their wide application in physics and other applied sciences. Problems like equation (3) is encountered naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics [2]-[7]. Let us review somework with related to the problem (3). There is a large body of works in the literature with logarithmic nonlinearity, see in this regard [1]-[4]-[6]-[8]-[9]. Although there have been a lot of works using potential well method, much of them are that on p-Laplacian parabolic equations and there are only a few works on p-Laplacian hyperbolic equations [3]-[5]-[11]-[12].

To best of our knowledge, the blow up and local existence of solution for p-Laplacian hyperbolic type with logaritmic nonlinearity has not been well studied. So that, we will interest with blow up of solution for problem (1).

2 Preliminaries

In order to state our main results, we define the corresponding energy to problem (1) as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|\, u^p dx + \frac{1}{p^2}\|u\|_p^p. \tag{4}$$

Lemma 1. *[10]. For any $u \in H_0^1(\Omega)$, we get*

$$\|u\|_q \leq C_q \|\nabla u\|_2,$$

for all $1 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$; $1 \leq q < \infty$ if $n \leq 2$, where C_p is the best embedding constant.

Lemma 2. *$E(t)$ is a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_k^k \leq 0. \tag{5}$$

Proof: Multiplying the equation (1) by u_t and integrating on Ω, we have the (5).

For reader's straightforwardness, we noticed the definition of weak solutions of problem (1). Following the proof lines in [11, 13] we can state the Theorem 4 for local existence of weak solutions. □

Definition 3. *A function $u(t)$ is called a weak solution of problem (1) on $\Omega \times [0,T)$, if*

$$u \in C\left((0,T); W_0^{1,p}(\Omega)\right) \cap C^1\left((0,T); L^2(\Omega)\right)$$

and

$$u_t \in L^\infty\left((0,T); L^2(\Omega)\right)$$

which satisfies

$$\begin{cases} \int\limits_\Omega u_{tt}(x,t)\, w(x)\, dx + \int\limits_\Omega |\nabla u(x,t)|^{p-2} \nabla u(x,t) \nabla w(x)\, dx + \int\limits_\Omega \nabla u(x,t) \nabla w(x)\, dx \\ \quad + \int\limits_\Omega \nabla u_t(x,t) \nabla w(x)\, dx + \int\limits_\Omega |u_t|^{k-2}(x,t)\, |u_t|(x,t)\, w(x)\, dx \\ \qquad = \int\limits_\Omega \ln|u(x,t)|\, u^{p-2}(x,t)\, w(x)\, dx, \quad \forall w \in H_0^1(\Omega) \\ \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases}$$

Theorem 4. *There exist $T > 0$, such that the problem (1) has a unique local solution weak solution on $[0,T]$.*

3 Blow up

In this part, we prove the blow up result of solution for the problem (1). We give some lemmas which be used in our proof. We establish a blow up result for solution with negative initial energy. Firstly, we give some useful lemmas. For proof of Lemma 5, Lemma 6 and Lemma 7 we refer the reader to Kafini [9].

Lemma 5. *Suppose that (2) holds. There exists a positive constant depending on Ω only such that*

$$\left(\int\limits_\Omega u^p \ln|u|\, dx\right)^{\frac{s}{p}} \leq C\left[\int\limits_\Omega u^p \ln|u|\, dx + \|\nabla u\|_2^2\right], \tag{6}$$

for any $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$, provided that $\int\limits_\Omega u^p \ln|u|\, dx \geq 0$.

Lemma 6. *Suppose that (2) holds. There exists a positive constant depending on Ω only such that*

$$\|u\|_p^p \leq C\left[\int\limits_\Omega u^p \ln|u|\, dx + \|\nabla u\|_2^2\right], \tag{7}$$

for any $u \in L^p(\Omega)$, provided that $\int\limits_\Omega u^p \ln|u|\, dx \geq 0$.

Corollary 1. *Let the assumptions of the Lemma 6 and $k < p$ hold. Using the fact that $\|u\|_k^k \leq C\|u\|_p^k \leq C\left(\|u\|_p^p\right)^{\frac{k}{p}}$. Then we obtain the following*

$$\|u\|_k^k \leq C\left[\left(\int\limits_\Omega u^p \ln|u|\, dx\right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}}\right]. \tag{8}$$

Lemma 7. *Suppose that (2) holds. There exists a positive constant depending on Ω only such that*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|\nabla u\|_2^2\right], \tag{9}$$

for any $u \in L^p(\Omega)$ and $2 \leq s \leq p$.

Theorem 8. *Assume that $E(0) < 0$. Let the conditions in Lemma 5 hold. Then the solution of (1) blows up in finite time*

$$T^* \leq \frac{1-\alpha}{\xi \frac{\alpha}{1-\alpha} L^{\frac{\alpha}{1-\alpha}}(0)} \tag{10}$$

where ξ and α positive constant.

Proof: For this purpose, we denote

$$H(t) = -E(t). \tag{11}$$

By using the definition of $H(t)$ and $E(t)$, (5), (11), we obtain

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \int_{\Omega} u^p \ln |u| \, dx. \tag{12}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx + \varepsilon \frac{1}{2} \|\nabla u\|^2, \tag{13}$$

for ε small to be chosen later and

$$\frac{2\left(p^2 - 2k\right)}{(k-1)p^3} < \alpha < \frac{p-k}{(k-1)}, \tag{14}$$

Now, differentiating $L(t)$ with respect to t, we obtain

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|_p^p \\ &\quad -\varepsilon \int_{\Omega} |u_t|^{k-2} u_t u dx + \varepsilon \int_{\Omega} u^p \ln |u| \, dx. \end{aligned} \tag{15}$$

Adding and subtracting $\varepsilon p(1-\alpha) H(t)$ for some $0 < \alpha < 1$ in (15), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1-\alpha)+2}{2}\right) \|u_t\|^2 \\ &\quad -\varepsilon \left(\frac{2-p(1-\alpha)}{2}\right) \|\nabla u\|^2 - \varepsilon\alpha \|\nabla u\|_p^p \\ &\quad +\varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p + \varepsilon\alpha \int_{\Omega} u^p \ln |u| \, dx \\ &\quad +\varepsilon p(1-\alpha) H(t) - \varepsilon \int_{\Omega} |u_t|^{k-2} u_t u dx. \end{aligned} \tag{16}$$

Exploiting Young's inequality the last term of the (16) for any $\delta > 0$, we get

$$\begin{aligned} L'(t) &= \left((1-\alpha) H^{-\alpha}(t) - \varepsilon \frac{k-1}{k} \delta^{-\frac{k}{k-1}}\right) H'(t) - \varepsilon \frac{\delta^k}{k} \|u\|_k^k \\ &\quad +\varepsilon \left(\frac{p(1-\alpha)+2}{2}\right) \|u_t\|^2 - \varepsilon \left(\frac{2-p(1-\alpha)}{2}\right) \|\nabla u\|^2 - \varepsilon\alpha \|\nabla u\|_p^p \\ &\quad +\varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p + \varepsilon\alpha \int_{\Omega} u^p \ln |u| \, dx + \varepsilon p(1-\alpha) H(t). \end{aligned} \tag{17}$$

Of course (17) holds even if δ is time dependent since the integral is taken over the x-variable. Therefore by choosing δ so that $\delta^{-\frac{k}{k-1}} = M_1 H^{-\alpha}(t)$, for M_1 to be specified later, and substituing in (17), we have

$$\begin{aligned} L'(t) &\geq \left(1-\alpha-\varepsilon \frac{k-1}{k} M_1\right) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1-\alpha)+2}{2}\right) \|u_t\|^2 - \varepsilon\alpha \|\nabla u\|_p^p \\ &\quad -\varepsilon \left(\frac{2-p(1-\alpha)}{2}\right) \|\nabla u\|^2 + \varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p - \varepsilon \frac{(M_1)^{1-k}}{k} H^{\alpha(k-1)}(t) \|u\|_k^k \\ &\quad +\varepsilon\alpha \int_{\Omega} u^p \ln |u| \, dx + \varepsilon p(1-\alpha) H(t). \end{aligned} \tag{18}$$

By using of the Corollary 1, embedding theorem and Young's inequality, we obtain

$$H^{\alpha(k-1)}\|u\|_k^k \le C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\alpha(k-1)+\frac{k}{p}} + \left(\int_\Omega u^p \ln|u|\,dx\right)^{\alpha(k-1)\frac{p^2}{p^2-2k}} + \|\nabla u\|_p^p\right]$$

Make use of the (14), we find

$$2<\alpha(k-1)p+k\le p \text{ and } 2<\frac{\alpha(k-1)p^3}{p^2-2k}\le p.$$

Therefore, by Lemma 5 we have

$$H^{\alpha(k-1)}\|u\|_k^k \le C\left[\int_\Omega u^p \ln|u|\,dx + \|\nabla u\|_p^p\right] \tag{19}$$

Inserting (19) into (18), we arrive at

$$\begin{aligned} L'(t) &\ge \left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2 \\ &\quad -\varepsilon\left[\alpha+\frac{(M_1)^{1-k}}{k}C\right]\|\nabla u\|_p^p-\varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p \\ &\quad +\varepsilon\left[\alpha-\frac{(M_1)^{1-k}}{k}C\right]\int_\Omega u^p\ln|u|\,dx+\varepsilon p(1-\alpha)H(t). \end{aligned} \tag{20}$$

Since $0<\frac{2}{p}<1$, now using the following inequality

$$x^v \le x+1 \le \left(1+\frac{1}{\beta}\right)(x+\beta),\ \forall x\ge 0, 0<v<1, \beta\ge 0 \tag{21}$$

espacially taking and (12), we have

$$\begin{aligned} \|\nabla u\|_p^p &\le \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_p^p+H(0)\right) \\ &\le d\left(\|\nabla u\|_p^p+H(t)\right) \end{aligned} \tag{22}$$

where $d=1+\frac{1}{H(0)}$.

Inserting (19)-(22) into (20) we deduce

$$\begin{aligned} L'(t) &\ge \left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p \\ &\quad +\varepsilon\left[d\left(\frac{p(1-\alpha)-2}{2}\right)-\alpha-\frac{(M_1)^{1-k}}{k}C\right]\|\nabla u\|_p^p+\varepsilon\left[\alpha-\frac{(M_1)^{1-k}}{k}C\right]\int_\Omega u^p\ln|u|\,dx \\ &\quad +\varepsilon\left[p(1-\alpha)+d\left(\frac{p(1-\alpha)-2}{2}\right)\right]H(t), \end{aligned} \tag{23}$$

where used $L^p(\Omega)\hookrightarrow L^2(\Omega)$, $2<p$.

At this point, we choose $\alpha>0$ small that

$$d\left(\frac{p(1-\alpha)-2}{2}\right)>0$$

and M_1 sufficiently large that

$$d\left(\frac{p(1-\alpha)-2}{2}\right)-\alpha-\frac{(M_1)^{1-k}}{k}C>0 \text{ and } \alpha-\frac{(M_1)^{1-k}}{k}C>0.$$

Once M_1 and α are fixed, we pick $0<\varepsilon<\frac{1-\alpha}{M_1}$ so that

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0u_1dx>0. \tag{24}$$

Therefore, (23) becomes form

$$L'(t)\ge\lambda\left[H(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|u\|_p^p+\int_\Omega u^p\ln|u|\,dx\right], \tag{25}$$

where $\lambda>0$.

Consequently we obtain

$$L(t) > L(0),\ t \geq 0.$$

On the other hand by $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$L(t)^{\frac{1}{1-\alpha}} \leq C\left[H(t) + \int_\Omega |uu_t dx|^{\frac{1}{1-\alpha}} + \|\nabla u\|^{\frac{2}{1-\alpha}}\right]. \tag{26}$$

Again by using of the (21)for, $v = \frac{2}{p(1-\alpha)} < 1$ since $\alpha < \frac{p-2}{2p}$, $d = 1 + \frac{1}{H(0)}$, we get

$$\left(\|\nabla u\|_p^p\right)^{\frac{2}{p(1-\alpha)}} \leq d\left(\|\nabla u\|_p^p + H(t)\right), \tag{27}$$

where used $L^p(\Omega) \hookrightarrow L^2(\Omega)$.

Hölder's inequality give us

$$\left|\int_\Omega uu_t dx\right| \leq c\left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$$

where c is the positive constant. This inequality implies that there exists a positive constant $C > 0$ such that

$$\left|\int_\Omega uu_t dx\right|^{1/(1-\alpha)} \leq \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2(1-\alpha)}}\left(\int_\Omega |u|^p dx\right)^{\frac{1}{p(1-\alpha)}}. \tag{28}$$

Applying Young's inequality to the right-hand side of the (26), we get

$$\left|\int_\Omega uu_t dx\right|^{1/(1-\alpha)} \leq C\left[\left(\int_\Omega |u_t|^2 dx\right)^{\frac{\kappa}{2(1-\alpha)}} + \left(\int_\Omega |u|^p dx\right)^{\frac{\mu}{p(1-\alpha)}}\right] \text{ for } \frac{1}{\mu} + \frac{1}{\kappa} = 1. \tag{29}$$

To be able to use Lemma 11, we take $\kappa = 2/(1-\alpha)$, which gives $\mu = 2(1-\alpha)/(1-2\alpha)$,(29) has the form

$$\left|\int_\Omega uu_t dx\right|^{1/(1-\alpha)} \leq C\left[\int_\Omega |u_t|^2 dx + \left(\int_\Omega |u|^p dx\right)^{\frac{2}{p(1-2\alpha)}}\right].$$

By using of Poincare's inequality we get

$$\left|\int_\Omega uu_t dx\right|^{1/(1-\alpha)} \leq C\left[\int_\Omega |u_t|^2 dx + \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{2}{p(1-2\alpha)}}\right]$$

With the re-use of the inequality (21) where $d = 1 + \frac{1}{H(0)}$, $v = \frac{2}{p(1-2\alpha)} < 1$,since $\alpha < \frac{p-2}{2p}$, we obtain

$$\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{2}{p(1-2\alpha)}} \leq d\left[\|\nabla u\|_p^p + H(t)\right]$$

Thus, (28) becomes

$$\left|\int_\Omega uu_t dx\right|^{1/(1-\alpha)} \leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p\right). \tag{30}$$

Inserting (27) and (30) into (26), it follows that

$$L(t)^{\frac{1}{1-\alpha}} \leq \left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|u\|_p^p + \int_\Omega u^p \ln|u|\, dx\right). \tag{31}$$

By associatining (31) and (25) we arrive at

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t) \tag{32}$$

where ξ is a positive constant. □

Integration of (32) over $(0,t)$ we reach

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi\alpha t}{1-\alpha}}$$

Therefore the solutions blows up within a time given by the estimate (10) above.

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