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Nonexistence of Solutions of a Delayed Wave Equation with Variable-Exponents

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Abstract: In this paper, we deal with a nonlinear Timoshenko equation with delay term and variable exponents. Under suitable conditions, we prove the blow-up of solutions in a finite time. Our results are more general than the earlier results. Time delays arise in many applications, for instance, it appears in physical, chemical, biological, thermal and economic phenomena. Also, delay is source of instability, a small delay can destabilize a system which is uniformly asymptotically stable. Several physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing are modelled by equations with variable exponents.

Keywords: Blow up, Delay term, Timoshenko equation, Variable-exponent.

1 Introduction

In this work, we study the following nonlinear Timoshenko equation with variable exponents and delay term

$$\begin{cases} u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u+\mu_1 u_t\left(x,t\right)\left|u_t\right|^{m(x)-2}\left(x,t\right) & \\ +\mu_2 u_t\left(x,t-\tau\right)\left|u_t\right|^{m(x)-2}\left(x,t-\tau\right) & \\ =bu\left|u\right|^{p(x)-2} & \text{in } \Omega\times R^+, \\ u\left(x,t\right)=0 & \text{in } \partial\Omega\times[0,\infty), \\ u\left(x,0\right)=u_0\left(x\right), u_t\left(x,0\right)=u_1\left(x\right) & \text{in } \Omega, \\ u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{1}$$

where $\Omega\subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\tau>0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number and $b\geq 0$ is a constant. $M(s)$ is a positive C^1-function like $M(s)=1+s^\gamma$, $\gamma>0$. The functions u_0, u_1, f_0 are the initial data to be specified later.

$p(\cdot)$ and $m(\cdot)$ are the variable exponents which given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{aligned} 2\leq p^-\leq p(x)\leq p^+\leq p^*, \\ 2\leq m^-\leq m(x)\leq m^+\leq m^* \end{aligned} \tag{2}$$

where

$$\begin{aligned} p^- &= ess\inf_{x\in\Omega} p(x),\quad p^+=ess\sup_{x\in\Omega} p(x), \\ m^- &= ess\inf_{x\in\Omega} m(x),\quad m^+=ess\sup_{x\in\Omega} m(x) \end{aligned}$$

and

$$p^*, m^*=\frac{2(n-1)}{n-2} \text{ if } n\geq 3.$$

- The Timoshenko equation is an approximate model describing the transversal motion of a rod [11].
- The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [3], [4].
- Time delays often appear in many practical problems such as thermal, economic phenomena, biological, chemical and physical. Also, time delay can be a source of instability [6].

In the absence of $\Delta^2 u$ term and when $M(s) \equiv 1$, the equation (1) becomes

$$u_{tt} - \Delta u + \mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2}. \tag{3}$$

Messaoudi and Kafini [8], obtained the decay estimates and global nonexistence of the equation (3).

The outline of this paper is as follows: In Sect. 2, the definition of the variable exponent Sobolev and Lebesgue spaces are stated. In Sect. 3, we establish the blow up of solutions.

2 Preliminaries

In this part, we denote some preliminary facts about Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see [2], [4], [5], [7], [10]).

Let $p : \Omega \to [1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to R;\ \text{measurable in } \Omega\ : \int_\Omega |u|^{p(\cdot)}\,dx < \infty \right\}.$$

We next, define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

We also assume that:

$$|p(x) - p(y)| \le -\frac{A}{\log|x-y|} \text{ and } |m(x) - m(y)| \le -\frac{B}{\log|x-y|} \text{ for all } x, y \in \Omega, \tag{4}$$

$A, B > 0$ and $0 < \delta < 1$ with $|x - y| < \delta$ (log-Hölder condition).

Lemma 1. *[2] (Poincare inequality) Assume that $p(\cdot)$ satisfies (4) and let Ω be a bounded domain of R^n. Then,*

$$\|u\|_{p(\cdot)} \le c\|\nabla u\|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c\left(p^-, p^+, |\Omega|\right) > 0$.

Lemma 2. *[2] If $p : \overline{\Omega} \to [1,\infty)$ is continuous,*

$$2 \le p^- \le p(x) \le p^+ \le \frac{2n}{n-2},\ n \ge 3, \tag{5}$$

satisfies, then the embedding $H_0^1(\Omega) \to L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 3. *[1] If $p^+ < \infty$ and $p : \Omega \to [1,\infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Lemma 4. *[1] (Hölder' inequality) Let $p, q, s \ge 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)},\ \text{for a.e. } y \in \Omega,$$

satisfies. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

Lemma 5. *[1] (Unit ball property) Let $p \ge 1$ be a measurable function on Ω. Then,*

$$\|f\|_{p(\cdot)} \le 1 \text{ if and only if } \varrho_{p(\cdot)}(f) \le 1,$$

where

$$\varrho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx.$$

Lemma 6. *[2] If $p \ge 1$ is a measurable function on Ω. Then,*

$$\min\left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\} \le \varrho_{p(\cdot)}(u) \le \max\left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and for a.e. $x \in \Omega$.

Remark 1. *We denote by c various positive constants which may be different at different occurrences. Also, throughout this paper, we use the embedding*

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which implies

$$\|u\|_p \le c\,\|\triangledown u\| \le c\,\|\triangle u\|,$$

where $2 \le p < \infty \ \ (n = 1, 2)\,, 2 \le p \le \frac{2n}{n-2} \ \ (n \ge 3)\,.$

3 Blow up

In this part, for the case $b > 0$, we establish the blow up of solutions for problem (1). As in [9], we introduce the new variable

$$z(x, \rho, t) = u_t(x, t - \tau\rho)\,,\ x \in \Omega,\ \rho \in (0, 1)\,,\ t > 0.$$

Thus, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0,\ x \in \Omega,\ \rho \in (0, 1)\,,\ t > 0.$$

Then, problem (1) takes the form

$$\begin{cases} u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t)\,|u_t(x,t)|^{m(x)-2} & \\ +\mu_2 z(x,1,t)\,|z(x,1,t)|^{m(x)-2} & \\ = bu\,|u|^{p(x)-2} & \text{in } \Omega \times (0,\infty)\,, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,\infty)\,, \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega \times (0,1)\,, \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x)\,, u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{6}$$

The energy functional associated with problem (6) is

$$\begin{aligned} E(t) = \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|\Delta u\|^2 + \tfrac{1}{2}\|\nabla u\|^2 + \tfrac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \\ + \int_0^1\int_\Omega \tfrac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho - b\int_\Omega \tfrac{|u|^{p(x)}}{p(x)}dx. \end{aligned} \tag{7}$$

Lemma 7. *Let (u, z) be a solution of (6). Then there exists some $C_0 > 0$ such that*

$$E'(t) \le -C_0 \int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)}\right)dx \le 0. \tag{8}$$

Lemma 8. *[8] Suppose that condition (2) holds.Then there exists a positive $C > 1$, depending on Ω only, such that*

$$\varrho^{s/p^-}(u) \le C\left(\|\Delta u\|^2 + \varrho(u)\right). \tag{9}$$

Then, we have following inequalities:

i)

$$\|u\|_{p^-}^s \le C\left(\|\Delta u\|^2 + \|u\|_{p^-}^{p^-}\right), \tag{10}$$

ii)

$$\varrho^{s/p^-}(u) \le C\left(|H(t)| + \|u_t\|^2 + \varrho(u) + \int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right), \tag{11}$$

iii)

$$\|u\|_{p^-}^s \le C\left(|H(t)| + \|u_t\|^2 + \|u\|_{p^-}^{p^-} + \int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right), \tag{12}$$

for any $u \in H_0^1(\Omega)$ and $2 \le s \le p^-$. Let (u, z) be a solution of (6), then

iv)

$$\varrho(u) \ge C\,\|u\|_{p^-}^{p^-}, \tag{13}$$

v)

$$\int_\Omega |u|^{m(x)}dx \le C\left(\varrho^{m^-/p^-}(u) + \varrho^{m^+/p^-}(u)\right). \tag{14}$$

Theorem 9. *Let conditions* (2) *and* (4) *be provided and assume that*

$$E(0) < 0.$$

Then the solution (6) *blows up in finite time.*

Proof: We establish

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx. \tag{15}$$

A direct differentiation of (15) using the first equation in (6) satisfies

$$\begin{aligned} L'(t) = (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \textstyle\int_{\Omega} \left[u_t^2 - |\Delta u|^2 - |\nabla u|^2 - |\nabla u|^{2(\gamma+1)} \right] dx \\ + \varepsilon b \textstyle\int_{\Omega} |u|^{p(x)} dx - \varepsilon \mu_1 \int_{\Omega} u u_t(x,t) |u_t(x,t)|^{m(x)-2} dx \\ - \varepsilon \mu_2 \textstyle\int_{\Omega} u z(x,1,t) |z(x,1,t)|^{m(x)-2} dx. \end{aligned} \tag{16}$$

We take δ such that

$$\delta^{-\frac{m(x)}{m(x)-1}} = k H^{-\alpha}(t).$$

Lemma 9 satisfies

$$H^{\alpha(m^+-1)}(t) \int_{\Omega} |u|^{m(x)} dx \le C \left(\|\Delta u\|^2 + \varrho(u) \right). \tag{17}$$

Consequently, we get

$$\begin{aligned} L'(t) \ge (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\tfrac{m^+-1}{m^+} \right) ck \right] \textstyle\int_{\Omega} |u_t|^{m(x)} dx \\ + (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\tfrac{m^+-1}{m^+} \right) ck \right] \textstyle\int_{\Omega} |z(x,1,t)|^{m(x)} dx \\ + \varepsilon \left(\tfrac{(p^- - 2) - ap^-}{2} - \tfrac{C}{m^- k^{1-m^-}} \right) \|\Delta u\|^2 \\ + \varepsilon (1-a) p^- H(t) + \varepsilon \tfrac{(1-a)p^- + 2}{2} \|u_t\|^2 + \varepsilon \tfrac{(1-a)p^- - 2}{2} \|\nabla u\|^2 \\ + \varepsilon \tfrac{(1-a)p^- - 2(\gamma+1)}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \varepsilon \left(ab - \tfrac{C}{m^- k^{1-m^-}} \right) \varrho(u) \\ + \varepsilon (1-a) p^- \textstyle\int_0^1 \int_{\Omega} \tfrac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned} \tag{18}$$

Choosing a small enough so that

$$\frac{(1-a)p^- - 2}{2} > 0 \text{ and } \frac{(1-a)p^- - 2(\gamma+1)}{2(\gamma+1)} > 0.$$

and k so large that

$$\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{1-m^-}} > 0 \text{ and } ab - \frac{C}{m^- k^{1-m^-}} > 0.$$

By (12), we obtain

$$\left| \int_{\Omega} u u_t(x,t) dx \right|^{1/(1-\alpha)} \le C \left[|H(t)| + \|u_t\|^2 + \|\triangle u\|^2 + \varrho(u) + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \right]. \tag{19}$$

For some $\Psi > 0$, we arrive at

$$L'(t) \ge \Psi L^{1/(1-\alpha)}(t).$$

A simple integration over $(0,t)$ yields, which implies that the solution blows up in a finite time T^*, with

$$T^* \le \frac{1-\alpha}{\Psi \alpha [L(0)]^{\alpha/(1-\alpha)}}.$$

As a result, the proof is completed. □

4 Conclusion

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for the nonlinear Timoshenko equation with delay term and variable exponents. We have been obtained the blow-up result under the sufficient conditions in a bounded domain.

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