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Decay and Blow up of Solutions for a Delayed Wave Equation with Variable-Exponents

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Abstract: This work deals with a nonlinear wave equation with delay term and variable exponents. Firstly, we prove the blow up of solutions in a finite time for negative initial energy. After, we obtain the decay results by applying an integral inequality due to Komornik. These results improve and extend earlier results in the literature. Generally, time delays arise in many applications. For instance, it appears in physical, chemical, biological, thermal and economic phenomena. Moreover, delay is source of instability. A small delay can destabilize a system which is uniformly asymptotically stable. Recently, several physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing are modelled by equations with variable exponents.

Keywords: Blow up, Decay, Delay term, Variable exponent.

1 Introduction

In this paper, we study the following nonlinear p-Laplacian equation with delay term and variable exponents

$$\begin{cases} u_{tt}-\Delta u-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+\mu_1 u_t(x,t)\,|u_t|^{m(x)-2}(x,t) & \\ +\mu_2 u_t(x,t-\tau)\,|u_t|^{m(x)-2}(x,t-\tau) & \\ =bu\,|u|^{q(x)-2} & \text{in } \Omega\times R^+, \\ u(x,t)=0 & \text{in } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau) & \text{in}\Omega\times(0,\tau), \end{cases} \tag{1}$$

where $\Omega\subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\tau>0$ is a time delay term, μ_1 is a positive constant, μ_2 is a real number and $b\geq 0$ is a constant. The term $\Delta_{p(\cdot)}u=\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$ is called $p(\cdot)$-Laplacian. The functions u_0, u_1, f_0 are the initial data to be specified later.

$p(\cdot)$, $q(\cdot)$ and $m(\cdot)$ are the variable exponents which given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{array}{l} 2\leq p^-\leq p(x)\leq p^+\leq p^*, \\ 2\leq q^-\leq q(x)\leq q^+\leq q^*, \\ 2\leq m^-\leq m(x)\leq m^+\leq m^* \end{array} \tag{2}$$

where

$$\begin{aligned} p^- &= ess\inf_{x\in\Omega} p(x),\quad p^+=ess\sup_{x\in\Omega} p(x), \\ q^- &= ess\inf_{x\in\Omega} q(x),\quad q^+=ess\sup_{x\in\Omega} q(x), \\ m^- &= ess\inf_{x\in\Omega} m(x),\quad m^+=ess\sup_{x\in\Omega} m(x) \end{aligned}$$

and

$$p^*=\begin{cases} \frac{Np(x)}{ess\sup_{x\in\Omega}(N-m(x))} & \text{if } p^+<n, \\ +\infty & \text{if } p^+\geq n. \end{cases}$$

Time delays often appear in many practical problems such as thermal, biological, economic phenomena, chemical, physical [5].

The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [3, 14].

In the absence of the p-Laplacian term div $\left(|\nabla u|^{p(x)-2}\nabla u\right)$, the problem (1) can be reduced to the following wave equation with delay term and variable exponents

$$u_{tt}-\Delta u+\mu_1 u_t\left(x,t\right)\left|u_t\left(x,t\right)\right|^{m(x)-2}+\mu_2 u_t\left(x,t-\tau\right)\left|u_t\right|^{m(x)-2}\left(x,t-\tau\right)=bu\left|u\right|^{p(x)-2}. \tag{3}$$

Messaoudi and Kafini [11], studied the decay estimates and the global nonexistence of the equation (3).

The plan of this paper is as follows. Firstly, in Sect. 2, some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}\left(\Omega\right)$ and $W^{1,p(x)}\left(\Omega\right)$ are given. In Sect. 3, we obtain the blow up of solutions. Finally, in Sect. 4, we establish the decay results.

2 Preliminaries

In this part, we give some preliminary facts about Lebesgue $L^{p(\cdot)}\left(\Omega\right)$ and Sobolev $W^{1,p(\cdot)}\left(\Omega\right)$ spaces with variable exponents (see [2]-[3]-[4]-[6]-[13]).

Let $p:\Omega\rightarrow[1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p\left(\cdot\right)$ by

$$L^{p(\cdot)}\left(\Omega\right)=\left\{u:\Omega\rightarrow R;\ \text{measurable in }\Omega\ :\int_{\Omega}|u|^{p(\cdot)}\,dx<\infty\right\}.$$

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}\left(\Omega\right)$ as follows

$$W^{1,p(\cdot)}\left(\Omega\right)=\left\{u\in L^{p(\cdot)}\left(\Omega\right):\ \nabla u\text{ exists and }\left|\nabla u\right|\in L^{p(\cdot)}\left(\Omega\right)\right\}.$$

We also assume that $p\left(\cdot\right)$, $q\left(\cdot\right)$ and $m\left(\cdot\right)$ satisfy the log-Hölder continuity condition:

$$\left|q\left(x\right)-q\left(y\right)\right|\leq-\frac{A}{\log|x-y|},\text{ for a.e. }x,y\in\Omega,\text{ with }|x-y|<\delta, \tag{4}$$

$A>0$ and $0<\delta<1$.

Lemma 1. *[2] (Poincare inequality) Assume that $q\left(\cdot\right)$ satisfies (4) and let Ω be a bounded domain of R^n. Then,*

$$\|u\|_{p(\cdot)}\leq c\left\|\nabla u\right\|_{p(\cdot)}\ \textit{for all } u\in W_0^{1,p(\cdot)}\left(\Omega\right),$$

where $c=c\left(q^-,q^+,|\Omega|\right)>0$.

Lemma 2. *[8] If $p\left(\cdot\right)\in C\left(\overline{\Omega}\right)$ and $q:\Omega\rightarrow[1,\infty)$ is measurable function such that*

$$es\sin f_{x\in\Omega}\left(p^*\left(x\right)-q\left(x\right)\right)>0\ \textit{with}\ p_*\left(x\right)=\begin{cases}\frac{np(x)}{ess\sup_{x\in\Omega}(n-p(x))} & \textit{if}\ p^+<n,\\ +\infty & \textit{if}\ p^+\geq n,\end{cases} \tag{5}$$

satisfies, then the embedding $W_0^{1,p(\cdot)}\left(\Omega\right)\rightarrow L^{q(\cdot)}\left(\Omega\right)$ is continuous and compact.

Lemma 3. *[1] If $p^+<\infty$ and $p:\Omega\rightarrow[1,\infty)$ is a measurable function, then $C_0^\infty\left(\Omega\right)$ is dense in $L^{p(\cdot)}\left(\Omega\right)$.*

Lemma 4. *[1] (Hölder' inequality) Let $p,q,s\geq1$ be measurable functions defined on Ω and*

$$\frac{1}{s\left(y\right)}=\frac{1}{p\left(y\right)}+\frac{1}{q\left(y\right)}\ ,\ \textit{for a.e. } y\in\Omega,$$

satisfies. If $f\in L^{p(\cdot)}\left(\Omega\right)$ and $g\in L^{q(\cdot)}\left(\Omega\right)$, then $fg\in L^{s(\cdot)}\left(\Omega\right)$ and

$$\left\|fg\right\|_{s(\cdot)}\leq2\left\|f\right\|_{p(\cdot)}\left\|g\right\|_{q(\cdot)}.$$

Lemma 5. *[1] (Unit ball property) Let $p\geq1$ be a measurable function on Ω. Then,*

$$\|f\|_{p(\cdot)}\leq1\ \textit{if and only if}\ \varrho_{p(\cdot)}\left(f\right)\leq1,$$

where

$$\varrho_{p(\cdot)}\left(f\right)=\int_{\Omega}\left|f\left(x\right)\right|^{p(x)}dx.$$

Lemma 6. *[2] If $p\geq1$ is a measurable function on Ω. Then,*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right\}\leq\varrho_{p(\cdot)}\left(u\right)\leq\max\left\{\|u\|_{p(\cdot)}^{p^-},\|u\|_{p(\cdot)}^{p^+}\right\}$$

for any $u\in L^{p(\cdot)}\left(\Omega\right)$ and for a.e. $x\in\Omega$.

3 Blow up

In this part, when $b > 0$, we prove the blow up of the solution for problem (1) with negative initial energy. Now, we introduce the new variable as in [12],

$$z(x,\rho,t) = u_t(x, t-\tau\rho),\ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Hence, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

Therefore, problem (1) is equivalent to

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} & \\ +\mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} & \\ = bu|u|^{q(x)-2} & \text{in } \Omega\times(0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \\ u(x,t) = 0 & \text{on } \partial\Omega\times[0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{6}$$

We define the energy of the solution of problem (6) by

$$\begin{aligned} E(t) = {} & \tfrac{1}{2}\|u_t\|^2 + \tfrac{1}{2}\|\nabla u\|^2 + \textstyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\ & + \textstyle\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho - b\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx. \end{aligned} \tag{7}$$

Lemma 7. *Let (u,z) be a solution of (6). Then there exists some $C_0 > 0$ such that*

$$E'(t) \le -C_0\int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)}\right)dx \le 0. \tag{8}$$

Lemma 8. *[9] Assume that the conditions of Lemma 2 hold.Then, there exists a constant $C > 1$, depending on Ω only, such that*

$$\varrho^{s/p^-}(u) \le C\left(\|\nabla u\|_{p(\cdot)}^{p^-} + \varrho(u)\right). \tag{9}$$

Then, we have following inequalities:
i)

$$\|u\|_{p^-}^s \le C\left(\|\nabla u\|_{p(\cdot)}^{p^-} + \|u\|_{q^-}^{q^-}\right), \tag{10}$$

ii)

$$\varrho^{s/q^-}(u) \le C\left(|H(t)| + \|u_t\|_2^2 + \varrho(u) + \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right), \tag{11}$$

iii)

$$\|u\|_{q^-}^s \le C\left(|H(t)| + \|u_t\|_2^2 + \|u\|_{q^-}^{q^-} + \int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right), \tag{12}$$

for any $u \in W_0^{1,p(\cdot)}(\Omega)$ and $p^- \le s \le q^-$. Let (u,z) be a solution of (6), then
iv)

$$\varrho(u) \ge C\|u\|_{q^-}^{q^-}, \tag{13}$$

v)

$$\int_\Omega |u|^{m(x)}dx \le C\left(\varrho^{m^-/q^-}(u) + \varrho^{m^+/q^-}(u)\right). \tag{14}$$

Theorem 9. *Let conditions (2) and (4) be provided and suppose that*

$$E(0) < 0.$$

Then the solution (6) blows up in finite time.

Proof: We define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} u u_t dx. \tag{15}$$

Then, we have

$$\begin{aligned} L'(t) \geq &\ (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\tfrac{m^+-1}{m^+}\right) ck\right] \textstyle\int_{\Omega} |u_t(t)|^{m(x)} dx \\ &+ (1-\alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\tfrac{m^+-1}{m^+}\right) ck\right] \textstyle\int_{\Omega} |z(x,1,t)|^{m(x)} dx \\ &+ \varepsilon \left(\tfrac{(1-a)q^- - p^+}{p^+} - \tfrac{C}{m^- k^{1-m^-}}\right) \textstyle\int_{\Omega} |\nabla u|^{p(x)} dx \\ &+ \varepsilon (1-a) q^- H(t) + \varepsilon \tfrac{(1-a)q^- + 2}{2} \|u_t\|^2 \\ &+ \varepsilon \tfrac{(1-a)q^- - 2}{2} \|\nabla u\|^2 + \varepsilon \left(ab - \tfrac{C}{m^- k^{1-m^-}}\right) \varrho(u) \\ &+ \varepsilon (1-a) q^- \textstyle\int_0^1 \int_{\Omega} \tfrac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned} \tag{16}$$

We choose a small enough so that

$$\frac{(1-a)q^- - 2}{2} > 0,$$

and k so large that

$$\frac{(1-a)q^- - p^+}{p^+} - \frac{C}{m^- k^{1-m^-}} > 0 \text{ and } ab - \frac{C}{m^- k^{1-m^-}} > 0.$$

Therefore, (16) becomes

$$L'(t) \geq \varepsilon \eta \left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|_{p(\cdot)}^{p^-} + \varrho_{p(\cdot)}(u) + \int_0^1 \int_{\Omega} \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right] \tag{17}$$

for a constant $\eta > 0$.

Thus, for some $\Psi > 0$, we conclude

$$L'(t) \geq \Psi L^{1/(1-\alpha)}(t).$$

A simple Integration over $(0,t)$ yields

$$T^* \leq \frac{1-\alpha}{\Psi \alpha [L(0)]^{\alpha/(1-\alpha)}}.$$

□

4 Decay

In this part, when $b = 0$, we prove our main decay results. We introduce the new variable as in [12]

$$z(x,\rho,t) = u_t(x, t - \tau\rho),\ x \in \Omega,\ \rho \in (0,1),\ t > 0;$$

hence, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Consequently, the original problem (1) can be transformed into the new system

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) + \mu_1 u_t(x,t) |u_t(x,t)|^{m(x)-2} & \\ +\mu_2 z(x,1,t) |z(x,1,t)|^{m(x)-2} = 0 & \text{in } \Omega \times (0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,\infty), \\ z(x,\rho,0) = f_0(x, -\rho\tau) & \text{in } \Omega \times (0,1), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,1), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{18}$$

We introduce the "modified" energy functional for the problem (18) by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \\ &\quad + \int_0^1 \int_{\Omega} \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned} \tag{19}$$

Lemma 10. *(Komornik, [7]) Let $E : R^+ \to R^+$ be a nonincreasing function and assume that there are constants σ, $\omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)\,dt \leq \frac{1}{\Omega}E^\sigma(0)\,E(s) = cE(s)\,,\ \forall s > 0.$$

Then, we have

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \leq cE(0)\,e^{-\omega t} & \text{if } \sigma = 0, \end{cases}$$

for all $t \geq 0$.

Lemma 11. *[11] The functional*

$$F(t) = \tau\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)\,|z(x,\rho,t)|^{m(x)}\,dxd\rho$$

satisfies

$$F'(t) \leq \int_\Omega \xi(x)\,|u_t|^{m(x)}\,dx - \tau e^{-\tau}\int_0^1\int_\Omega \xi(x)\,|z(x,\rho,t)|^{m(x)}\,dxd\rho$$

along the solution of (18).

Lemma 12. *[10] Let u be a solution of (18). Then for some $C > 0$,*

$$\varrho_{p(x)}(\nabla u) \geq C\,\|\nabla u\|_{p^-}^{p^+}. \tag{20}$$

Theorem 13. *Assume that conditions (4) is satisfied and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq m^- \leq m(x) \leq m^+ \leq p^*,\ \forall x \in C\left(\overline{\Omega}\right).$$

Then there exist two constants c, $\alpha > 0$ independent of t such that any global solution of (18) satisfies,

$$\begin{cases} E(t) \leq ce^{-\alpha t} & \text{if } m(x) = 2, \\ E(t) \leq cE(0)/(1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

Proof: We multiply the first equation of (18) by $uE^r(t)$, for $q > 0$ to be specified later, and integrate over $\Omega \times (s,T)$, $s < T$, we get

$$\int_s^T E^r(t)\int_\Omega \begin{pmatrix} \frac{d}{dt}(uu_t) - u_t^2 + |\nabla u|^2 + |\nabla u|^{p(x)} + \mu_1 uu_t(x,t)\,|u_t(x,t)|^{m(x)-2} \\ +\mu_2 uz(x,1,t)\,|z(x,1,t)|^{m(x)-2} \end{pmatrix} dxdt = 0. \tag{21}$$

From the definition of $E(t)$, given in (19), and the relation

$$\frac{d}{dt}\left(E^r(t)\int_\Omega uu_t dx\right) = rE^{r-1}(t)\,E'(t)\int_\Omega uu_t dx + E^r(t)\,\frac{d}{dt}\int_\Omega uu_t dx,$$

the equation (21) becomes

$$\begin{aligned} 2\textstyle\int_s^T E^{r+1}(t)\,dt \leq & -\textstyle\int_s^T \frac{d}{dt}\left(E^r(t)\int_\Omega uu_t dx\right)dt + r\int_s^T E^{r-1}(t)\,E'(t)\int_\Omega uu_t dxdt \\ & +2\textstyle\int_s^T E^r(t)\int_\Omega u_t^2 dxdt - \mu_1\int_s^T E^r(t)\int_\Omega uu_t\,|u_t|^{m(x)-2}\,dxdt \\ & -\mu_2\textstyle\int_s^T E^r(t)\int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dxdt \\ & +2\textstyle\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt. \end{aligned} \tag{22}$$

We estimate the right-hand side terms of equation (22), respectively.

As a result, we arrive at

$$\begin{aligned} \int_s^T E^{r+1}(t)\,dt & \leq cE(s) + cM\int_s^T E^r(t)\int_\Omega |z(x,1,t)|^{m(x)}\,dxdt \\ & \leq cE(s). \end{aligned} \tag{23}$$

Hence, by taking $T \to \infty$, we obtain

$$\int_s^\infty E^{\frac{m^+}{2}}(t)\,dt \leq cE(s).$$

Thus, Komornik's Lemma (with $\sigma = r = m^+/2 - 1$) implies the desired result. $\square$

5 Conclusion

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there were no blow-up and decay result for the nonlinear p-Laplacian equation with delay term and variable exponents. Firstly, we have been obtained the blow-up result. Later, we have been proved the decay result for our problem under the sufficient conditions in a bounded domain.

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