



Volterra operators between limits of Bergman-type weighted spaces of analytic functions

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Abstract

We characterize continuity and compactness of the Volterra integral operator T_g with the non-constant analytic symbol g between certain weighted Fréchet or (LB)-spaces of analytic functions on the open unit disc, which arise as projective (resp. inductive) limits of intersections (resp. unions) of Bergman spaces of order $1 < p < \infty$ induced by the standard radial weight $(1 - |z|^2)^\alpha$ for $0 < \alpha < \infty$. Motivated from the earlier results obtained for weighted Bergman spaces of standard weight, we also establish several results concerning the spectrum of the Volterra operators acting on the weighted Bergman Fréchet space $A_{\alpha+}^p$, and acting on the weighted Bergman (LB)-space $A_{\alpha-}^p$.

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1. Introduction

The aim of this note is to investigate continuity, compactness and spectrum of the Volterra integral operator defined on certain weighted Fréchet or (LB)-spaces of analytic functions on the open unit disc \mathbb{D} of the complex plane. For a non-constant analytic function $g \in H(\mathbb{D})$, the Volterra operator T_g with symbol g is defined on the space $H(\mathbb{D})$ of analytic functions on the unit disc by

$$T_g(f)(z) := \int_0^z f(\zeta)g'(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

We investigate the Volterra operator T_g when it is defined on spaces which appear as intersections or unions of the weighted Bergman spaces of analytic functions described below. The Bergman space $A_\alpha^p = A_\alpha^p(\mathbb{D})$ of order $1 < p < \infty$ induced by the standard radial weight $(1 - |z|^2)^\alpha$ for $-1 < \alpha < \infty$ is given by

$$A_\alpha^p := \{f \in H(\mathbb{D}) : \|f\|_{p,\alpha} = \left((\alpha + 1) \int_{\mathbb{D}} |f(z)|^p ds_\alpha(z) \right)^{1/p} < \infty\}, \quad (1.1)$$

$ds_\alpha(z) = (1 - |z|^2)^\alpha ds(z)$, and $ds(z) = \frac{1}{\pi} dx dy$. One might also use the weight $(1 - |z|)^\alpha$ instead of $(1 - |z|^2)^\alpha$. Since $1 - |z| \leq 1 - |z|^2 \leq 2(1 - |z|)$, these weights induce the same space and the norms are equivalent. Each A_α^p is a closed subspace of $L^p(\mathbb{D}, ds(z))$ in which the polynomials are dense [15, Section 1.1]. The weighted Bergman space A_α^p is a Banach space with the norm $\|\cdot\|_{p,\alpha}$. Classical Bergman space $A^p(\mathbb{D})$ corresponds to the case $\alpha = 0$. We denote by A_ω^p a weighted Bergman space of order $1 < p < \infty$ with the weight function $\omega(z)$ such that it becomes a Banach space with the norm

$$\|f\|_{p,\omega} := \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) ds(z) \right)^{1/p}.$$

The aim of this paper is to investigate the Volterra operator T_g on spaces that arise as intersections and unions of Bergman spaces of order $1 < p < \infty$ induced by the standard radial weights $(1 - |z|^2)^\alpha$ for $0 < \alpha < \infty$. The weighted Bergman Fréchet space $A_{\alpha+}^p$ is given by

$$\begin{aligned} A_{\alpha+}^p &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p ds_\mu(z) \right)^{1/p} < \infty, \forall \mu > \alpha\} \\ &= \bigcap_{\mu > \alpha} A_\mu^p = \bigcap_{n \in \mathbb{N}} A_{(\alpha + \frac{1}{n})}^p = \text{proj}_{n \in \mathbb{N}} A_{(\alpha + \frac{1}{n})}^p. \end{aligned} \tag{1.2}$$

The weighted Bergman (LB)-space $A_{\alpha-}^p$ is defined by

$$\begin{aligned} A_{\alpha-}^p &:= \{f \in H(\mathbb{D}) : \left(\int_{\mathbb{D}} |f(z)|^p ds_\mu(z) \right)^{1/p} < \infty, \text{ for some } \mu < \alpha\} \\ &= \bigcup_{\mu < \alpha} A_\mu^p = \bigcup_{n \in \mathbb{N}} A_{(\alpha - \frac{1}{n})}^p = \text{ind}_{n \in \mathbb{N}} A_{(\alpha - \frac{1}{n})}^p, \end{aligned} \tag{1.3}$$

where the inductive limit is taken over all $n \in \mathbb{N}$ such that $(\alpha - \frac{1}{n}) > 0$. The monograph [15] presents an investigation of Bergman type spaces (for $p = \infty$) in that fashion with relevance to interpolation and sampling of analytic functions. The paper [18] gives a description of intersections and unions of weighted Bergman spaces of order $0 < p < \infty$. Unlike those, we treat the space $A_{\alpha+}^p$ as a Fréchet space when equipped with the locally convex topology generated by the increasing system of norms

$$\|f\|_{p,\alpha,n} := \left(\int_{\mathbb{D}} |f(z)|^p ds_{(\alpha + \frac{1}{n})}(z) \right)^{1/p}, \tag{1.4}$$

for $f \in A_{\alpha+}^p$ and each $n \in \mathbb{N}$. Analogously, $A_{\alpha-}^p$ is an (LB)-space endowed with the finest locally convex topology such that each natural inclusion map from A_μ^p into A_γ^p , for $0 < \mu < \gamma$ is continuous. It is also regular, since every bounded set $B \subseteq A_{\alpha-}^p$ is contained and bounded in the Banach space A_μ^p , for some $0 < \mu < \alpha$. We also note that, for $1 < p < \infty$ and $0 < \alpha < \infty$ we have $A_{\alpha-}^p \subset A_\alpha^p \subset A_{\alpha+}^p$, where each inclusion is continuous. Since for each pair $0 < \mu < \gamma < \infty$ the canonical inclusion map $\iota: A_\mu^p \rightarrow A_\gamma^p$ is compact (see, for instance, [17, Proposition 3.1] for a proof), both $A_{\alpha+}^p$ and $A_{\alpha-}^p$ are Schwartz spaces by [16, §21.1, Example 1(b)] and [20, Proposition 25.20]. We characterize continuity and compactness of T_g between different weighted Bergman Fréchet and (LB)-spaces in Section 2. In connection with previous significant results obtained for Banach spaces, Section 3 is devoted to the spectral properties of T_g acting on a weighted Bergman Fréchet or (LB)-space.

If $g(z) = z$, for $z \in \mathbb{D}$, then the Volterra operator coincides with the integration operator J . If we let $g(z) = -\log(1 - z)$, for $z \in \mathbb{D}$, then $C(f)(z) := \frac{1}{z} T_g(f)(z)$, $z \in \mathbb{D}$, $z \neq 0$, $C(f)(0) := f(0)$ is the Cesàro operator. This operator acting on $A_{\alpha+}^p$ and $A_{\alpha-}^p$ was investigated by the author in [17].

The Volterra operator for holomorphic functions on the unit disc was introduced by Pommerenke [21]. He showed that T_g is bounded on the Hardy space H^2 if and only if $g \in BMOA$. Aleman and Siskakis extended this result to H^p , $1 \leq p < \infty$ [8]. Later they also considered the case of weighted Bergman spaces [9]. We also refer to [4, 10]. Aleman and Constantin [5], Aleman and Persson [7], Aleman and Paláez [6] studied the spectra of Volterra and Cesàro operators on several spaces of holomorphic functions on the unit disc. For a survey regarding Volterra operators defined on several spaces of analytic functions, see [23].

The motivation of the present research is due to the investigation on the spectrum of Volterra-type operators on weighted Bergman spaces carried out by Aleman and Constantin in [5], and the study of the author [17] where continuity, compactness and spectrum of Cesàro operator in the same context of unions and intersections of weighted Bergman spaces considered here inspired by the works of Albanese, Bonet and Ricker [1, 3]. The Volterra operator is closely related to Cesàro operator. Our results, however, requires different approaches and new ingredients. For an analogue study concerning the Volterra operator in limits of growth spaces of analytic functions, we refer to Bonet [12] in relation with [19].

2. Continuous of Volterra operators on $A_{\alpha+}^p$ and $A_{\alpha-}^p$

The Volterra operator T_g , where g belongs to $H(\mathbb{D})$ is continuous on the weighted Bergman space A_{α}^p if and only if g belongs to the Bloch space \mathcal{B}_1 . This is proved in [9, Theorem 1]. Let us remind that a holomorphic function g belongs to \mathcal{B}_{α} for $0 \leq \alpha < \infty$ if and only if $\sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2)^{\alpha} < \infty$. The function $g \in H(\mathbb{D})$ belongs to the little Bloch space $\mathcal{B}_{\alpha}^0 \subset \mathcal{B}_{\alpha}$ if and only if $\lim_{|z| \rightarrow 1^-} |g'(z)|(1 - |z|^2)^{\alpha} = 0$. We refer the reader to [24] for further information on Bloch spaces. The following result is due to [22, Theorem 2]. It shall be quoted frequently in our proofs.

Proposition 2.1. *Let $g \in H(\mathbb{D})$ be an analytic function. Suppose that $1 < p, q < \infty$, and $-1 < \alpha, \beta < \infty$.*

- (i) *Let $p \leq q$ and $\frac{2+\alpha}{p} - \frac{2+\beta}{q} \leq 1$. Then, the Volterra operator $T_g: A_{\alpha}^p \rightarrow A_{\beta}^q$ is continuous if and only if $g \in \mathcal{B}_{1+\frac{2+\beta}{q}-\frac{2+\alpha}{p}}$. In particular, if $p = q$ and $\alpha = \beta$, $T_g: A_{\alpha}^p \rightarrow A_{\alpha}^p$ is continuous if and only if $g \in \mathcal{B}_1$.*
- (ii) *Let $q < p$ and $\frac{2+\alpha}{p} - \frac{2+\beta}{q} \leq 1$. Then, the Volterra operator $T_g: A_{\alpha}^p \rightarrow A_{\beta}^q$ is continuous if and only if $g' \in A_{s+\gamma}^s$, where $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$ and $\gamma = \frac{p\beta - q\alpha}{p - q}$.*
- (iii) *Let $\frac{2+\alpha}{p} - \frac{2+\beta}{q} > 1$, and suppose that $T_g: A_{\alpha}^p \rightarrow A_{\beta}^q$ be continuous. Then, g is constant.*

Proposition 2.2. *Let $1 < p \leq q < \infty$ and $0 < \alpha, \beta < \infty$ satisfy $\gamma := 1 + \frac{2+\beta}{q} - \frac{2+\alpha}{p} \geq 0$. Let $g \in H(\mathbb{D})$ be an analytic function.*

- (1) *The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is continuous if and only if*

$$g \in \bigcap_{\tau \in (\gamma, x)} \mathcal{B}_{\tau}, \tag{2.1}$$

for every $x > \gamma$. In particular,

- (i) *The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\alpha+}^p$ is continuous if and only if*

$$g \in \bigcap_{\tau \in (1, x)} \mathcal{B}_{\tau}, \tag{2.2}$$

for every $x > 1$.

- (ii) *Let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$. The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is continuous if g satisfies (2.2) for every $x > 1$.*

- (iii) Let $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$. If the Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is continuous, then g satisfies (2.2) for every $x > 1$.
- (2) The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is continuous if and only if g satisfies (2.1) for $x = \gamma + \min\{(\alpha + 1)\frac{q-1}{pq}, (\beta + 1)\frac{q-1}{q}\}$. In particular,
 - (i) The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\alpha-}^p$ is continuous if and only if g satisfies (2.2) for $x = 1 + \frac{\alpha+1}{2p}$.
 - (ii) Let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$. The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is continuous if g satisfies (2.2) for $x = 1 + \frac{\alpha+1}{2p}$.
 - (iii) Let $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$. If the Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is continuous, then g satisfies (2.2) for $x = 1 + \frac{\min\{\alpha,\beta\}+1}{p}$.

Proof. (1) Since the polynomials are dense in each of A_{μ}^p , for every $\mu > \alpha$, by (1.2), the Fréchet space $A_{\alpha+}^p$ is a reduced projective limit of the Banach spaces A_{μ}^p , $\mu > \alpha$. This implies, $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is continuous if and only if for each $\mu > \beta$ there is $\alpha < \eta < \alpha + \mu - \beta$ such that $T_g: A_{\eta}^p \rightarrow A_{\mu}^q$ is continuous. Now let g satisfy (2.1) for every $x > \gamma$. For any $\varepsilon > 0$, find $\delta \in (0, \varepsilon)$ such that $g \in \mathcal{B}_{\gamma+\varepsilon-\delta}$. Given $\mu := \beta + q\varepsilon$, define $\eta := \alpha + p\delta$ and let us observe that for every $z \in \mathbb{D}$,

$$|g'(z)|(1 - |z|^2)^{1+\frac{2+\mu}{q}-\frac{2+\eta}{p}} = |g'(z)|(1 - |z|^2)^{\gamma+\varepsilon-\delta}. \tag{2.3}$$

Hence $g \in \mathcal{B}_{1+\frac{2+\mu}{q}-\frac{2+\eta}{p}}$. So $T_g: A_{\eta}^p \rightarrow A_{\mu}^q$ is continuous by Proposition 2.1(i). Therefore $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is also continuous. For the converse, let $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ be continuous. Then, for every $\varepsilon > 0$, given $\mu := \beta + q\varepsilon$ there exists $\alpha < \eta < \alpha + q\varepsilon$ such that $T_g: A_{\eta}^p \rightarrow A_{\mu}^q$ is continuous. By Proposition 2.1(i), $g \in \mathcal{B}_{1+\frac{2+\mu}{q}-\frac{2+\eta}{p}}$. Then, observe that for every $z \in \mathbb{D}$,

$$|g'(z)|(1 - |z|^2)^{1+\frac{2+\mu}{q}-\frac{2+\eta}{p}} > |g'(z)|(1 - |z|^2)^{\gamma+\varepsilon}. \tag{2.4}$$

Hence, $g \in \mathcal{B}_{\gamma+\varepsilon}$ for every $\varepsilon > 0$. So (2.1) holds for all $x > \gamma$. In particular, if we let $p = q$ and $\alpha = \beta$, part (i) follows immediately. If we let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$, then by (2.3), part (ii) follows. If we suppose $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$, then by (2.4), part (iii) holds.

(2) Let $x = \min\{(\alpha+1)\frac{q-1}{pq}, (\beta+1)\frac{q-1}{q}\}$. By Grothendieck’s factorization theorem (see e.g. [20, Theorem 24.33]), $T_g: A_{\alpha-}^p \rightarrow A_{\alpha-}^p$ is continuous if and only if for every $-1 < \mu < \alpha$, there exists $\mu < \eta < \alpha$ such that $T_g: A_{\mu}^p \rightarrow A_{\eta}^p$ is continuous. Suppose that for every $\varepsilon \in (0, x)$ we have $g \in \mathcal{B}_{\gamma+\varepsilon}$. Then, given $-1 < \mu := \alpha - \frac{pq}{q-1}\varepsilon$ define $-1 < \eta := \beta - \frac{q}{q-1}\varepsilon$. Observe that for every $z \in \mathbb{D}$,

$$\begin{aligned} |g'(z)|(1 - |z|^2)^{1+\frac{2+\eta}{q}-\frac{2+\mu}{p}} &= |g'(z)|(1 - |z|^2)^{\gamma+\frac{q}{q-1}\varepsilon-\frac{1}{q-1}\varepsilon} \\ &= |g'(z)|(1 - |z|^2)^{\gamma+\varepsilon}. \end{aligned} \tag{2.5}$$

Thus, $g \in \mathcal{B}_{1+\frac{2+\eta}{q}-\frac{2+\mu}{p}}$. By Proposition 2.1(i), $T_g: A_{\mu}^p \rightarrow A_{\eta}^q$ is continuous. Then, $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is also continuous. For the converse, let $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ be continuous. Then, for every $-1 < \mu := \alpha - p\varepsilon$ there exists $-1 < \beta - p\varepsilon < \eta := \beta - p\delta$ such that $T_g: A_{\mu}^p \rightarrow A_{\eta}^q$ is continuous. Then, by Proposition 2.1(i), $g \in \mathcal{B}_{1+\frac{2+\eta}{q}-\frac{2+\mu}{p}}$. Thus, for every $z \in \mathbb{D}$,

$$|g'(z)|(1 - |z|^2)^{1+\frac{2+\eta}{q}-\frac{2+\mu}{p}} = |g'(z)|(1 - |z|^2)^{\gamma+\varepsilon-\frac{p}{q}\delta}. \tag{2.6}$$

Hence, there exists $\hat{\delta} \in (0, \frac{p}{q}\varepsilon]$ such that $g \in \mathcal{B}_{\gamma+\varepsilon-\hat{\delta}}$. This happens when (2.1) is satisfied. In particular, if we let $p = q$ and $\alpha = \beta$, part (i) follows immediately. If we let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$, then by (2.5), part (ii) follows. If we suppose $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$, then by (2.6) we get part (iii). \square

Remark 2.3. Let $\gamma := 1 + \frac{2+\beta}{q} - \frac{2+\alpha}{p} \geq 0$. Concerning Proposition 2.2, suppose that $g \in \bigcap_{\tau > \gamma} \mathcal{B}_\tau \setminus \mathcal{B}_\gamma$. Then, for every $\varepsilon > 0$, we have $g \in \mathcal{B}_{\gamma+\varepsilon} \setminus \mathcal{B}_\gamma$. Then, given $\mu := \beta + q\varepsilon$, for every $z \in \mathbb{D}$,

$$|g'(z)|(1 - |z|^2)^{1 + \frac{2+\mu}{q} - \frac{2+\alpha}{p}} = |g'(z)|(1 - |z|^2)^{\gamma+\varepsilon}.$$

So, $g \in \mathcal{B}_{1 + \frac{2+\mu}{q} - \frac{2+\alpha}{p}}$. Hence $T_g: A_\alpha^p \rightarrow A_\mu^q$ is continuous. Thus, the continuity estimate

$$\|T_g f\|_{q, \beta+q\varepsilon} \lesssim \|f\|_{p, \alpha}, \quad f \in A_{\alpha+}^p. \tag{2.7}$$

is optimal. That is, we cannot get rid of the term $q\varepsilon$, by the density argument we also used in Proposition 2.2.

The following proposition directly follows from Proposition 2.1(ii).

Proposition 2.4. *Let $1 < q < p < \infty$ and $0 < \alpha, \beta < \infty$ satisfy $\frac{2+\alpha}{p} - \frac{2+\beta}{q} \leq 1$. Define $\frac{1}{s} = \frac{1}{q} - \frac{1}{p}$. Then,*

- (1) *The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is continuous if and only if for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon]$ such that $g' \in A_{s+\gamma}^s$, where $\gamma = \frac{p(\beta+\varepsilon) - p(\alpha+\delta)}{p-q}$.*
- (2) *The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is continuous if and only if for every $\varepsilon \in (0, \beta + 1)$ there exists $\delta \in (0, \varepsilon]$ such that $g' \in A_{s+\mu}^s$, where $\mu = \frac{p(\beta-\delta) - p(\alpha-\varepsilon)}{p-q}$.*

An operator $T: X \rightarrow X$ on a space X is called *compact* (resp. *bounded*) if there exists a zero neighborhood U such that $T(U)$ is a relatively compact (resp. bounded) subset of X . If the bounded subsets of X are relatively compact, i.e., X is a Schwartz space, then bounded operators coincides with compact operators. By [17, Corollary 3.2], this is the case for $A_{\alpha+}^p$ and $A_{\alpha-}^p$. Therefore, Proposition 2.1 will play a crucial role in characterizing compactness of the Volterra operator T_g on $A_{\alpha+}^p$ and $A_{\alpha-}^p$. The following well-known abstract lemma will help us prove our main result on compactness.

Lemma 2.5. (i) *Let $E = \text{proj}_m E_m$ and $F = \text{proj}_n F_n$ be Fréchet spaces such that E (resp. F) is the intersection of the sequence of Banach spaces E_m (resp. F_n), E is dense in E_m and $E_{m+1} \subset E_m$ with continuous inclusion for each m (resp. F is dense in F_n and $F_{n+1} \subset F_n$ with continuous inclusion for each n). Let $T: E \rightarrow F$ be a linear operator. Assume T is continuous. Then T is bounded if and only if there is m such that for each n , T has a unique continuous linear extension $T_{m,n}: E_m \rightarrow F_n$.*

(ii) *Let $X = \text{ind } X_n$ and $Y = \text{ind } Y_m$ be two (LB)-spaces which are increasing unions of Banach spaces $X = \bigcup_{n=1}^\infty X_n$ and $Y = \bigcup_{m=1}^\infty Y_m$. Let $T: X \rightarrow Y$ be a continuous linear map. Assume that Y is a regular (LB)-space. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(X_n) \subset Y_m$ and $T: X_n \rightarrow Y_m$ is continuous for all $n \geq m$.*

Proposition 2.6. *Let $1 < p \leq q < \infty$, and $0 < \alpha, \beta < \infty$ satisfy $\gamma := 1 + \frac{2+\beta}{q} - \frac{2+\alpha}{p} > 0$. Let $g \in H(\mathbb{D})$ be an analytic function.*

- (1) *The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact if and only if*

$$g \in \bigcup_{\tau < \gamma} \mathcal{B}_\tau. \tag{2.8}$$

In particular,

- (i) *The Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\alpha+}^p$ is compact if and only if*

$$g \in \bigcup_{\tau < x} \mathcal{B}_\tau, \tag{2.9}$$

for $x = 1$.

- (ii) Let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$. Then, the Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact if g satisfies (2.9) for $x = 1$.
- (iii) Let $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$. If the Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact, then g satisfies (2.9) for $x = 1$.
- (2) The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is compact if and only if g satisfies (2.8). In particular,
 - (i) The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\alpha-}^p$ is compact if and only if g satisfies (2.9) for $x = \min\{1, \frac{\alpha+1}{p}\}$.
 - (ii) Let $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$. If the Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is compact, then g satisfies (2.9) for $x = \frac{\min\{\alpha, \beta\}+1}{q}$.
 - (iii) Let $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$. The Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is compact if g satisfies (2.9) for $x = \frac{\alpha+1}{q}$.

Proof. (1) Suppose that $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact. Since $A_{\alpha+}^p$ is a Fréchet-Schwartz space for any fixed $1 < p < \infty$ and $0 < \alpha < \infty$, the Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is bounded. By Lemma 2.5(i), there exists $\varepsilon > 0$ such that for $\mu := \alpha + p\varepsilon$, every $\eta := \beta + p\delta < \beta + p\varepsilon$ yields $T_g: A_{\mu}^p \rightarrow A_{\eta}^q$ is continuous. If $1 + \frac{2+\eta}{q} - \frac{2+\mu}{p} < 0$, then by Proposition 2.1(iii), g is constant, so $g \in \mathcal{B}_a$, for any $a \in [0, \infty)$. Now let $1 + \frac{2+\eta}{q} - \frac{2+\mu}{p} \geq 0$. By Proposition 2.1(i), $g \in \mathcal{B}_{1+\frac{2+\eta}{q}-\frac{2+\mu}{p}} = \mathcal{B}_{\gamma+\frac{p}{q}\delta-\varepsilon}$. Furthermore,

$$|g'(z)|(1 - |z|^2)^{\gamma+\frac{p}{q}\delta-\varepsilon} > |g'(z)|(1 - |z|^2)^{\gamma+\delta-\varepsilon}. \tag{2.10}$$

Therefore, $g \in \mathcal{B}_{\gamma-\varepsilon+\delta}$. This gives the desired result. For the converse, let there exist $\varepsilon \in (0, \gamma]$ such that $g \in \mathcal{B}_{\gamma-\varepsilon}$. For any $\delta \in (0, \varepsilon]$ we have

$$|g'(z)|(1 - |z|^2)^{\gamma+\frac{p}{q}\delta-\varepsilon} < |g'(z)|(1 - |z|^2)^{\gamma-\varepsilon}. \tag{2.11}$$

So, $g \in \mathcal{B}_{\gamma+\frac{p}{q}\delta-\varepsilon}$. Then, by Proposition 2.1(i), $T_g: A_{\alpha+p\varepsilon}^p \rightarrow A_{\beta+p\delta}^q$ is continuous. By Lemma 2.5(i), $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact. In particular, if $\alpha = \beta$ and $p = q$, then part (i) is also true. If suppose that $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$, then by (2.10) part (iii) follows. If we suppose that $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$, then (2.11) yields part (ii).

(2) Define $x = \frac{\min\{\alpha, \beta\}+1}{q}$. Let $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ be compact, equivalently bounded. Then, by Lemma 2.5(ii), there exists $\varepsilon \in (0, x)$ such that for $-1 < \mu := \beta - q\varepsilon$, every $-1 < \alpha - q\varepsilon < \eta < \alpha$ yields $T_g: A_{\eta}^p \rightarrow A_{\mu}^q$ is continuous. If $1 + \frac{2+\mu}{q} - \frac{2+\eta}{p} < 0$, by Proposition 2.1(iii), g is constant, and there is nothing to prove. Suppose $1 + \frac{2+\mu}{q} - \frac{2+\eta}{p} \geq 0$. Note that for any $\delta \in (0, \varepsilon]$, $T_g: A_{\alpha-p\delta}^p \rightarrow A_{\mu}^q$ is continuous. By Proposition 2.1(i), $g \in \mathcal{B}_{\gamma-\varepsilon+\delta}$. Hence, (2.8) is satisfied. For the converse, let there exist $\varepsilon \in (0, x)$ such that for all $\delta \in (0, \varepsilon]$ we have $g \in \mathcal{B}_{\gamma-\varepsilon+\delta}$. Note that

$$|g'(z)|(1 - |z|^2)^{\gamma+\frac{q}{p}\delta-\varepsilon} < |g'(z)|(1 - |z|^2)^{\gamma-\varepsilon+\delta}. \tag{2.12}$$

This means $g \in \mathcal{B}_{\gamma+\frac{q}{p}\delta-\varepsilon}$. By Proposition 2.1(i), $T_g: A_{\alpha-q\delta}^p \rightarrow A_{\beta-q\varepsilon}^q$ is continuous. By Lemma 2.5(ii), $T_g: A_{\alpha-}^p \rightarrow A_{\beta-}^q$ is compact. In particular, letting $p = q$ and $\alpha = \beta$ yields part (i). If we let $\frac{2+\beta}{q} < \frac{2+\alpha}{p}$, then $g \in \mathcal{B}_{1-\varepsilon+\delta}$, and part (ii) follows. If we take $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$, then by (2.12) part (iii) holds. □

Remark 2.7. Let $\gamma := 1 + \frac{2+\beta}{q} - \frac{2+\alpha}{p} \geq 0$. Note that by [22, Corollary 7(i)], the Volterra operator $T_g: A_\alpha^p \rightarrow A_\beta^q$ is compact if and only if $g \in \mathcal{B}_\gamma^0$. Concerning Proposition 2.6, let $g \in \bigcup_{\tau < \gamma} \mathcal{B}_\tau$. So there exists $\varepsilon > 0$ with $g \in \mathcal{B}_{\gamma-\varepsilon}$. This means there exists $M > 0$ such that $|g'(z)|(1 - |z|^2)^{\gamma-\varepsilon} \leq M$, for every $z \in \mathbb{D}$. Let us observe that

$$\begin{aligned} |g'(z)|(1 - |z|^2)^\gamma &= |g'(z)|(1 - |z|^2)^\varepsilon(1 - |z|^2)^{\gamma-\varepsilon} \\ &\leq M(1 - |z|^2)^\varepsilon \rightarrow 0, \end{aligned}$$

as $|z| \rightarrow 1^-$. So $g \in \mathcal{B}_\gamma^0$. Hence $T_g: A_\alpha^p \rightarrow A_\beta^q$ is compact. Now let $g \in \mathcal{B}_\gamma^0 \setminus \bigcup_{\tau < \gamma} \mathcal{B}_\tau$. Then, $T_g: A_\alpha^p \rightarrow A_\beta^q$ is compact. Suppose that $T_g: A_{\alpha+}^p \rightarrow A_{\beta+}^q$ is compact as well. Then, there exists $\varepsilon \in (0, \gamma]$ such that for $\mu := \alpha + p\varepsilon$ one has $T_g: A_\mu^p \rightarrow A_\beta^q$ is continuous. This means,

$$\infty > \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2)^{1 + \frac{2+\beta}{q} - \frac{2+\mu}{p}} = \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2)^{\gamma-\varepsilon} = \infty.$$

This is a contradiction.

3. Spectra of Volterra operators on $A_{\alpha+}^p$ and $A_{\alpha-}^p$

Let E be a locally convex Hausdorff space, and Γ_E a system of continuous seminorms determining the topology of E . Let E' denote the space of all continuous linear functionals on E . Denote the identity operator on E by I . Let $\mathcal{L}(E)$ denote the space of all continuous linear operators from E into itself. For $T \in \mathcal{L}(E)$, the *resolvent set* $\rho(T; E)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ exists in $\mathcal{L}(E)$. The set $\sigma(T; E) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T . The *point spectrum* $\sigma_{pt}(T; E)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)$ is not injective. Contrary to Banach spaces, concerning the spectrum of an operator T on the Fréchet space E , one may encounter that $\rho(T; E) = \emptyset$ or $\rho(T; E)$ fails to be an open set in \mathbb{C} . For this reason, some authors prefer to consider the subset $\rho^*(T; E)$ of $\rho(T; E)$ consisting of $\lambda \in \mathbb{C}$ such that there exists $\delta > 0$ such that $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\} \subseteq \rho(T; E)$ and $\{R(\mu, T) : \mu \in B(\lambda, \delta)\}$ is equicontinuous in $\mathcal{L}(E)$. Define the *Waelbroeck spectrum* $\sigma^*(T; E) := \mathbb{C} \setminus \rho^*(T; E)$, which is a closed set containing $\sigma(T; E)$. If $T \in \mathcal{L}(E)$ with E a Banach space, then $\sigma^*(T; E) = \sigma(T; E)$.

Given $\lambda \in \mathbb{C} \setminus \{0\}$, and $h \in H(\mathbb{D})$, the equation $f - \frac{1}{\lambda}T_g f = h$ has a unique solution given by

$$f(z) = R_{\lambda,g}h(z) := h(0)e^{g(z)/\lambda} + e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} h'(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Lemma 3.1 and Lemma 3.2 are taken from [12]. Their proofs are very close to the case of entire functions which were considered in [11].

Lemma 3.1. *Let E be a locally convex space continuously contained in $H(\mathbb{D})$ that contains the constants. Assume that $T_g: E \rightarrow E$ is continuous for some non-constant entire function g such that $g(0) = 0$. Then, the following statements are equivalent:*

- (i) $\lambda \in \rho(T_g; E)$.
- (ii) $R_{\lambda,g}: E \rightarrow E$ is continuous.
- (iii) $e^{g(z)/\lambda} \in E$ and $S_{\lambda,g}h := e^{g(z)/\lambda} \int_0^z e^{-g(\zeta)/\lambda} h'(\zeta) d\zeta, z \in \mathbb{C}$ is continuous on the subspace E_0 of E of all functions $h \in E$ with $h(0) = 0$.

In connection with Lemma 3.1, we denote A_α^{p0} as the subspace of A_α^p which stands for functions $h \in A_\alpha^p$ satisfying $h(0) = 0$. Similar notation shall be adapted for $A_{\alpha+}^p$ and $A_{\alpha-}^p$ throughout.

Lemma 3.2. *Let E be a locally convex space continuously contained in $H(\mathbb{D})$ that contains the constants. Assume that $T_g: E \rightarrow E$ is continuous for some non-constant entire function g such that $g(0) = 0$. Then,*

$$\{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\}: e^{g/\lambda} \notin E\} \subset \sigma(T_g; E).$$

The following abstract spectral results for Fréchet and (LB)-spaces will be useful in proving our assertions.

Lemma 3.3 ([2, Lemma 2.1]). *Let $E = \bigcap_{n \in \mathbb{N}} E_n$ be a Fréchet space which is the intersection of a sequence of Banach spaces $((E_n, \|\cdot\|_n))_{n \in \mathbb{N}}$ satisfying $E_{n+1} \subseteq E_n$ with $\|x\|_n \leq \|x\|_{n+1}$, for all $n \in \mathbb{N}$ and $x \in E_{n+1}$. Let $T \in \mathcal{L}(E)$. Suppose that for all $n \in \mathbb{N}$, there exists $T_n \in \mathcal{L}(E_n)$ such that the restriction of T_n to E (resp. of T_n to E_{n+1}) coincides with T (resp. T_{n+1}). Then, the following statements hold:*

(i) $\sigma(T; E) \subseteq \bigcup_{n \in \mathbb{N}} \sigma(T_n; E_n)$ and $R(\lambda, T)$ coincides with the restriction of $R(\lambda, T_n)$ to E , for all $n \in \mathbb{N}$ and each $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(T_n; E_n)$.

(ii) If $\bigcup_{n \in \mathbb{N}} \sigma(T_n; E_n) \subseteq \overline{\sigma(T; E)}$, then

$$\sigma^*(T; E) = \overline{\sigma(T; E)}.$$

Lemma 3.4 ([3, Lemma 5.2]). *Let $E = \text{ind}_{n \in \mathbb{N}}(E_n, \|\cdot\|_n)$ be a Hausdorff inductive limit of Banach spaces. Let $T \in \mathcal{L}(E)$. Suppose that for each $n \in \mathbb{N}$, the restriction T_n of T to E_n maps E_n into itself and $T_n \in \mathcal{L}(E_n)$. Then, the following properties are satisfied:*

(i) $\sigma_{pt}(T; E) = \bigcup_{n \in \mathbb{N}} \sigma_{pt}(T_n; E_n)$.

(ii) $\sigma(T; E) \subseteq \bigcap_{m \in \mathbb{N}} (\bigcup_{n=m}^{\infty} \sigma(T_n; E_n))$. Moreover, if $\lambda \in \bigcap_{n=m}^{\infty} \rho(T_n; E_n)$ for some $m \in \mathbb{N}$, then $R(\lambda, T_n)$ coincides with the restriction of $R(\lambda, T)$ to E_n , for every $n \geq m$.

(iii) If $\bigcup_{n=m}^{\infty} \sigma(T_n; E_n) \subseteq \overline{\sigma(T; E)}$, for some $m \in \mathbb{N}$, then

$$\sigma^*(T; E) = \overline{\sigma(T; E)}.$$

For every $z \in \mathbb{D}$, we define the Möbius transformation

$$\varphi_z(w) := \frac{z - w}{1 - \bar{z}w}, \quad w \in \mathbb{D}.$$

It is routine to check that the change of variables $u = \varphi_z(w)$ gives

$$1 - |u|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}, \quad J_u = \frac{(1 - |z|^2)^2}{|1 - \bar{z}w|^4}, \tag{3.1}$$

and $|1 - \bar{z}w| = \frac{1 - |z|^2}{|1 - \bar{z}u|}.$

where J_u is the real Jacobian term. For $1 < p < \infty$, and $-1 < \alpha < \infty$ we have the relation

$$\|f\|_{p,\alpha} \asymp \|f'\|_{p,p+\alpha} + |f(0)|, \quad \forall f \in H(\mathbb{D}). \tag{3.2}$$

One inequality is a classical result due to Hardy-Littlewood [14, Theorem 5.6], and the reverse inequality is easily proved by the methods in [14, Chapter 5]. Now we state our first result concerning the spectrum of the Volterra operator.

Proposition 3.5. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. If $g \in \mathcal{B}_1$, then*

(1) *The spectrum of Volterra operator $T_g: A_{\alpha+}^p \rightarrow A_{\alpha+}^p$ satisfies*

$$\sigma(T_g; A_{\alpha+}^p) = \bigcup_{\mu > \alpha} \sigma(T_g; A_{\mu}^p).$$

(2) *The spectrum of Volterra operator $T_g: A_{\alpha-}^p \rightarrow A_{\alpha-}^p$ satisfies*

$$\sigma(T_g; A_{\alpha-}^p) = \bigcap_{\mu < \alpha} \left(\bigcup_{\eta \in (\mu, \alpha)} \sigma(T_g; A_{\eta}^p) \right).$$

Proof. (1) One inclusion follows by Lemma 3.3. For the other inclusion, assume that $\lambda \notin \sigma(T_g; A_{\alpha+}^p)$. Then, by Lemma 3.1 $e^{g/\lambda} \in A_{\mu}^p$, for every $\mu > \alpha$. We show $\lambda \notin \bigcup_{\mu > \alpha} \sigma(T_g; A_{\mu}^p)$. Suppose not. Then, there exists $\beta > \alpha$ such that $\lambda \in \sigma(T_g; A_{\beta}^p)$. Applying Lemma 3.1 once again for A_{β}^p , we see that $S_{\lambda, g} \notin \mathcal{L}(A_{\beta}^{p_0})$. On the other hand, using the change of variables $u = \varphi_z(\zeta)$ for an arbitrary $z \in \mathbb{D}$, by (3.1) we obtain

$$\begin{aligned} \infty &> \left\| e^{g/\lambda} \right\|_{p, \beta}^p = (\beta + 1) \int_{\mathbb{D}} \left| e^{\frac{g(u)}{\lambda}} \right|^p (1 - |u|^2)^{\beta} ds(u) \\ &\asymp \int_{\mathbb{D}} \left| e^{\frac{g(\zeta)}{\lambda}} \right|^p \left(\frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2} \right)^{2+\beta} (1 - |\zeta|^2)^{\beta} ds(\zeta). \end{aligned}$$

Then, by [13, Theorem 1], the multiplication operator $M_{e^{g/\lambda}}$ is continuous on A_{β}^p . Thus, $\omega(z) := e^{p \operatorname{Re} \frac{g(z)}{\lambda}} (1 - |z|^2)^{\beta}$ is a weight function in \mathbb{D} , and as illustrated in [5, p. 202] the weighted Bergman space norm induced by ω satisfies the property $\left\| e^{-g/\lambda} S_{\lambda, g} h \right\|_{p, \omega} \asymp \left\| (e^{-g/\lambda} S_{\lambda, g} h)' \right\|_{p, \hat{\omega}} = \left\| e^{-g/\lambda} h' \right\|_{p, \hat{\omega}}$, for $h \in A_{\beta}^{p_0}$ if we denote $\hat{\omega}(z) = \omega(z)(1 - |z|^2)^p$. But then, for every $h \in A_{\beta}^{p_0}$ we obtain by (3.2) that (cf. [5, Theorem 5.1])

$$\begin{aligned} \|S_{\lambda, g} h\|_{p, \beta}^p &= \left\| \frac{1}{e^{g/\lambda}} S_{\lambda, g} h \right\|_{p, \omega}^p \asymp \left\| \left(\frac{1}{e^{g/\lambda}} S_{\lambda, g} h \right)' \right\|_{p, \hat{\omega}}^p = \left\| \frac{1}{e^{g/\lambda}} h' \right\|_{p, \hat{\omega}}^p \\ &= (\beta + p + 1) \int_{\mathbb{D}} |h'(z)|^p (1 - |z|^2)^{\beta+p} ds(z) \\ &= \|h'\|_{p, \beta+p} \asymp \|h\|_{p, \beta}^p, \end{aligned}$$

which implies that $S_{\lambda, g}$ is continuous on $A_{\beta}^{p_0}$. This is a contradiction.

(2) Similarly, one of the inclusions follow by Lemma 3.4. For the other inclusion, assume that $\lambda \notin \sigma(T_g; A_{\alpha-}^p)$. We show $\lambda \notin \bigcap_{\mu < \alpha} (\bigcup_{\eta \in (\mu, \alpha)} \sigma(T_g; A_{\eta}^p))$. Assume contrary. Then, for every $\mu < \alpha$ there exists $\eta \in (\mu, \alpha)$ such that $\lambda \in \sigma(T_g; A_{\eta}^p)$. Since $\lambda \in \rho(T_g; A_{\alpha-}^p)$, by Lemma 3.1 we have $e^{g/\lambda} \in A_{\alpha-}^p$ and so there exists $\beta < \alpha$ such that $e^{g/\lambda} \in A_{\mu}^p$ for every $\mu \in [\beta, \alpha)$. Without loss of any generality, suppose $\eta \geq \beta$. The rest is very similar to part (1). □

Given an analytic function $g \in \mathbb{D}$, we denote by g_z , $z \in \mathbb{D}$ its hyperbolic translate $g_z(\zeta) := g(\varphi_z(\zeta)) - g(z)$, $\zeta \in \mathbb{D}$. The following important result shall be quoted for several occasions.

Theorem 3.6 ([5, Theorem A; B; 5.1]). *Let $p > 0$ and let $\alpha > -1$. Let $g \in \mathcal{B}_1$. Then, the following statements hold:*

- (i) $\sigma_{pt}(T_g; A_\alpha^p) = \emptyset$.
- (ii) *For every $\gamma > -2\alpha - 3$, the spectrum of the Volterra operator $\sigma(T_g; A_\alpha^p)$ includes*

$$\{0\} \cup \left\{ \lambda \in \mathbb{C} : \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^\gamma (1 - |\zeta|^2)^\alpha ds(\zeta) = \infty \right\}. \tag{3.3}$$

If g' is rational, then this condition is equivalent to $e^{g/\lambda} \notin A_\alpha^p$.

- (iii) *There exists $g \in \mathcal{B}_1$ such that $e^{g/\lambda} \in A_\alpha^p$, for all $\lambda \in \mathbb{C} \setminus \{0\}$ but*

$$\sigma(T_g; A_\alpha^p) \supset \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p}{2(2+\alpha)} \right| \leq \frac{p}{2(2+\alpha)} \right\}.$$

- (iv) *If $g'(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}$, $z \in \mathbb{D}$ for some finite measure μ on the unit circle,*

$$\begin{aligned} \sigma(T_g; A_\alpha^p) &= \{0\} \cup \overline{\{\lambda \in \mathbb{C} \setminus \{0\} : e^{g/\lambda} \notin A_\alpha^p\}} \\ &= \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2+\alpha}{p} \right\}. \end{aligned}$$

Theorem 3.7. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. Let $g \in \mathcal{B}_1$. Then, the following statements hold.*

- (1) $\sigma_{pt}(T_g; A_{\alpha+}^p) = \emptyset$.
- (2) *For every $\gamma \geq -2\alpha - 3$, the set $\sigma(T_g; A_{\alpha+}^p)$ contains (3.3).*
- (3) *There exists $g \in \mathcal{B}_1$ such that $e^{g/\lambda} \in A_{\alpha+}^p$, for all $\lambda \in \mathbb{C} \setminus \{0\}$ but*

$$\sigma(T_g; A_{\alpha+}^p) \supset \{0\} \cup \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p}{2(2+\alpha)} \right| < \frac{p}{2(2+\alpha)} \right\}.$$

- (4) *If $g'(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}$, $z \in \mathbb{D}$ for some finite measure μ on the unit circle,*

- (i) *The spectrum of the Volterra operator on $A_{\alpha+}^p$ reads as*

$$\begin{aligned} \sigma(T_g; A_{\alpha+}^p) &= \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} > \frac{2+\alpha}{p} \right\} \\ &= \{0\} \cup \{ \lambda \in \mathbb{C} \setminus \{0\} : e^{g/\lambda} \notin A_{\alpha+}^p \}. \end{aligned}$$

- (ii) $\sigma^*(T_g; A_{\alpha+}^p) = \overline{\sigma(T_g; A_{\alpha+}^p)}$.

Proof. (1) Let $T_g: A_{\alpha+}^p \rightarrow A_{\alpha+}^p$ have an eigenvalue $\lambda \in \mathbb{C}$. Suppose that $f \in A_{\alpha+}^p \setminus \{0\}$ is its corresponding eigenvector. But then, there exists $\mu > \alpha$ with $f \in A_\mu^p$ such that $T_g f = \lambda f$ in A_μ^p hence $\lambda \in \sigma_{pt}(T_g; A_\mu^p)$. This contradicts Theorem 3.6(i).

(2) By Lemma 3.2 we directly have $0 \in \sigma(T_g; A_{\alpha+}^p)$. By Proposition 3.5 and Theorem 3.6(ii) we already know that for every $\gamma > -2\mu - 3$ we have

$$\begin{aligned} \sigma(T_g; A_{\alpha+}^p) &= \bigcup_{\mu > \alpha} \sigma(T_g; A_\mu^p) \\ &\supset \bigcup_{\mu > \alpha} \left\{ \lambda \in \mathbb{C} : \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^\gamma (1 - |\zeta|^2)^\mu ds(\zeta) = \infty \right\}. \end{aligned}$$

We show that the latter includes (3.3). Take any $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying the equation in (3.3), and assume contrary. Then, for every $\varepsilon > 0$, given $\mu := \alpha + \varepsilon$ there exist $\gamma_\mu > -2\mu - 3$ and $c_\mu > 0$ such that

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma_\mu} (1 - |\zeta|^2)^\mu ds(\zeta) < c_\mu. \tag{3.4}$$

Since $\gamma_\mu + 2\varepsilon > -2\alpha - 3$, by (3.1), the change of variables $w = \varphi_z(\zeta)$ yields

$$\begin{aligned} \infty &= \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma_\mu + 2\varepsilon} (1 - |\zeta|^2)^\alpha \, ds(\zeta) \\ &= \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\alpha + 2 + \gamma_\mu + 2\varepsilon}}{e^{p \operatorname{Re} \frac{g(z)}{\lambda}}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{g(w)}{\lambda}} \frac{(1 - |w|^2)^\alpha}{|1 - \bar{z}w|^{2\alpha + 4 + \gamma_\mu + 2\varepsilon}} \frac{(1 - |w|^2)^\varepsilon}{(1 - |w|^2)^\varepsilon} \, ds(w) \\ &\leq \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\mu + 2 + \gamma_\mu}}{e^{p \operatorname{Re} \frac{g(z)}{\lambda}}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{g(w)}{\lambda}} \frac{(1 - |w|^2)^\mu}{|1 - \bar{z}w|^{2\mu + 4 + \gamma_\mu}} \, ds(w). \end{aligned}$$

By the change of variables $u = \varphi_z(w)$ we obtain

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(u)}{\lambda}} |1 - \bar{z}u|^{\gamma_\mu} (1 - |u|^2)^\mu \, ds(u) = \infty.$$

That contradicts (3.4).

(3) Let us choose $g \in \mathcal{B}_1$ as in Theorem 3.6(iii). Since $A_\alpha^p \subseteq A_{\alpha+}^p$, we obtain $e^{g/\lambda} \in A_{\alpha+}^p$, for all $\lambda \in \mathbb{C} \setminus \{0\}$. We know by Lemma 3.2 that $0 \in \sigma(T_g; A_{\alpha+}^p)$. Now let us take $\lambda \in \mathbb{C} \setminus \{0\}$ with $\left| \lambda - \frac{p}{2(2+\alpha)} \right| < \frac{p}{2(2+\alpha)}$. Then, there exists $N \in \mathbb{N}$ such that $\left| \lambda - \frac{p}{2(2+(\alpha+\frac{1}{n}))} \right| \leq \frac{p}{2(2+(\alpha+\frac{1}{n}))}$, for all $n \geq N$. Suppose that $\lambda \in \rho(T_g; A_{\alpha+}^p)$. By Theorem 3.6(iii), $\lambda \in \sigma(T_g; A_{(\alpha+\frac{1}{n})}^p)$, for all $n \geq N$. On the other hand, by Proposition 3.5, we have $\lambda \in \bigcap_{n \in \mathbb{N}} \rho(T_g; A_{(\alpha+\frac{1}{n})}^p)$. This is a contradiction.

(4) By Proposition 3.5 and Theorem 3.6 (iv),

$$\begin{aligned} \sigma(T_g; A_{\alpha+}^p) &= \bigcup_{n \in \mathbb{N}} \sigma(T_g; A_{(\alpha+\frac{1}{n})}^p) \\ &= \{0\} \cup \bigcup_{n \in \mathbb{N}} \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2 + (\alpha + \frac{1}{n})}{p} \right\} \\ &= \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} > \frac{2 + \alpha}{p} \right\}. \end{aligned}$$

For the second equality suppose $\lambda \in \sigma(T_g; A_{\alpha+}^p)$. Then by Lemma 3.3(i) $\lambda \in \bigcup_{\mu > \alpha} \sigma(T_g; A_\mu^p)$, hence there exists $\beta > \alpha$ with $\lambda \in \sigma(T_g; A_\beta^p)$. So by Theorem 3.6(iv), we obtain $e^{g/\lambda} \notin A_\beta^p$. Hence $e^{g/\lambda} \notin A_{\alpha+}^p$. Thus, $\sigma(T_g; A_{\alpha+}^p) \subset \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : e^{g/\lambda} \notin A_{\alpha+}^p\}$. Combining with Lemma 3.2 we prove that (i) holds. For part (ii), thanks to Lemma 3.3(ii), it suffices to make sure that $\bigcup_{\mu > \alpha} \sigma(T_g; A_\mu^p) \subseteq \overline{\sigma(T_g; A_{\alpha+}^p)}$. This is true by part (i). □

Theorem 3.8. *Let $1 < p < \infty$, and $0 < \alpha < \infty$. Let $g \in \mathcal{B}_1$. Then, the following statements hold.*

- (1) $\sigma_{pt}(T_g; A_{\alpha-}^p) = \emptyset$.
- (2) For every $\gamma > -2\alpha - 3$, the set $\sigma(T_g; A_{\alpha-}^p)$ contains (3.3).
- (3) There exists $g \in \mathcal{B}_1$ such that $e^{g/\lambda} \in A_{\alpha-}^p$, for all $\lambda \in \mathbb{C} \setminus \{0\}$ but

$$\sigma(T_g; A_{\alpha-}^p) \supset \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p}{2(2+\alpha)} \right| \leq \frac{p}{2(2+\alpha)} \right\}.$$

- (4) If $g'(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{\zeta - z}$, $z \in \mathbb{D}$ for some finite measure μ on the unit circle,

(i) The spectrum of T_g on $A_{\alpha-}^p$ reads

$$\sigma(T_g; A_{\alpha-}^p) = \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2 + \alpha}{p} \right\}.$$

(ii) $\sigma^*(T_g; A_{\alpha-}^p) = \sigma(T_g; A_{\alpha-}^p)$.

Proof. (1) This is a direct consequence of Lemma 3.4 and Theorem 3.6(i).

(2) By Proposition 3.5 and Theorem 3.6(ii), for every $\gamma > -2\eta - 3$ we have

$$\begin{aligned} &\sigma(T_g; A_{\alpha-}^p) \\ &= \bigcap_{\mu < \alpha} \left(\bigcup_{\eta \in (\mu, \alpha)} \sigma(T_g; A_{\eta}^p) \right) \\ &\supset \bigcap_{\mu < \alpha} \bigcup_{\eta \in (\mu, \alpha)} \{ \lambda \in \mathbb{C} : \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma} (1 - |\zeta|^2)^{\eta} d s(\zeta) = \infty \}. \end{aligned}$$

We show that the latter includes (3.3). So we take any $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying the equality in (3.3), and assume contrary. Then, there exists $-1 < \mu := \alpha - \varepsilon$ such that for every $\mu < \eta := \alpha - \delta < \alpha$ there are $\gamma_{\eta} > -2\eta - 3$ and $c_{\eta} > 0$ satisfying

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma_{\eta}} (1 - |\zeta|^2)^{\eta} d s(\zeta) < c_{\eta}.$$

Note that for every $z, \zeta \in \mathbb{D}$ we have

$$\frac{1}{|1 - \bar{z}\zeta|} \leq \frac{1}{1 - |\zeta|} \leq \frac{2}{1 - |\zeta|^2}.$$

Since we have $\gamma_{\eta} - \delta > -2\alpha - 3$, we obtain

$$\begin{aligned} \infty &= \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma_{\eta} - \delta} (1 - |\zeta|^2)^{\alpha} d s(\zeta) \\ &\leq 2 \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} e^{p \operatorname{Re} \frac{gz(\zeta)}{\lambda}} |1 - \bar{z}\zeta|^{\gamma_{\eta}} (1 - |\zeta|^2)^{\eta} d s(\zeta) < c_{\eta}. \end{aligned}$$

This is a contradiction.

(3) In the selection of $g \in \mathcal{B}_1$, we follow Theorem 3.6(iii). In [5, p. 229] it is shown that with this particular choice of g , the function $e^{g/\lambda}$ belongs to the Hardy space H^p , for every $p > 0$. In particular, for every $\lambda \in \mathbb{C} \setminus \{0\}$ we have $e^{g/\lambda} \in A_{(\alpha - \frac{1}{n})}^p$, for all $n \in \mathbb{N}$. Let us take $\lambda \in \mathbb{C} \setminus \{0\}$ satisfying $\left| \lambda - \frac{p}{2(2+\alpha)} \right| \leq \frac{p}{2(2+\alpha)}$ and suppose that $\lambda \in \rho(T_g; A_{\alpha-}^p)$. It is clear that for any $n \in \mathbb{N}$ we have $\left| \lambda - \frac{p}{2(2+(\alpha - \frac{1}{n}))} \right| \leq \frac{p}{2(2+(\alpha - \frac{1}{n}))}$ and hence by Theorem 3.6(iii) we obtain $\lambda \in \sigma(T_g; A_{(\alpha - \frac{1}{n})}^p)$, for any $n \in \mathbb{N}$. By Proposition 3.5(2), this is a contradiction.

(4) By Proposition 3.5 and Theorem 3.6(iv) we obtain

$$\begin{aligned} \sigma(T_g; A_{\alpha-}^p) &= \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \sigma(T_g; A_{(\alpha - \frac{1}{n})}^p) \\ &= \{0\} \cup \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2 + (\alpha - \frac{1}{n})}{p} \right\} \\ &= \{0\} \cup \bigcap_{m \in \mathbb{N}} \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} > \frac{2 + (\alpha - \frac{1}{m})}{p} \right\} \\ &= \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2 + \alpha}{p} \right\}. \end{aligned}$$

This proves part (i). For part (ii), let us adapt the argument from [3, Proposition 2.9] (also used in [12, Proposition 3.9] and [17, Theorem 2.7(3)]). Let $\lambda \notin \sigma(T_g; A_{\alpha-}^p)$.

Then there exist $r > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\overline{B(\lambda, r)} \cap \left(\{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : \sup_{\zeta \in \mathbb{T}} \operatorname{Re} \frac{\mu(\{\zeta\})}{\lambda} \geq \frac{2 + \alpha}{p} \right\} \right) = \emptyset.$$

Therefore, $\overline{B(\lambda, r)} \subset \rho(T_g; A_{\alpha-}^p)$, as well as $\overline{B(\lambda, r)} \subset \rho(T_g; A_{(\alpha-\frac{1}{n})}^p)$, for all $n \geq N$.

In particular, the set $\{R(\delta, T_g) : \delta \in \overline{B(\lambda, r)}\}$ is equicontinuous, that is, operator norm bounded in $\mathcal{L}(A_{(\alpha-\frac{1}{n})}^p)$, for all $n \geq N$. It remains to show that $\{R(\delta, T_g) : \delta \in \overline{B(\lambda, r)}\}$ is equicontinuous in $A_{\alpha-}^p$. Since $A_{\alpha-}^p$ is a (DFS)-space, we can apply Banach-Steinhaus principle. So it suffices to show that $\{R(\delta, T_g)f : \delta \in \overline{B(\lambda, r)}\}$ is a bounded set in $A_{\alpha-}^p$, for every $f \in A_{\alpha-}^p$. Let us fix $f \in A_{\alpha-}^p$. Then, $f \in A_{(\alpha-\frac{1}{n})}^p$, for some $n \geq N$. So $\{R(\delta, T_g)f : \delta \in \overline{B(\lambda, r)}\}$ is a bounded set in $A_{(\alpha-\frac{1}{n})}^p$, and hence in $A_{\alpha-}^p$. \square

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