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Decay Estimate for the Time-Delayed Fourth-Order Wave Equations

Müge Meyvacı*

Abstract
The objective of this article is to analyze the stability of solutions for the following fourth- order nonlinear wave equations with an internal delay term:

$$u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t) + \sigma_2(t)|u_t(x,t-\tau)|^{2m-2}u_t(x,t-\tau) = 0.$$

We obtain appropriate conditions on $\sigma_1(t)$ and $\sigma_2(t)$ for the decay properties of the solutions. The multiplier technique and nonlinear integral inequalities are used in the proof.

Keywords: *Energy decay rate; Fourth order wave; Asymptotic behavior.*

AMS Subject Classification (2020): *Primary: 35B30; Secondary: 35B35; 35G25.*

1. Introduction

In this study, we examine the following initial boundary value problem for the nonlinear fourth-order time-delayed wave equations:

$$\begin{aligned} &u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t) \\ &\qquad + \sigma_2(t)|u_t(x,t-\tau)|^{2m-2}u_t(x,t-\tau) = 0, \text{in} \quad \Omega \times (0,\infty), \end{aligned} \tag{1.1}$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0, \quad \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$u_t(x,t-\tau) = f(x,t-\tau) \quad \text{in} \quad \Omega \times (0,\tau), \tag{1.4}$$

where $m > 1$ is a constant; σ_1 and σ_2 are positive functions; $\Omega \subset \mathbb{R}^n (n > 4)$ is a bounded domain; $\partial\Omega$ is a smooth boundary of Ω; τ is the time delay and initial function (u_0, u_1, f_0) in a suitable space.
Without the delay term ($\sigma_2 = 0$), the behaviors of the solutions of the fourth-order wave equations have been broadly analyzed in the literature (see [5],[7], [8], [14] and the references therein). Moreover, there are fewer results

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on the stability analysis of the solutions of time- delayed wave equations (see [1], [6], [11], [13] and the references therein). However, there is no detection of the decay rate of the nonlinear fourth-order wave equations with a delay term.
In [2], Benaissa, Benaissa and Messaoudi considered a nonlinear wave equation,

$$u_{tt} - \Delta u + \mu_1 \sigma(t) g_1 \left(u_t(x,t)\right) + \mu_2 \sigma(t) g_2 \left(u_t(x, t-\tau(t))\right) = 0,$$

where $\tau(t) > 0$ is a time dependent delay term, and μ_1 and μ_2 are positive constants. The existence and decay estimates for the solutions of the initial boundary value problem were proven.
In [3], Benaissa and Messaoudi analyzed the following nonlinear wave equation:

$$u_{tt} - \Delta u + \mu_1 \sigma(t) u_t(x,t) + \mu_2 \sigma(t) u_t(x, t-\tau(t)) + \theta(t) h \left(\nabla u(x,t)\right) = 0,$$

and the decay properties of the solutions were determined.
In [12], Ning, Shen and Zhao examined a wave equation of the form

$$u_{tt} + \mathcal{A}u + a(x)\left[\mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau)\right] = 0,$$

where $\mathcal{A}u = -divA(x) = (a_{ij}(x))$ is a symmetric matrix, $a(x)$ is a positive bounded function, and μ_1 and μ_2 are positive constants. The well-posedness of the system and exponential decay of the solutions were established.
In [4], Benaissa, Benguessoum and Messaoudi analyzed the following linear wave equation:

$$u_{tt} - \Delta u + \mu_1(t) u_t(x,t) + \mu_2(t) u_t(x, t-\tau(t)) = 0,$$

under assumptions about $\mu_1(t)$ and $\mu_2(t)$, the existence and decay properties of the solutions of the above equation with the initial boundary values were investigated.
In [9], Li and Chai examined the following damped plate equation:

$$u_{tt} + \mathcal{A}^2 u + b(x)\left[\mu_1 \beta\left(u_t(x,t)\right) + \mu_2 \phi\left(u_t(x, t-\tau)\right)\right] = 0,$$

where $\mathcal{A}u = div\left(A(x)\nabla u\right)$. The existence of solutions was proven, and the decay rate estimates for the energy were obtained.
The main goal of the present study is to deduce the decay properties of the solutions of the time-delayed fourth-order problem (1.1)-(1.4). To the best of our insight, this problem has not been considered in this respect.
The proof of our principle result is founded on the following Lemma which was demonstrated by Martinez in ([10]).

Lemma 1.1. *([10]) Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a non increasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad and \quad \phi(t) \to \infty \quad as \quad t \to \infty. \tag{1.5}$$

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_S^{+\infty} E(t)^{1+\sigma} \phi^{'}(t) dt \leq \frac{1}{\omega} E(0)^{\sigma} E(S), \quad 0 \leq S < \infty, \tag{1.6}$$

then $E(t)$ has the following decay properties:

$$if \quad \sigma = 0, \quad then \quad E(t) \leq E(0) e^{1-\omega\phi(t)}, \forall t \geq 0, \tag{1.7}$$

$$if \quad \sigma > 0, \quad then \quad E(t) \leq E(0) \Big(\frac{1+\sigma}{1+\omega\sigma\phi(t)} \Big)^{\frac{1}{\sigma}}, \forall t \geq 0. \tag{1.8}$$

2. asymptotic behavior

In the present section, we aim to constitute a decay property of the solutions of the problem (1.1)-(1.4) using multiplier method and integral inequalities. We use the following variable as in [11].

$$z(x, \rho, t) = u_t(x, t - \tau\rho). \tag{2.1}$$

Hence, we change problem (1.1)-(1.4) to the following problem:

$$\begin{aligned} &u_{tt} + \Delta^2 u + u + \sigma_1(t)|u_t(x,t)|^{2m-2}u_t(x,t) \\ &\qquad + \sigma_2(t)|z(x,1,t)|^{2m-2}z(x,1,t) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \end{aligned} \tag{2.2}$$

$$\tau z_t + z_\rho = 0 \quad \text{in} \quad \Omega \times (0,1) \times (0,\infty), \tag{2.3}$$

$$u(x,t) = \frac{\partial u}{\partial \nu} = 0 \quad \partial\Omega \times (0,\infty), \tag{2.4}$$

$$z(x,0,t) = u_t(x,t) \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.5}$$

$$z(x,\rho,0) = u_t(x,-\tau\rho) = f(x,-\tau\rho) \quad \text{in} \quad \Omega \times (0,1). \tag{2.6}$$

Lemma 2.1. *Assume that (u,z) is a solution of the new problem (2.2)-(2.6) and $\sigma_1(t)$, $\sigma_2(t)$ satisfy the following properties A1: $\sigma_1(t) : \mathbb{R}^+ \to (0,\infty)$ is a non-increasing function on $C^1(\mathbb{R}^+)$ such that*

$$|\sigma_1(t)| \le M.$$

A2: $\sigma_2(t) : \mathbb{R}^+ \to (0,\infty)$ is a function on $C^1(\mathbb{R}^+)$ such that

$$|\sigma_2(t)| < M_2\sigma_1(t),$$

where M and M_2 are positive constants. Then, the positive energy of problem (2.2)-(2.6) satisfies the following inequality:

$$\frac{dE(t)}{dt} \le -\sigma_1(t)(1-\theta_1)\int_\Omega |u_t|^{2m}\,dx - \sigma_1(t)\theta_2\int_\Omega |z(x,1,t)|^{2m}\,dx,$$

where

$$\begin{gathered} \theta_1 = \frac{M_2 + \tilde{M}}{2m}, \theta_2 = \frac{\tilde{M} + (2m-1)M_2}{2m}, \\ (2m-1)M_2 < \tilde{M} < 2m - M_2. \end{gathered} \tag{2.7}$$

Proof. By multiplying equation (2.2) by u_t and integrating over Ω we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2\right) = &-\int_\Omega \sigma_1(t)u_t|u_t|^{2m-1}dx \\ &-\int_\Omega \sigma_2(t)u_t|z(x,1,t)|^{2m-2}z(x,1,t)dx. \end{aligned} \tag{2.8}$$

Furthermore, multiplying equation (2.3) by function $\gamma_1(t)\,|z(x,\rho,t)|^{2m-2}\,z(x,\rho,t)$ and integrating over $(0,1)\times\Omega$ we derive

$$\begin{aligned} \frac{d}{dt}\left(\frac{\tau}{2m}\int_\Omega\int_0^1 \gamma_1(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx\right) = &\frac{\tau}{2m}\int_\Omega\int_0^1 \gamma_1'(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx \\ &-\frac{1}{2m}\int_\Omega \gamma_1(t)\left(|z(x,1,t)|^{2m} - |z(x,0,t)|^{2m}\right)dx, \end{aligned} \tag{2.9}$$

where

$$\gamma_1(t) = \tilde{M}\sigma_1(t), \tag{2.10}$$

$$\gamma_1'(t) < 0. \tag{2.11}$$

We define

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2 + \frac{\tau}{2m}\int_\Omega \gamma_1(t)\int_0^1 |z(x,\rho,t)|^{2m}\,d\rho dx. \tag{2.12}$$

Hence, by combining equations (2.8) and (2.9), we have

$$\begin{aligned} \frac{d}{dt}E(t) = &-\int_\Omega \sigma_1(t)u_t|u_t|^{2m-1}dx - \int_\Omega \sigma_2(t)u_t|z(x,1,t)|^{2m-1}dx \\ &+\frac{\tau}{2m}\int_\Omega\int_0^1 \gamma_1'(t)\,|z(x,\rho,t)|^{2m}\,d\rho dx - \frac{1}{2m}\int_\Omega \gamma_1(t)\left(|z(x,1,t)|^{2m} - |u_t|^{2m}\right)dx, \end{aligned}$$

Then, using the definition of $\gamma_1(t)$ (2.10), condition (2.5) and property (2.11) we get

$$\frac{d}{dt}E(t) \leq -\int_{\Omega} \sigma_1(t)u_t|u_t|^{2m-1}dx - \int_{\Omega} \sigma_2(t)u_t|z(x,1,t)|^{2m-2}z(x,1,t)dx$$
$$-\frac{1}{2m}\int_{\Omega} \tilde{M}\sigma_1(t)\left(|z(x,1,t)|^{2m} - |u_t|^{2m}\right)dx.$$

To estimate the second integral of the above equation, we use Young Inequality to obtain

$$\int_{\Omega} \sigma_2(t)u_t|z(x,1,t)|^{2m-1}dx \leq \frac{1}{2m}\int_{\Omega} |\sigma_2(t)|\,|u_t|^{2m}\,dx + \frac{2m-1}{2m}\int_{\Omega} |\sigma_2(t)|\,|z|^{2m}\,dx. \tag{2.13}$$

Moreover, we have

$$|\sigma_2(t)| < M_2\sigma_1(t). \tag{2.14}$$

Thus, we deduce that inequality

$$\frac{d}{dt}E(t) \leq -\sigma_1(t)\left[1-\theta_1\right]\int_{\Omega} |u_t|^{2m}\,dx - \sigma_1(t)\theta_2\int_{\Omega} |z|^{2m}\,dx, \tag{2.15}$$

where

$$\theta_1 = \frac{\tilde{M}+M_2}{2m}, \tag{2.16}$$

and

$$\theta_2 = \frac{\tilde{M}+(2m-1)M_2}{2m}. \tag{2.17}$$

Recalling the property of $\tilde{M}$ (2.7) we have

$$\frac{d}{dt}E(t) \leq 0. \tag{2.18}$$

Hence, the positive energy is non-increasing.

□

Now, we are ready to obtain the decay rate of the solutions of problem (2.2)-(2.6).

Theorem 2.1. *Assume that A1 and A2 hold. Then, there exist positive constants q and ω such that the energy of problem (2.2)-(2.6) satisfies the following property*

$$E(t) \leq E(0)\left(\frac{1+q}{1+\omega q\int_0^t \sigma_1(s)ds}\right)^{\frac{1}{q}}, \quad \forall t>0,$$

where

$$q > \frac{2m-1}{2},$$

and

$$\omega^{-1} = \frac{2e^{2\tau}}{3}\max\Bigg\{2M, \frac{2}{(q+1)(1-\theta_1)}, \left(\frac{4qe^{2\tau}}{E(0)}\right)^q\left(\frac{1-\theta_1}{2(q+1)}\right)^{q+1}, \frac{qM}{M+1}$$
$$, \frac{(2m-1)E(0)^{\frac{m-1}{2m-1}}}{2m}\left(\frac{2^{m+2}e^{2\tau}(M^2c_1^2)^m}{m}\right)^{\frac{1}{2m-1}}\left(\frac{1}{1-\theta_1}+\frac{M_2^{\frac{2}{2m-1}}}{\theta_2}\right)$$
$$, \frac{M\tilde{M}}{2m(q+1)\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)}\Bigg\}.$$

Proof. To establish a decay rate estimate of the positive energy; by multiplying the equation (2.2) by function $\phi'(t)E^q(t)u(x,t)$ and integrating over $(S,T)\times\Omega$ we deduce the following equation,

$$0=\int_S^T\int_\Omega \phi' E^q u\Big[u_{tt}+\Delta^2 u+u+\sigma_1(t)|u_t|^{2m-2}u_t+\sigma_2(t)|z(x,1,t)|^{2m-2}z(x,1,t)\Big]dxdt,$$

where $\phi(t)$ satisfies the hypothesis of Lemma 1.1. Using the boundary conditions, we have

$$\begin{aligned}0=&\int_S^T\int_\Omega\left[\frac{d}{dt}\left(\phi' E^q u u_t\right)-\left(\phi' E^q\right)' u u_t-\phi' E^q u_t^2\right]dxdt\\&+\int_S^T\phi' E^q\int_\Omega(\Delta u)^2dxdt+\int_S^T\phi' E^q\sigma_1(t)\int_\Omega u\left|u_t\right|^{2m-2}u_t dxdt\\&+\int_S^T\phi' E^q\int_\Omega u^2dxdt+\int_S^T\phi' E^q\sigma_2(t)\int_\Omega u\left|z\right|^{2m-2}z(x,1,t)dxdt.\end{aligned}\tag{2.19}$$

Furthermore, by multiplying equation (2.3) by $\phi'(t)E^q(t)\gamma_1(t)e^{-2\tau\rho}\left|z\right|^{2m-2}z(x,\rho,t)$ and integrating over $(S,T)\times\Omega\times(0,1)$, we obtain

$$0=\int_S^T\phi'(t)E^q(t)\int_\Omega\int_0^1\gamma_1(t)e^{-2\tau\rho}\left|z\right|^{2m-2}z(x,\rho,t)\left(\tau z_t+z_\rho\right)d\rho dxdt,$$

with the boundary conditions, we get

$$\begin{aligned}0=&\frac{\tau}{2m}\int_\Omega\int_0^1\phi' E^q\gamma_1 e^{-2\tau\rho}\left|z\right|^{2m}\Big|_S^T d\rho dx+\frac{1}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega e^{-2\tau\rho}\left|z\right|^{2m}\Big|_0^1 dxdt\\&+\frac{\tau}{m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt\\&-\frac{\tau}{2m}\int_S^T\left(\phi' E^q\gamma_1\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt.\end{aligned}\tag{2.20}$$

By taking the sum of equations (2.19) and (2.20), we have

$$\begin{aligned}0=&\int_\Omega\phi' E^q u u_t\Big|_S^T-\int_S^T\int_\Omega\left[\left(\phi' E^q\right)' u u_t-\phi' E^q u_t^2\right]dxdt\\&+\int_S^T\phi' E^q\int_\Omega(\Delta u)^2dxdt+\int_S^T\phi' E^q\int_\Omega u^2dxdt\\&+\int_S^T\phi' E^q\sigma_1(t)\int_\Omega u\left|u_t\right|^{2m-2}u_t dxdt\\&+\int_S^T\phi' E^q\sigma_2(t)\int_\Omega u\left|z\right|^{2m-2}z(x,1,t)dxdt\\&+\frac{\tau}{2m}\int_\Omega\int_0^1\phi' E^q\gamma_1 e^{-2\tau\rho}\left|z\right|^{2m}\Big|_S^T d\rho dx\\&-\frac{\tau}{2m}\int_S^T\left(\phi' E^q\gamma_1\right)'\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt\\&+\frac{1}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega e^{-2\tau\rho}\left|z\right|^{2m}\Big|_0^1 dxdt\\&+\frac{\tau}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt.\end{aligned}\tag{2.21}$$

Because of the definition of $E(t)$, we get the following inequality,

$$\begin{aligned}&\frac{\tau}{2m}\int_S^T\phi' E^q\gamma_1\int_\Omega\int_0^1 e^{-2\tau\rho}\left|z\right|^{2m}d\rho dxdt>\\&\qquad\qquad\int_S^T\phi' E^q e^{-2\tau}\left(2E(t)-\left\|u_t\right\|^2-\left\|\Delta u\right\|^2-\left\|u\right\|^2\right)dt.\end{aligned}$$

By combining the last inequality with equation (2.21), we obtain

$$
\begin{aligned}
2\int_S^T \phi' E^{q+1} e^{-2\tau} dt < &-\left[\phi' E^q \int_\Omega u u_t dx\right]\Big|_S^T + 2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt \\
&- \int_S^T \phi' E^q \sigma_1(t) \int_\Omega u\,|u_t|^{2m-1} dx dt \\
&- \int_S^T \phi' E^q \sigma_2(t) \int_\Omega u\,|z|^{2m-2} z(x,1,t) dx dt \\
&+ \int_S^T (\phi' E^q)' \int_\Omega u u_t dx dt \\
&- \frac{\tau}{2m}\int_\Omega \int_0^1 \phi' E^q \gamma_1 e^{-2\tau\rho} |z|^{2m} \Big|_S^T d\rho dx \\
&+ \frac{\tau}{2m}\int_S^T (\phi' E^q \gamma_1)' \int_\Omega \int_0^1 e^{-2\tau\rho} |z|^{2m} d\rho dx dt \\
&- \frac{1}{2m}\int_S^T \phi' E^q \gamma_1 \int_\Omega e^{-2\tau\rho} |z|^{2m} \Big|_0^1 dx dt.
\end{aligned}
\tag{2.22}
$$

By virtue of Young's Inequality, Sobolev inequality, the definition of function $\gamma_1(t)$ (2.10), hypothesis of theorem 2.1 , conclusion of lemma 2.1 and assumption that $\phi'(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function ($0 < |\phi'| < M$) we reach the following inequalities;

$$\left|\phi' E^q \int_\Omega u u_t dx\right| \leq 2ME^{q+1}(t). \tag{2.23}$$

$$\left|\phi' E^q \int_\Omega u u_t dx\right|_S^T \leq 2ME^{q+1}(S). \tag{2.24}$$

$$\left|\int_S^T (\phi' E^q)' \int_\Omega u u_t dx dt\right| \leq 2ME^{q+1}(S) + \frac{2Mq}{q+1}E^{q+1}(S). \tag{2.25}$$

$$
\begin{aligned}
\int_S^T \phi' E^q \sigma_1(t) \int_\Omega u\,|u_t|^{2m-1} dx dt \leq &\frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m}\int_S^T [-E'(t)] dt \\
&+ \frac{\epsilon_1^{2m}}{2m}\int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} c_1 \phi'(t) 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m} \left(E^{q+\frac{1}{2}}\right)^{2m} dt.
\end{aligned}
\tag{2.26}
$$

$$
\begin{aligned}
\int_S^T \phi' E^q \sigma_2(t) \int_\Omega u\,|z|^{2m-2} z(x,1,t) dx dt \leq &\frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m}\int_S^T \left[-E'(t)\right] dt \\
&+ \frac{\epsilon_2^{2m}}{2m}\int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} M_2 c_1 \phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m} \left(E^{q+\frac{1}{2}}\right)^{2m} dt.
\end{aligned}
\tag{2.27}
$$

$$-\frac{\tau}{2m}\int_\Omega \int_0^1 \phi' E^q \gamma_1 e^{-2\tau\rho} |z|^{2m} \Big|_S^T d\rho dx \leq ME^{q+1}(S). \tag{2.28}$$

$$\frac{\tau}{2m}\int_S^T (\phi' E^q \gamma_1)' \int_\Omega \int_0^1 e^{-2\tau\rho} |z|^{2m} d\rho dx dt \leq \frac{Mq}{q+1}E^{q+1}(S). \tag{2.29}$$

$$\frac{1}{2m}\int_S^T \phi' E^q \gamma_1 \int_\Omega e^{-2\tau} |z(x,1,t)|^{2m} dx dt \leq \frac{M\tilde{M}}{2m\theta_2(q+1)}E^{q+1}(S). \tag{2.30}$$

$$\frac{1}{2m}\int_S^T \phi' E^q \gamma_1 \int_\Omega |z(x,0,t)|^{2m} dx dt \leq \frac{M\tilde{M}}{2m(1-\theta_1)(q+1)}E^{q+1}(S). \tag{2.31}$$

Based on the estimates (2.23)-(2.31) and equation (2.21),

$$
\begin{aligned}
2\int_S^T \phi' E^{q+1} e^{-2\tau} dt \leq\ & 4ME^{q+1}(S) + 2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt \\
&+ \frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m} \int_S^T [-E'(t)] dt \\
&+ \frac{\epsilon_1^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} c_1 \phi' 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}} \right)^{2m} \left(E^{q+\frac{1}{2}} \right)^{2m} dt \\
&+ \frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m} \int_S^T [-E'(t)]\, dt \\
&+ \frac{\epsilon_2^{2m}}{2m} \int_S^T \left(\frac{\sigma_1^{\frac{1}{2m}} M_2 c_1 \phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}} \right)^{2m} \left(E^{q+\frac{1}{2}} \right)^{2m} dt \\
&+ \frac{2Mq}{q+1} E^{q+1}(S) + \frac{M\tilde{M}}{2m\theta_2(q+1)} E^{q+1}(S) \\
&+ \frac{M\tilde{M}}{2m(1-\theta_1)(q+1)} E^{q+1}(S),
\end{aligned} \tag{2.32}
$$

where ϵ_1 and ϵ_2 are positive constants, which will be later selected. Let us define $\phi(t)$ as follows;

$$
\phi(t) = \int_0^t \sigma_1(s) ds,
$$

Dividing region Ω such that $\Omega_1 = \{x; |u_t| \geq 1\}$ and $\Omega_2 = \{x; |u_t| < 1\}$, we get

$$
2\int_S^T \phi' E^q \int_\Omega u_t^2 dx dt = 2\int_S^T \phi' E^q \int_{\Omega_1} u_t^2 dx dt + 2\int_S^T \phi' E^q \int_{\Omega_2} u_t^2 dx dt.
$$

Moreover, using Young's inequality, lemma 2.1 and the definition of $\phi(t)$ we infer the following inequalities

$$
2\int_S^T \phi' E^q \int_{\Omega_1} u_t^2 dx dt \leq \frac{2}{(1-\theta_1)(q+1)} E^{q+1}(S), \tag{2.33}
$$

and

$$
\begin{aligned}
2\int_S^T \phi' E^q \int_{\Omega_2} u_t^2 dx dt \leq\ & \frac{2\epsilon_3^{\frac{k+1}{2}}}{k+1} \int_S^T [-E'(t)]\, dt \\
&+ \frac{k-1}{(k+1)\epsilon_3^{\frac{k+1}{k-1}}} \left(\frac{2}{1-\theta_1} \right)^{\frac{k+1}{k-1}} \int_S^T \phi' E^{\frac{q(k+1)}{k-1}} dt,
\end{aligned} \tag{2.34}
$$

where $k > 2m$ and ϵ_3 is a positive constant, which will be later selected. From the estimates (2.33), (2.34) and inequality (2.32), we obtain

$$
\begin{aligned}
2e^{-2\tau}\int_S^T\int_\Omega \phi' E^{q+1}dt \leq & 2ME^{q+1}(S)+\frac{2}{(1-\theta_1)(q+1)}E^{q+1}(S)\\
&+\frac{(k-1)\epsilon_3^{-\frac{k+1}{k-1}}}{k+1}\left(\frac{2}{1-\theta_1}\right)^{\frac{k+1}{k-1}}\int_S^T \phi' E^{\frac{q(k+1)}{k-1}}dt+\frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m}E(S)\\
&+\frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m}E(S)+\frac{\epsilon_1^{2m}}{2m}\int_S^T\left(\frac{\sigma_1^{\frac{1}{2}}c_1\phi' 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m}\left(E^{q+\frac{1}{2}}\right)^{2m}dt\\
&+\frac{\epsilon_2^{2m}}{2m}\int_S^T\left(\frac{\sigma_1^{\frac{1}{2m}}M_2c_1\phi' 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m}\left(E^{q+\frac{1}{2}}\right)^{2m}dt+\frac{2Mq}{q+1}E^{q+1}(S)\\
&+\frac{2\epsilon_3^{\frac{k+1}{2}}}{k+1}E(S)+\frac{M\tilde{M}}{2m(q+1)}\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)E^{q+1}(S).
\end{aligned}
$$

Selecting $k=2q+1$, we deduce the following inequality,

$$
\begin{aligned}
2e^{-2\tau}\int_S^T\int_\Omega \phi' E^{q+1}dt \leq & 4ME^q(0)E(S)+\frac{2}{(1-\theta_1)(q+1)}E^q(0)E(S)\\
&+\frac{\epsilon_3^{(q+1)}}{q+1}E(S)+\frac{q\epsilon_3^{-\frac{q+1}{q}}}{q+1}\left(\frac{2}{1-\theta_1}\right)^{\frac{q+1}{q}}\int_S^T \phi' E^{q+1}dt\\
&+\frac{(2m-1)\epsilon_2^{-\frac{2m}{2m-1}}}{2m}E(S)+\frac{(2m-1)\epsilon_1^{-\frac{2m}{2m-1}}}{2m}E(S)\\
&+\frac{M^{2m}\epsilon_1^{2m}}{2m}\left(\frac{c_1 2^{\frac{1}{2}}}{(1-\theta_1)^{\frac{2m-1}{2m}}}\right)^{2m}E(0)^\theta\int_S^T \phi' E^{q+1}dt\\
&+\frac{\epsilon_2^{2m}}{2m}\left(\frac{M_2c_1 2^{\frac{1}{2}}}{\theta_2^{\frac{2m-1}{2m}}}\right)^{2m}E(0)^\theta\int_S^T \phi' E^{q+1}dt+\frac{2Mq}{q+1}E^q(0)E(S)\\
&+\frac{M\tilde{M}}{2m(q+1)}\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)E^q(0)E(S),
\end{aligned}
$$

where $\theta=m(2q+1)-(q+1)$. Selecting $\epsilon_1=\left(\frac{me^{-2\tau}(1-\theta_1)^{2m-1}}{E(0)^\theta 2^{m+2}(M^2c_1^2)^m}\right)^{\frac{1}{2m}}$, $\epsilon_2=\left(\frac{me^{-2\tau}\theta_2^{2m-1}}{E(0)^\theta 2^{m+2}(M_2^2M^2c_1^2)^m}\right)^{\frac{1}{2m}}$ and $\epsilon_3=\frac{1-\theta_1}{2}\left(\frac{(q+1)e^{-2\tau}}{4q}\right)^{\frac{q}{q+1}}$ we get

$$
\int_S^T \phi'(t)E^{q+1}(t)dt \leq \frac{1}{\omega}E^q(0)E(S),
$$

where

$$
\begin{aligned}
\omega^{-1}=\frac{2e^{2\tau}}{3}\max\Big\{&2M,\frac{2}{(q+1)(1-\theta_1)},\left(\frac{4qe^{2\tau}}{E(0)}\right)^q\left(\frac{1-\theta_1}{2(q+1)}\right)^{q+1},\frac{qM}{M+1}\\
&,\frac{(2m-1)E(0)^{\frac{m-1}{2m-1}}}{2m}\left(\frac{2^{m+2}e^{2\tau}(M^2c_1^2)^m}{m}\right)^{\frac{1}{2m-1}}\left(\frac{1}{1-\theta_1}+\frac{M_2^{\frac{2}{2m-1}}}{\theta_2}\right)\\
&,\frac{M\tilde{M}}{2m(q+1)\left(\frac{1}{\theta_2}+\frac{1}{1-\theta_1}\right)}\Big\}.
\end{aligned}
$$

and

$$
q>\frac{2m-1}{2}.
$$

Hence, we deduce the following result from the conclusion of Lemma 1.1

$$E(t) \leq E(0)\left(\frac{1+q}{1+\omega q\int_0^t \sigma_1(s)ds}\right)^{\frac{1}{q}}, \quad \forall t > 0.$$

□

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Affiliations

MÜGE MEYVACI
ADDRESS: Mimar Sinan Fine Art University, Dept. of Mathematics, 34360, İstanbul-Türkiye.
E-MAIL: muge.meyvaci@msgsu.edu.tr
ORCID ID: https://orcid.org/000-0002-4772-3869