Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (1) (2022), 156 – 171
DOI : 10.15672/hujms.778148

Research Article

Coefficient inequalities for certain starlike and convex functions

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Abstract

In this paper, we consider two Ma–Minda-type subclasses of starlike and convex functions associated with the normalized analytic function $\varphi_{Ne}(z) = 1 + z - z^3/3$ that maps an open unit disk onto the Nephroid shaped bounded domain in the right–half of the complex plane. We investigate convolution and quasi-Hadamard product properties for the functions belonging to such classes. In addition, we compute best possible estimates on third order Hermitian–Toeplitz determinant and non-sharp estimates on certain third order Hankel determinants for the starlike functions associated with the interior region of Nephroid.

Mathematics Subject Classification (2020). 30C45, 30C50

Keywords. starlike functions, convex functions, nephroid, convolution properties, quasi-Hadamard product properties, Hermitian–Toeplitz and Hankel determinants

1. Introduction

The coefficient inequalities of the normalized analytic univalent functions yield it's geometric properties related information. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk and let $\mathcal{A}$ be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined in $\mathbb{D}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing all the univalent functions in $\mathbb{D}$. Let Ω be the family of analytic functions w satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f and g are analytic functions in $\mathbb{D}$, then we say f is subordinate to g, written as $f \prec g$, if there exists a function $w \in \Omega$ such that $f = g \circ w$. In particular, if $g \in \mathcal{S}$, the equivalence condition $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ holds [8]. The function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin and the function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex. In terms of subordination, the function $f \in \mathcal{A}$ is starlike and convex if and only if the subordination relations $zf'(z)/f(z) \prec (1 + z)/(1 - z)$ and $zf''(z)/f'(z) \prec 2z/(1 - z)$ for

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Received: 08.08.2020; Accepted: 06.09.2021

all $z \in \mathbb{D}$ respectively hold. Several subclasses of the starlike and convex functions were studied by many authors [13, 15, 29, 34–36] in the literature.

Using the concept of subordination, Ma and Minda [28] introduced and studied the unified classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ of starlike and convex functions, where φ is the analytic function satisfying $Re(\varphi(z)) > 0$ for all $z \in \mathbb{D}$. These classes contain various subclasses of starlike and convex functions. In recent past, several Ma–Minda-type classes of starlike and convex functions have been introduced and studied by various authors [16, 23, 37, 38]. In this paper, we consider two subclasses $\mathcal{S}^*_{Ne}$ and $\mathcal{C}_{Ne}$ of Ma–Minda classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ respectively which are associated with the analytic function $\varphi_{Ne}(z) = 1 + z - z^3/3$ that is univalent, starlike with respect to 1 and maps $\mathbb{D}$ onto a Nephroid shaped bounded symmetric region with respect to real axis in the right–half plane. Analytically, these classes are defined as

$$\mathcal{S}^*_{Ne} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi_{Ne}(z) \right\} \text{ and } \mathcal{C}_{Ne} = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi_{Ne}(z) \right\}$$

for all $z \in \mathbb{D}$. Recently, these classes were introduced by Wani and Swaminathan [40]. They studied several properties of these classes such as the structural formula, growth and distortion theorems, Fekete-Szegö functionals, radius estimates [41] and subordination results.

If $f, g \in \mathcal{A}$, where f is given by (1.1) and g is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then the *convolution* or *Hadamard product* of f and g, denoted by $f * g$, is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

It is noted that if $g(z) = z/(1-z)$, then $f * g = f$ and if $g(z) = z/(1-z)^2$, then $f * g = zf'$ for all $f \in \mathcal{A}$. Further, let $\mathcal{T}$ be the class of analytic functions with negative coefficients of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_1 > 0; a_n \geq 0) \tag{1.2}$$

defined in $\mathbb{D}$. For the functions, f defined by (1.2) and $g(z) = b_1 z - \sum_{n=1}^{\infty} b_n z^n$, the quasi-Hadamard product (or convolution) is given by

$$f(z) * g(z) = a_1 b_1 z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

The quasi-Hadamard of two or more functions were defined by Owa [30] and Kumar [19]. Let the functions f_i $(i = 1, ..., m)$ and g_j $(j = 1, ..., s)$ of the form

$$f_i(z) = a_{1,i} z - \sum_{n=2}^{\infty} a_{n,i} z^n, \quad (a_{1,i} > 0; a_{n,i} \geq 0) \tag{1.3}$$

$$g_j(z) = b_{1,j} z - \sum_{n=2}^{\infty} b_{n,j} z^n, \quad (b_{1,j} > 0; b_{n,j} \geq 0) \tag{1.4}$$

be analytic in $\mathbb{D}$. Denote by h the quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_s$ is defined by

$$h(z) = \left\{ \prod_{i=1}^{m} a_{1,i} \prod_{j=1}^{s} b_{1,j} \right\} z - \sum_{n=2}^{\infty} \left\{ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{s} b_{n,j} \right\} z^n. \tag{1.5}$$

In 2000, Hossen [12] established certain results related to quasi-Hadamard product for $p-$valent starlike and $p-$valent convex functions. Aouf [3] proved a theorem concerning to quasi-Hadamard product for certain analytic functions. Using uniformly starlikeness and uniformly convexity, Breaz and El-Ashwah [5] studied quasi-Hadamard product between some $p-$valent and uniformly analytic functions with negative coefficients.

Hankel and Hermitian-Toeplitz determinants have important role in various branches of pure and applied mathematics. Let $\langle a_k \rangle_{k\geq 1}$ denotes a sequence of coefficients of the normalized analytic function $f \in \mathcal{A}$. The coefficient estimates of normalized univalent functions in the disk $\mathbb{D}$ give many useful information regarding the geometric properties. For instance, the estimate on second coefficient of the function $f \in \mathcal{S}$ yields the growth and distortion theorems. This idea inspires researchers to determine the estimates on the coefficient functionals such as the Hermitian-Toepltiz and Hankel determinants. For $q, n \in \mathbb{N}$, the Hankel determinant of order n associated with the sequence $\langle a_k \rangle_{k\geq 1}$ is defined by

$$H_q^{(n)}(f) := \det\{a_{n+i+j-2}\}_{i,j}^{q}, \;\; 1 \leq i, \, j \leq q, \, a_1 = 1. \tag{1.6}$$

For the functions $f \in \mathcal{S}$ and $f \in \mathcal{S}^*$, Hankel determinants were discussed initially by Pommerenke [31, 32]. Later, Hayman (1968) [11] computed the best possible bound $\kappa n^{1/2}$ on Hankel determinant $|H_{2,n}(f)|$ for general univalent functions, where κ as an absolute constant. In 2013, authors [26] determined sharp estimates on second Hankel determinant for Ma-Minda starlike and convex functions. In 2010, Babalola [4] first computed bounds on the third Hankel determinant for analytic functions with bounded-turning as well as starlike and convex functions. Later on, Zaprawa [43] obtained improved bounds for third order Hankel determinant obtained by Babalola [4] but these bounds were not sharp. Kowalczyk *et al.* [17] established sharp inequality $|H_3^{(1)}(f)| \leq 4/135$ for convex functions. Recently, Kumar *et al.* [22] improved certain existing bound on the third Hankel determinant for some classes of close-to-convex functions. For recent results on third Hankel determinant, see [10, 24, 25, 39]. Hankel determinants are closely related to Hermitian-Toeplitz determinants [18, 42]. The third order Hermitian-Toeplitz determinant $T_{3,1}(f)$ for the function $f \in \mathcal{A}$ is given by

$$|T_{3,1}(f)| := 2Re(a_2^2\,\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1. \tag{1.7}$$

The sharp estimates on certain symmetric Toeplitz determinants were evaluated for univalent functions and typically real functions by Ali *et al.* [2]. Further, the best possible lower and upper bounds for the second and third-order Hermitian–Toeplitz determinants are estimated over the classes of starlike and convex functions of order α [7]. Jastrzębski [14] computed best possible upper and lower bounds of second and third order Hermitian–Toeplitz determinants for some close-to-star functions. Recently, Kumar and Kumar [21] investigate sharp upper and lower bounds on third order Hermitian-Toeplitz determinant for the classes of strongly starlike functions.

Motivated by the above stated research work, second section provides convolution properties of the classes $\mathcal{S}^*_{Ne}$ and $\mathcal{C}_{Ne}$. Further, certain results associated with quasi-Hadamard product for such classes are established in Section 3. In the last section, we obtain best possible lower and upper bounds on the third-order Hermitian–Toeplitz determinant for starlike functions in the class $\mathcal{S}^*_{Ne}$. In addition, non-sharp estimates on third–order Hankel determinants $H_3^{(1)}(f)$, $H_3^{(2)}(f)$ and $H_3^{(3)}(f)$ for the functions f belonging to the class $\mathcal{S}^*_{Ne}$ are also computed.

2. Convolution properties

In view of the work done in [6, 9], we derive convolution properties of the classes $\mathcal{S}^*_{Ne}$ and $\mathcal{C}_{Ne}$. We first begin with necessary and sufficient convolution conditions of the class $\mathcal{S}^*_{Ne}$.

Theorem 2.1. *The function f defined by* (1.2) *is in the class $\mathcal{S}^*_{Ne}$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \neq 0 \tag{2.1}$$

for all $L = \frac{3+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}$, where $\theta \in [0, 2\pi]$ and also $L = 1$.

Proof. Suppose the function $f \in \mathcal{S}^*_{Ne}$, then we have

$$\frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3}. \tag{2.2}$$

Since the function $zf'(z)/f(z)$ is analytic in $\mathbb{D}$, it follows $f(z) \neq 0, z \in \mathbb{D}^* = \mathbb{D}\backslash\{0\}$; that is, $(1/z)f(z) \neq 0$ and this is equivalent to the fact that (2.1) holds for $L = 1$. In view of relation (2.2), we have

$$\frac{zf'(z)}{f(z)} = \frac{3 + 3w(z) - w^3(z)}{3}, \tag{2.3}$$

where $w \in \Omega$. The expression (2.3) is equivalent to

$$\frac{zf'(z)}{f(z)} \neq \frac{3 + 3e^{i\theta} - e^{3i\theta}}{3} \tag{2.4}$$

so that

$$\frac{1}{z}\Big[3zf'(z) - (3 + 3e^{i\theta} - e^{3i\theta})f(z)\Big] \neq 0. \tag{2.5}$$

Since we have convolution relations $f(z) * \frac{z}{1-z} = f(z)$ and $f(z) * \frac{z}{(1-z)^2} = zf'(z)$, then expression (2.5) is written as

$$\frac{1}{z}\left[f(z) * \left(\frac{3z}{(1-z)^2} - \frac{(3 + 3e^{i\theta} - e^{3i\theta})z}{(1-z)}\right)\right] \neq 0.$$

Therefore, we have

$$\frac{e^{3i\theta} - 3e^{i\theta}}{z}\left[f(z) * \frac{z - \frac{3+3e^{i\theta} - e^{3i\theta}}{3e^{i\theta} - e^{3i\theta}}z^2}{(1-z)^2}\right] \neq 0, \tag{2.6}$$

which completes the necessary part of Theorem 2.1.

Conversely, because assumption (2.1) holds for $L = 1$, it follows that $(1/z)f(z) \neq 0$ for all $z \in \mathbb{D}$, hence the function $\psi(z) = zf'(z)/f(z)$ is analytic in $\mathbb{D}$, and it is regular at $z = 0$ with $\psi(0) = 1$. Since it was shown in the first part of the proof that assumption (2.1) is equivalent to (2.4), we have

$$\frac{zf'(z)}{f(z)} \neq \frac{3 + 3e^{i\theta} - e^{3i\theta}}{3} \tag{2.7}$$

and if we denote

$$\varphi_{Ne}(z) = \frac{3 + 3z - z^3}{3} \tag{2.8}$$

relation (2.7) shows that the simply connected domain $\psi(\mathbb{D})$ is included in a connected component of $\mathbb{C}\backslash\varphi_{Ne}(\partial\mathbb{D})$. Using the fact $\psi(0) = \varphi_{Ne}(0)$ together with the univalence of the function φ_{Ne}, it follows that $\psi \prec \varphi_{Ne}$, which represents (2.2). Thus, $f \in \mathcal{S}^*_{Ne}$ which completes the proof of Theorem 2.1. □

Theorem 2.2. *A necessary and sufficient condition for the function f defined by (1.2) to be in the class $\mathcal{S}^*_{Ne}$ is that*

$$a_1 - \sum_{n=2}^{\infty} \frac{3 - 3n + 3e^{i\theta} - e^{3i\theta}}{3e^{i\theta} - e^{3i\theta}} a_n z^{n-1} \neq 0. \tag{2.9}$$

Proof. From Theorem 2.1, $f \in \mathcal{S}^*_{Ne}$ if and only if

$$\frac{1}{z}\left[f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \neq 0 \tag{2.10}$$

for all $L = \frac{3+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}$ and also $L = 1$. The left-hand side of (2.10) is written as

$$\begin{aligned}\frac{1}{z}\left[f(z) * \left(\frac{z}{(1-z)^2} - \frac{Lz^2}{(1-z)^2}\right)\right] &= \frac{1}{z}\{zf'(z) - L(zf'(z) - f(z))\} \\ &= a_1 - \sum_{n=2}^{\infty}(n(1-L)+L)a_n z^{n-1} \\ &= a_1 - \sum_{n=2}^{\infty}\frac{3-3n+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}a_n z^{n-1},\end{aligned}$$

which completes the desired proof. □

We next determine coefficient estimate for a function of form (1.2) to be in the class $\mathcal{S}^*_{Ne}$.

Theorem 2.3. *If the function f defined by (1.2) satisfies the following inequality*

$$\sum_{n=2}^{\infty}(3n-1)|a_n| \leq 2a_1, \tag{2.11}$$

then $f \in \mathcal{S}^*_{Ne}$.

Proof. According to the expression(2.9), a simple computation gives

$$\begin{aligned}\left|a_1 - \sum_{n=2}^{\infty}\frac{3-3n+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}a_n z^{n-1}\right| &\geq a_1 - \sum_{n=2}^{\infty}\left|\frac{3-3n+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}\right||a_n| \\ &= a_1 - \sum_{n=2}^{\infty}\frac{|-(3n-3)+(3e^{i\theta}-e^{3i\theta})|}{|3e^{i\theta}-e^{3i\theta}|}|a_n| \\ &\geq a_1 - \sum_{n=2}^{\infty}\frac{3n-1}{2}|a_n| \geq 0,\end{aligned}$$

if the inequality (2.11) holds. Hence, the desired proof is completed. □

By making use of the well-known Alexander relation between starlike and convex functions and in view of Theorem 2.1, following necessary and sufficient convolution conditions for the class $\mathcal{C}_{Ne}$ are given.

Theorem 2.4. *The function f defined by (1.2) is in the class $\mathcal{C}_{Ne}$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z+[1-2L]z^2}{(1-z)^3}\right] \neq 0 \tag{2.12}$$

for all $L = \frac{3+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}$, where $\theta \in [0, 2\pi]$, and also $L = 1$.

Reasoning along the similar lines as the proof of the Theorem 2.2 and Theorem 2.3, we establish following results for the class $\mathcal{C}_{Ne}$. We are omitting the details.

Theorem 2.5. *A necessary and sufficient condition for the function f defined by (1.2) to be in the class $\mathcal{C}_{Ne}$ is that*

$$a_1 - \sum_{n=2}^{\infty} n\frac{3-3n+3e^{i\theta}-e^{3i\theta}}{3e^{i\theta}-e^{3i\theta}}a_n z^{n-1} \neq 0. \tag{2.13}$$

Theorem 2.6. *If the function f defined by (1.2) satisfies the following inequality*

$$\sum_{n=2}^{\infty} n(3n-1)|a_n| \leq 2a_1, \tag{2.14}$$

then $f \in \mathcal{C}_{Ne}$.

3. Quasi-Hadamard product properties

In this section, we obtain quasi-Hadamard product of the classes $\mathcal{S}^*_{Ne}$ and $\mathcal{C}_{Ne}$. In order to prove further results in this section, we need to define a class $\mathcal{S}_{(c)Ne}$ which as follows: A function f of the form (1.2) is in $\mathcal{S}_{(c)Ne}$ if and only if the inequality

$$\sum_{n=2}^{\infty} n^c(3n-1)a_n \leq 2a_1$$

holds for any fixed non-negative real number c. It is noted that for $c=1$, $\mathcal{S}_{(1)Ne} \equiv \mathcal{C}_{Ne}$, and for $c=0$, $\mathcal{S}_{(0)Ne} \equiv \mathcal{S}^*_{Ne}$. Therefore for any positive integer c, following inclusion relation holds:

$$\mathcal{S}_{(c)Ne} \subset \mathcal{S}_{(c-1)Ne} \subset ... \subset \mathcal{S}_{(2)Ne} \subset \mathcal{C}_{Ne} \subset \mathcal{S}^*_{Ne}.$$

Theorem 3.1. *Let the functions f_i defined by (1.3) be in the class $\mathcal{S}^*_{Ne}$ for every $i = 1, 2, ...m$. Then the quasi-Hadamard product $f_1 * f_2 * ... * f_m$ belongs to the class $\mathcal{S}_{(m-1)Ne}$.*

Proof. To prove the theorem, we need to show that

$$\sum_{n=2}^{\infty}\left[n^{m-1}(3n-1)\prod_{i=1}^{m} a_{n,i}\right] \leq 2\prod_{i=1}^{m} a_{1,i}.$$

Since $f_i \in \mathcal{S}^*_{Ne}$, we have

$$\sum_{n=2}^{\infty}(3n-1)a_{n,i} \leq 2a_{1,i} \tag{3.1}$$

for every $i = 1, 2, ...m$. Thus,

$$(3n-1)a_{n,i} \leq 2a_{1,i}$$

or

$$a_{n,i} \leq \frac{2}{(3n-1)}a_{1,i}$$

for every $i = 1, 2, ...m$. Since $\frac{3n-1}{2} > n$ for every $n \geq 2$, thus $\frac{2}{3n-1} < \frac{1}{n}$. Hence, the right side of the last inequality not greater than $n^{-1}a_{1,i}$. Thus, we obtain

$$a_{n,i} \leq n^{-1}a_{1,i}. \tag{3.2}$$

By making use of the inequality (3.2) for $i = 1, 2, ...m-1$ and the inequality (3.1) for $i = m$, we get

$$\begin{aligned}\sum_{n=2}^{\infty}\left[n^{m-1}(3n-1)\prod_{i=1}^{m} a_{n,i}\right] &\leq \sum_{n=2}^{\infty}\left[n^{m-1}(3n-1)a_{n,m}\left\{n^{-(m-1)}\prod_{i=1}^{m-1} a_{1,i}\right\}\right]\\ &= \sum_{n=2}^{\infty}(3n-1)a_{n,m}\left\{\prod_{i=1}^{m-1} a_{1,i}\right\}\\ &\leq 2\prod_{i=1}^{m} a_{1,i}.\end{aligned}$$

Since $\mathcal{S}_{(m-1)Ne} \subset \mathcal{S}_{(m-2)Ne} \subset ... \subset \mathcal{S}_{(0)Ne} \equiv \mathcal{S}^*_{Ne}$ and therefore, $f_1 * f_2 * ... * f_m \in \mathcal{S}_{(m-1)Ne}$. This completes the proof. □

Theorem 3.2. *Let the functions f_i defined by (1.3) be in the class $\mathcal{C}_{Ne}$ for every $i = 1, 2, ...m$. Then the quasi-Hadamard product $f_1 * f_2 * ... * f_m$ belongs to the class $\mathcal{S}_{(2m-1)Ne}$.*

Proof. To prove the theorem, we need to show that

$$\sum_{n=2}^{\infty}\left[n^{2m-1}(3n-1)\prod_{i=1}^{m} a_{n,i}\right] \leq 2\prod_{i=1}^{m} a_{1,i}.$$

Since $f_i \in \mathcal{C}_{Ne}$, we have

$$\sum_{n=2}^{\infty} n(3n-1)a_{n,i} \leq 2a_{1,i} \tag{3.3}$$

for every $i = 1, 2, ...m$. Thus

$$n(3n-1)a_{n,i} \leq 2a_{1,i}$$

or

$$a_{n,i} \leq \frac{2}{n(3n-1)} a_{1,i}$$

for every $i = 1, 2, ...m$. Since $\frac{n(3n-1)}{2} > n^2$ for every $n \geq 2$, thus $\frac{2}{n(3n-1)} < \frac{1}{n^2}$. Then the right side of the last inequality not greater than $n^{-2}a_{1,i}$. Thus,

$$a_{n,i} \leq n^{-2}a_{1,i} \tag{3.4}$$

for every $i = 1, 2, ...m$. By making use of the inequality (3.4) for $i = 1, 2, ...m-1$ and the inequality (3.3) for $i = m$, we get

$$\begin{aligned}\sum_{n=2}^{\infty}\left[n^{2m-1}(3n-1)\prod_{i=1}^{m} a_{n,i}\right] &\leq \sum_{n=2}^{\infty}\left[n^{2m-1}(3n-1)a_{n,m}\left\{n^{-2(m-1)}\prod_{i=1}^{m-1} a_{1,i}\right\}\right]\\ &= \sum_{n=2}^{\infty} n(3n-1)a_{n,m}\left\{\prod_{i=1}^{m-1} a_{1,i}\right\}\\ &\leq 2\prod_{i=1}^{m} a_{1,i}.\end{aligned}$$

Since $\mathcal{S}_{(2m-1)Ne} \subset \mathcal{S}_{(2m-2)Ne} \subset ... \subset \mathcal{S}_{(1)Ne} \equiv \mathcal{C}_{Ne}$, thus, $f_1 * f_2 * ... * f_m \in \mathcal{S}_{(2m-1)Ne}$. This completes the proof. □

Theorem 3.3. *Let the functions f_i defined by (1.3) be in the class $\mathcal{C}_{Ne}$ for every $i = 1, 2, ...m$; and let the functions g_j defined by (1.4) be in the class $\mathcal{S}^*_{Ne}$ for every $j = 1, 2, ...s$. Then the quasi-Hadamard product $f_1 * f_2 * ... * f_m * g_1 * g_2 * ... * g_s$ belongs to the class $\mathcal{S}_{(2m+s-1)Ne}$.*

Proof. To prove the theorem, we need to show that

$$\sum_{n=2}^{\infty}\left[n^{2m+s-1}(3n-1)\left\{\prod_{i=1}^{m} a_{n,i}\prod_{j=1}^{s} b_{n,j}\right\}\right] \leq 2\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{s} b_{1,j}\right\}.$$

Since $f_i \in \mathcal{C}_{Ne}$, we have

$$\sum_{n=2}^{\infty} n(3n-1)a_{n,i} \leq 2a_{1,i}$$

for every $i = 1, 2, ...m$, thus it is noted that

$$n(3n-1)a_{n,i} \leq 2a_{1,i}$$

or

$$a_{n,i} \leq \frac{2}{n(3n-1)} a_{1,i}.$$

The right side of the last inequality not greater than $n^{-2}a_{1,i}$. Thus,

$$a_{n,i} \leq n^{-2}a_{1,i} \tag{3.5}$$

for every $i = 1, 2, ...m$. Similarly, since $g_j \in \mathcal{S}^*_{Ne}$, we have

$$\sum_{n=2}^{\infty}(3n-1)b_{n,j} \leq 2b_{1,j} \tag{3.6}$$

for every $j = 1, 2, ...s$. Hence, we obtain

$$b_{n,j} \leq n^{-1} b_{1,j}. \tag{3.7}$$

By using the inequality (3.5) for $i = 1, 2, ...m$, the inequality(3.7) for $j = 1, 2, ...s-1$ and the inequality (3.6) for $j = s$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[n^{2m+s-1}(3n-1)\left\{\prod_{i=1}^{m} a_{n,i}\prod_{j=1}^{s} b_{n,j}\right\}\right]\\
&\leq \sum_{n=2}^{\infty}\left[n^{2m+s-1}(3n-1)b_{n,s}\left\{n^{-2m}n^{-(s-1)}\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{s-1} b_{1,j}\right\}\right]\\
&= \sum_{n=2}^{\infty}(3n-1)b_{n,s}\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{s-1} b_{1,j}\right\}\\
&\leq 2\left\{\prod_{i=1}^{m} a_{1,i}\prod_{j=1}^{s} b_{1,j}\right\}.
\end{aligned}$$

Since $\mathcal{S}_{(2m+s-1)Ne} \subset \mathcal{S}_{(2m+s-2)Ne} \subset ... \subset \mathcal{S}_{(2)Ne} \subset \mathcal{C}_{Ne} \subset \mathcal{S}^*_{Ne}$, we conclude the required result. □

4. Third order Hermitian–Toeplitz and Hankel determinants

The first result of this section provides the best possible lower and upper bounds for the Hermitian–Toeplitz determinants of third order for the class $\mathcal{S}^*_{Ne}$. In order to prove this result, we need the following lemma due to Libera and Zlotkiewicz:

Lemma 4.1. [27, Lemma 3, p. 254] Let $\mathcal{P}$ be the class of analytic functions having the Taylor series of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{4.1}$$

satisfying the condition $Re(p(z)) > 0$ $(z \in \mathbb{D})$. Then

$$2p_2 = p_1^2 + (4 - p_1^2)\xi$$

for some $\xi \in \overline{\mathbb{D}}$.

Theorem 4.2. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{S}^*_{Ne}$. Then the best possible bounds on third order Hermitian–Toeplitz are given by*

$$-\frac{1}{4} \leq |T_{3,1}(f)| \leq 1.$$

Proof. Let the function $f \in \mathcal{S}^*_{Ne}$. Then, we have $zf'(z)/f(z) = 1 + w(z) - w^3(z)/3$, where $w(z) = c_1 z + c_2 z^2 \cdots \in \Omega$. Therefore, for some $p \in \mathcal{P}$ of the form (4.1), it is noted that

$$\frac{zf'(z)}{f(z)} = \frac{5(p(z))^3 + 15(p(z))^2 + 3p(z) + 1}{3(p(z)+1)^3}. \tag{4.2}$$

On equating the coefficients of like power terms, we get

$$a_2 = \frac{p_1}{2} \quad \text{and} \quad a_3 = \frac{p_2}{4}. \tag{4.3}$$

In view of (4.3) and Lemma 4.1, for some $\xi \in \overline{\mathbb{D}}$, we have

$$\begin{aligned}
2Re(a_2^2\,\overline{a_3}) &= 2Re(\frac{p_1^2}{4}.\frac{1}{4}\overline{p_2})\\
&= \frac{1}{16}p_1^2(p_1^2 + (4-p_1^2)Re(\overline{\xi}))\\
&= \frac{1}{16}(p_1^4 + (4-p_1^2)p_1^2 Re(\overline{\xi})),
\end{aligned} \tag{4.4}$$

$$2|a_2|^2 = \frac{1}{2}p_1^2, \tag{4.5}$$

and

$$\begin{aligned}|a_3|^2 &= \left|\frac{1}{4}(p_2)\right|^2 \\ &= \frac{1}{16}\left(p_1^4 + (4-p_1^2)^2|\xi|^2 + 2(4-p_1^2)p_1^2 Re(\bar{\xi})\right). \end{aligned} \tag{4.6}$$

In view of expressions (1.7), (4.4), (4.5) and (4.6), we have

$$\begin{aligned}|T_{3,1}(f)| :&= 1 + \frac{1}{64}(3p_1^4 - 32p_1^2 - (4-p_1^2)^2|\xi|^2 + 2(4-p_1^2)p_1^2 Re(\bar{\xi})) \\ &= F(p_1^2,\, |\xi|,\, Re(\bar{\xi})). \end{aligned} \tag{4.7}$$

Making use of inequality $-Re\xi \leq |\xi| \geq Re\xi$, above expression is written as

$$|T_{3,1}(f)| : \leq 1 + \frac{1}{64}(3x^2 - 32x - (4-x)^2y^2 + 2(4-x)xy)) = F(x,\, y)$$

and

$$|T_{3,1}(f)| : \geq 1 + \frac{1}{64}(3x^2 - 32x - (4-x)^2y^2 - 2(4-x)xy)) = G(x,\, y),$$

where $p^2 =: x \in [0,4]$ and $|\xi| =: y \in [0,1]$. By making use of second derivative test for function of two variable, we note that $F(x,\, y)$ has no extreme point in the interior region of the rectangular domain $S = [0,4] \times [0,1]$. Therefore, the function $F(x,\, y)$ has maximum value on the boundary of domain S that is 1. In similar way, the function $G(x,\, y)$ has the minimum value in the domain S that is $-1/4$. The analysis done on the functions F and G for getting extreme values gives the desired inequality. The upper and the lower bounds are sharp for the function f_u and f_l, respectively, which are defined by

$$\frac{zf_u'(z)}{f_u(z)} = 1 + z^3 - \frac{1}{3}z^9 \quad \text{and} \quad \frac{zf_l'(z)}{f_l(z)} = 1 + z - \frac{1}{3}z^3.$$

□

Next, we provide non-sharp upper bounds on some Hankel determinants of third order for the functions in the class $\mathcal{S}^*_{Ne}$. In order to prove results related to Hankel determinants, we need following lemmas.

Lemma 4.3. [1, Lemma 3, p. 66] Let the function $p \in \mathcal{P}$, $0 \leq \beta \leq 1$ and $\beta(2\beta-1) \leq \delta \leq \beta$. Then

$$|p_3 - 2\beta p_1 p_2 + \delta p_1^3| \leq 2.$$

Lemma 4.4. [33, Lemma 2.3, p. 507] Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$,

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \text{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, then the inequality is sharp for the function $p(z) = (1+z^{m+n})/(1-z^{m+n})$. In the other cases, the inequality is sharp for the function $p_0(z) = (1+z)/(1-z)$.

Lemma 4.5. [20] Let $p \in \mathcal{P}$. Then, for any real number μ, the following holds:

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq \frac{4}{3}; \\ 2\mu\sqrt{\frac{\mu}{\mu-1}}, & \mu > \frac{4}{3}. \end{cases}$$

The result is sharp. If $\mu \leq \frac{4}{3}$, equality holds for the function $p_0(z) := (1+z)/(1-z)$, and if $\mu > \frac{4}{3}$, then equality holds for the function

$$p_1(z) := \frac{1-z^2}{z^2 - 2\sqrt{\frac{\mu}{\mu-1}}\, z + 1}.$$

Theorem 4.6. *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{S}^*_{Ne}$. Then,*

(i) $|H_3^{(1)}(f)| \leq 0.925696$,
(ii) $|H_3^{(2)}(f)| \leq 1.6225$,
(iii) $|H_3^{(3)}(f)| \leq 1.34575$.

Proof. In view of (1.6), the third order Hankel determinants $H_3^{(1)}(f)$, $H_3^{(2)}(f)$ and $H_3^{(3)}(f)$ for the functions $f \in \mathcal{A}$ are given by

$$H_3^{(1)}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \tag{4.8}$$

$$H_3^{(2)}(f) = a_2(a_2a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2), \tag{4.9}$$

$$H_3^{(3)}(f) = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2). \tag{4.10}$$

Since the function $f \in \mathcal{S}^*_{Ne}$, then from expression (4.2), we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} = 1 &+ a_2z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 + (-a_2^4 + 4a_2^2a_3 - 4a_2a_4 \\ &- 2a_3^2 + 4a_5)z^4 + (a_2^5 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2 - 5a_2a_5 - 5a_3a_4 + 5a_6)z^5 \\ &+ (-a_2^6 + 6a_2^4a_3 - 6a_2^3a_4 - 9a_2^2a_3^2 + 6a_2^2a_5 + 12a_2a_3a_4 - 6a_2a_6 + 2a_3^3 \\ &- 6a_3a_5 - 3a_4^2 + 6a_7)z^6 + \cdots\end{aligned}$$

and

$$\begin{aligned}\frac{5(p(z))^3 + 15(p(z))^2 + 3p(z) + 1}{3(p(z)+1)^3} =& 1 + \frac{p_1z}{2} + (\frac{p_2}{2} - \frac{p_1^2}{4})z^2 + \frac{1}{12}(p_1^3 - 6p_1p_2 + 6p_3)z^3 \\ &+ \frac{1}{4}(p_1^2p_2 - 2p_1p_3 - p_2^2 + 2p_4)z^4 + \frac{1}{32}(-p_1^5 + 8p_1^2p_3 \\ &+ 8p_1p_2^2 - 16p_1p_4 - 16p_2p_3 + 16p_5)z^5 + \frac{1}{192}(7p_1^6 \\ &- 30p_1^4p_2 + 48p_1^2p_4 + 96p_1p_2p_3 - 96p_1p_5 + 16p_2^3 \\ &- 96p_2p_4 - 48p_3^2 + 96p_6)z^6 + \cdots .\end{aligned}$$

On equating the coefficients of like power of z, we have

$$a_4 = \frac{1}{72}(-p_1^3 - 3p_1p_2 + 12p_3), \tag{4.11}$$

$$a_5 = \frac{1}{576}(5p_1^4 - 12p_1^2p_2 - 18p_2^2 - 24p_1p_3 + 72p_4), \tag{4.12}$$

$$a_6 = \frac{1}{5760}(-27p_1^5 + 160p_1^3p_2 - 72p_1^2p_3 - 336p_2p_3 - 6p_1(7p_2^2 + 36p_4) + 576p_5), \tag{4.13}$$

$$\begin{aligned}a_7 = \frac{1}{103680}&(262p_1^6 - 2235p_1^4p2 + 2352p_1^3p_3 + 36p_1^2\left(97p_2^2 - 24p_4\right) \\ &- 72p_1(7p_2p_3 + 48p_5) + 90\left(p_2^3 - 60p_2p_4 - 32p_3^2 + 96p_6\right)).\end{aligned} \tag{4.14}$$

After rearrangement of terms and on applying triangle inequality, the expressions given by (4.11) and (4.12) are written as

$$|a_4| \leq \frac{1}{6}\left|p_3 - \frac{1}{4}p_1p_2 - \frac{1}{12}p_1^3\right|, \tag{4.15}$$

and

$$576|a_5| \leq 12||p_1|^2\left|\frac{5}{12}p_1^2 - p_2\right| + 74\left|-\frac{12}{37}p_1p_3 + p_4\right| + 18|p_2|^2. \tag{4.16}$$

In view of the fact $|p_n| \leq 2$ and by making use of Lemma 4.3 and Lemma 4.4 in inequalities (4.15) and (4.16), respectively, we have

$$|a_4| \leq \frac{1}{3} \quad \text{and} \quad |a_5| \leq \frac{79}{144}. \tag{4.17}$$

(i) For the function $f \in \mathcal{S}^*_{Ne}$, using (4.3), (4.11), (4.12), (4.13) and (4.8), we have

$$\begin{aligned} 20736 H_3^{(1)}(f) &= -49p_1^6 + 57p_1^4p_2 - 198p_1^2p_2^2 - 486p_2^3 + 312p_1^3p_3 \\ &\quad + 936p_1p_2p_3 - 576p_3^2 - 648p_1^2p_4 + 648p_2p_4 \\ &= 57p_1^4\big(-\frac{49}{57}p_1^2 + p_2\big) + 936p_1p_2\big(-\frac{11}{52}p_1p_2 + p_3\big) \\ &\quad + 648p_4(-p_1^2 + p_2) + 312p_3\big(p_1^3 - \frac{24}{13}p_3\big) - 486p_2^3. \end{aligned} \tag{4.18}$$

By making use of triangle inequality, Lemmas 4.4, 4.5 and the fact $|p_n| \leq 2$, the expression (4.18) becomes

$$\begin{aligned} 20736|H_3^{(1)}(f)| &\leq 57(2)^5 + 936(2)^3 + 648(2)^2 + 312(2)^2\left(\frac{24}{13}\right)\sqrt{\frac{24}{11}} + 486(2)^3 \\ &= 15792 + 4608\sqrt{\frac{6}{11}}, \end{aligned}$$

which implies

$$|H_3^{(1)}(f)| \leq \frac{329}{432} + \frac{2}{3}\sqrt{\frac{2}{33}} \approx 0.925696.$$

(ii) Further, if $f \in \mathcal{S}^*_{Ne}$, using (4.3), (4.11), (4.12), (4.13) and (4.9), we have

$$\begin{aligned} 29859840 H_3^{(2)}(f) &= -34992p_1^7 - 1045p_1^9 + 207360p_1^5p_2 + 4320p_1^7p_2 - 54432p_1^3p_2^2 + 11448p_1^5p_2^2 \\ &\quad - 49680p_1^3p_2^2 + 18468p_1p_2^4 - 93312p_1^4p_3 + 7920p_1^6p_3 - 435456p_1^2p_2p_3 \\ &\quad - 12960p_1^4p_2p_3 - 67392p_1^2p_2^2p_3 + 31104p_2^2p_3 + 8640p_1^3p_3^2 - 138240p_3^3 \\ &\quad - 32400p_1^5p_4 + 51840p_1^3p_2p_4 + 108864p_1p_2^2p_4 + 155520p_1^2p_3p_4 \\ &\quad + 311040p_2p_3p_4 - 233280p_1p_4^2 + 746496p_1^2p_5 - 186624p_2^2p_5 - 279936p_1^3p_4. \end{aligned}$$

After rearrangement of terms and using triangle inequality, above expression can be written as

$$\begin{aligned} 29859840|H_3^{(2)}(f)| &\leq 207360|p_1|^5\left|-\frac{27}{160}p_1^2 + p_2\right| + 4320|p_1|^7\left|-\frac{209}{864}p_1^2 + p_2\right| \\ &\quad + 49680|p_2|^2|p_1|^3\left|\frac{53}{230}p_1^2 - p_2\right| + 12960|p_1|^4|p_3|\left|\frac{11}{18}p_1^2 - p_2\right| \\ &\quad + 31104|p_3||p_2|^2\left|-\frac{13}{6}p_1^2 + p_2\right| + 8640|p_3|^2\left|p_1^3 - 16p_3\right| \\ &\quad + 746496|p_1|^2\left|-\frac{3}{8}p_1p_4 + p_5\right| + 51840|p_1|^3|p_4|\left|-\frac{5}{8}p_1^2 + p_2\right| \\ &\quad + 186624|p_2|^2\left|\frac{7}{12}p_1p_4 - p_5\right| + 233280|p_1||p_4|\left|\frac{2}{3}p_1p_3 - p_4\right| \\ &\quad + 18468|p_1||p_2|^4 + 311040|p_2||p_3||p_4| + 93312|p_1|^4|p_3| \\ &\quad + 435456|p_1|^2|p_2||p_3| + 54432|p_1|^3|p_2|^2. \end{aligned}$$

Using Lemmas 4.4, 4.5 and the fact $|p_n| \leq 2$, above inequality becomes

$$\begin{aligned}
29859840|H_3^{(2)}(f)| \leq\; & 207360(2)^6 + 4320(2)^8 + 54432(2)^5 + 49680(2)^6 + 18468(2)^5 \\
& + 12960(2)^6 + 93312(2)^5 + 31104(2)^4(\frac{10}{3}) + 435456(2)^4 \\
& + 8640(2)^3(16)\sqrt{\frac{16}{15}} + 746496(2)^3 + 51840(2)^5 \\
& + 186624(2)^3 + 233280(2)^3 + 311040(2)^3 \\
=\; & 256(184787 + 1152\sqrt{15}),
\end{aligned}$$

which implies that

$$|H_3^{(2)}(f)| \leq \frac{184787}{116640} + \frac{4}{27\sqrt{15}} \approx 1.6225.$$

(iii) In view of (4.11), (4.12) and (4.13), a simple calculation yields

$$\begin{aligned}
1658880(a_4a_6 - a_5^2) =& -5(-5p_1^4 + 12p_1^2p_2 + 24p_1p_3 + 18(p_2^2 - 4p_4))^2 \\
& + 4(p_1^3 + 3p_1p_2 - 12p_3)(27p_1^5 - 160p_1^3p_2 + 42p_1p_2^2 + 72p_1^2p_3 \\
& + 336p_2p_3 + 216p_1p_4 - 576p_5) \\
=& -17p_1^8 + 284p_1^6p_2 + 192p_1^5p_3 - 1572p_1^4p_2^2 - 2736p_1^4p_4 + 7008p_1^3p_2p_3 \\
& - 2304p_1^3p_5 - 1656p_1^2p_2^3 + 11232p_1^2p_2p_4 - 6336p_1^2p_3^2 - 2304p_1p_2^2p_3 \\
& - 6912p_1p_2p_5 + 6912p_1p_3p_4 - 1620p_2^4 + 12960p_2^2p_4 - 16128p_2p_3^2 \\
& + 27648p_3p_5 - 25920p_4^2.
\end{aligned}$$

On rearrangement of terms, above expression becomes

$$\begin{aligned}
1658880(a_4a_6 - a_5^2) =& 284p_1^6\left(-\frac{17}{284}p_1^2 + p_2\right) + 27648p_5\left(-\frac{1}{4}p_1p_2 + p_3\right) \\
& + 25920\left(\frac{4}{15}p_1p_3 - p_4\right) + 11232p_1^2p_2\left(-\frac{23}{156}p_2^2 + p_4\right) \\
& + 12960p_2^2\left(-\frac{581}{648}p_2^2 + p_4\right) + 2736p_1^4\left(\frac{4}{57}p_1p_3 - p_4\right) \\
& + 7008p_1^3p_3 - 2304p_1p_2^2p_3 - 6336p_1^2p_3^2 - 16128p_2p_3^2 \\
& - 2304p_1^3p_5 - 1572p_1^4p_2^2. \qquad (4.19)
\end{aligned}$$

Using triangle inequality and Lemma 4.4 in (4.19), we get

$$1658880|a_4a_6 - a_5^2| \leq 1098432,$$

which implies

$$|a_4a_6 - a_5^2| \leq \frac{1907}{2880}. \qquad (4.20)$$

In view of (4.11), (4.12), (4.13) and (4.14), a simple calculation yields

$$\begin{aligned}
29859840(a_4a_7 - a_5a_6) =& 167p_1^9 - 4320p_1^7p_2 + 576p_1^6p_3 + 27648p_1^5p_2^2 + 30672p_1^5p_4 \\
& - 91584p_1^4p_2p_3 - 12096p_1^4p_5 - 20880p_1^3p_2^3 - 95040p_1^3p_2p_4 \\
& + 108864p_1^3p_3^2 - 34560p_1^3p_6 + 116640p_1^2p_2^2p_3 + 103680p_1^2p_2p_5 \\
& - 41472p_1^2p_3p_4 - 7884p_1p_2^4 + 57024p_1p_2^2p_4 - 62208p_1p_2p_3^2 \\
& - 103680p_1p_2p_6 - 41472p_1p_3p_5 + 139968p_1p_4^2 - 50112p_2^3p_3 \\
& + 93312p_2^2p_5 - 41472p_2p_3p_4 - 138240p_3^3 + 414720p_3p_6 \\
& - 373248p_4p_5 \\
=& 4320p_1^7\left(\frac{167}{4320}p_1^2 - p_2\right) + 20880p_1^3p_2^2\left(\frac{192}{145}p_1^2 - p_2\right) \\
& + 91584p_1^4p_3\left(-\frac{1}{159}p_1^2 + p_2\right) + 108864p_3^2\left(-\frac{80}{63}p_3 + p_1^3\right) \\
& + 62208p_1p_2p_3\left(\frac{3645}{19444}p_1p_2 - p_3\right) + 93312p_2^2\left(-\frac{29}{54}p_2p_3 + p_5\right) \\
& - 12096p_1^4\left(\frac{1417}{56}p_1p_4 - p_5\right) + 41472p_2p_4\left(\frac{11}{8}p_1p_2 - p_3\right) \\
& + 139968p_1p_4\left(-\frac{8}{27}p_1p_3 + p_4\right) + 103680p_1p_2\left(p_1p_5 - p_6\right) \\
& + 34560\left(12p_3 - p_1^3\right) - 7884p_1p_2^4 - 95040p_1^3p_2p_5 \\
& - 41472p_1p_3p_5 - 373248p_4p_5.
\end{aligned}$$

By making use of triangle inequality, Lemmas 4.4, 4.5 and the fact $|p_n| \leq 2$ in above inequality, we get

$$29859840|a_4a_7 - a_5a_6| \leq \frac{1152}{187}\left(6193253 + 48960\sqrt{33} + 42240\sqrt{85}\right)$$

implies that

$$|a_4a_7 - a_5a_6| \leq \frac{4\sqrt{\frac{5}{17}}}{27} + \frac{33119}{25920} + \frac{1}{3\sqrt{33}}. \tag{4.21}$$

In view of (4.12), (4.13) and (4.14), a simple calculation yields

$$
\begin{aligned}
298598400(a_5a_7 - a_6^2) = {} & 5\left(5p_1^4 - 12p_1^2p_2 - 24p_1p_3 - 18p_2^2 + 72p_4\right)(262p_1^6 - 2235p_1^4p_2 \\
& + 2352p_1^3p_3 + 36p_1^2(97p_2^2 - 24p_4) - 72p_1(7p_2p_3 + 48p_5) \\
& + 90(p_2^3 - 60p_2p_4 - 32p_3^2 + 96p_6)) - 9(27p_1^5 - 160p_1^3p_2 \\
& + 72p_1^2p_3 + 42p_1p_2^2 + 216p_1p_4 + 336p_2p_3 - 576p_5)^2 \\
= {} & -11p_1^{10} + 6165p_1^8p_2 - 7632p_1^7p_3 - 52992p_1^6p_2^2 - 32256p_1^6p_4 \\
& + 158544p_1^5p_2p_3 + 193536p_1^5p_5 + 114840p_1^4p_2^3 - 265680p_1^4p_2p_4 \\
& - 400896p_1^4p_3^2 + 216000p_1^4p_6 + 312768p_1^3p_2^2p_3 - 1451520p_1^3p_2p_5 \\
& + 670464p_1^3p_3p_4 - 335556p_1^2p_2^4 + 1495584p_1^2p_2^2p_4 - 202176p_1^2p_2p_3^2 \\
& - 518400p_1^2p_2p_6 + 1161216p_1^2p_3p_5 - 730944p_1^2p_4^2 - 219456p_1p_2^3p_3 \\
& + 746496p_1p_2^2p_5 - 839808p_1p_2p_3p_4 + 345600p_1p_3^3 - 1036800p_1p_3p_6 \\
& + 995328p_1p_4p_5 - 8100p_2^5 + 518400p_2^3p_4 - 756864p_2^2p_3^2 - 777600p_2^2p_6 \\
& + 3483648p_2p_3p_5 - 1944000p_2p_4^2 - 1036800p_3^2p_4 \\
& + 3110400p_4p_6 - 2985984p_5^2.
\end{aligned}
$$

In similar way, on rearrangement of terms and on applying triangle inequality, Lemmas 4.4, 4.5 and the fact $|p_n| \leq 2$ in above expression, we get

$$|a_5a_7 - a_6^2| \leq \frac{264619}{259200}. \tag{4.22}$$

On applying triangle inequality in (4.10), we have

$$|H_3^{(3)}(f)| \leq |a_3||a_5a_7 - a_6^2| + |a_4||a_4a_7 - a_5a_6| + |a_5||a_4a_6 - a_5^2|$$

and on putting the desired values from (4.17), (4.20), (4.21), (4.22) in above inequality, we have

$$|H_3^{(3)}(f)| \leq \frac{4}{81}\sqrt{\frac{5}{17}} + \frac{8084743}{6220800} + \frac{1}{9\sqrt{33}} \approx 1.34575.$$

□

Acknowledgement

The authors would like to express their gratitude to the referees for many valuable suggestions and insights that helped to improve quality and clarity of this manuscript.

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