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On the Exact Solutions of a Nonlinear Conformable Time Fractional Equation via IBSEFM

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Abstract: Investigating the solutions of fractional differential equations are essential to understand the nonlinear process that appears in some branch of physical phenomena such as optics, quantum electrons, control theory of dynamical systems. Several computational techniques for the solutions of these equations have been developed. In this study, we implement the Improved Bernoulli Sub-Equation Function Method (IBSEFM) to construct the exact solutions of conformable time fractional KdV equation. We obtain the exact solutions of KdV equation via IBSEFM. We plot the 2D,3D graphs and contourplots by the aid of mathematics software that acquired from the values of the solutions.

Keywords: Conformable Time Fractional Derivative, Exact Solutions, Improved Bernoulli Sub-Equation Function Method.

1 Introduction

In the last decades, the fractional differential equations have become a useful tool for describing nonlinear phenomena of science and engineering models. Many of techniques which applied to the nonlinear partial differential equations have been adapted for fractional nonlinear partial differential equations to find exact solutions. For example fractional Riccati expansion [1], functional variable [2], exp-function [3], first integral [4], Improved Bernoulli Sub-Equation function method [5] and many others.

Korteweg-de Vries (KdV) equation is the most famous nonlinear partial differential equation with soliton type wave solutions in shallow water surface. Although it was firstly introduced by Boussinesq [6] in 1877, it is named after the study of Korteweg and de Vries [7]. The KdV equation is completely integrable and has wave solutions of classical solitary wave type shapes. In addition to this, it has infinitely many conservation laws representing various physical quantities such as energy, momentum, mass, etc. This equation can describe many types of physical phenomena particularly waves covering internal ocean waves in changing density layers, plasma ion-acoustic waves and acoustic-type waves over crystal lattice.

The KdV equation is similar to the nonlinear Schrödinger equation due to the fact that both are solvable by inverse scattering transform approach. It has stable N-soliton solutions that behave like particles, too [8]. These solutions are valid for multiple collisions, i.e. more than two well-separated solitons, even when their heights are different from each other [9]. Recent developments in computer algebra lead various solution techniques to appear [10]-[11]-[12].

In this study, we consider conformable time fractional KdV equation as follows:

$$D_t^{\gamma}U + qUU_z + pU_{zzz} = 0, \quad t \geq 0, \quad \gamma \in (0,1], \tag{1}$$

where U is function of the independent variables t, z and p is real parameter. The operator D_t^{γ} represents conformal fractional derivative operator defined only for positive region of t [13]. Different from some classical methods such as various forms of Kudryashov approach,rational exponential approach [15],[16], etc., the fractional form of the Sine-Gordon equation method is implemented to both equations to derive exact solutions in traveling wave forms.

During last few years, a straightforward in [14] definition of the fractional derivative called conformable fractional derivative is given. It is given a new definition of the fractional derivative as conformable fractional derivative in [13]. They show that conformable fractional derivative has simple definition and theoretically easier than fractional derivative to handle. Furthermore some properties of conformable fractional derivative are included in the study.

The conformable derivative of order α with respect to the independent variable t is defined as [14]:

$$D_t^{\alpha}(y(t)) = \lim_{\tau \to 0} \frac{y(t + \tau t^{1-\alpha}) - y(t)}{\tau}, \ t > 0, \ \alpha \in (0,1],$$

for a function $y = y(t) : [0, \infty) \to \mathbb{R}$.

Theorem 1. *Assume that the order of the derivative $\alpha \in (0,1]$ and suppose that $u = u(t)$ and $v = v(t)$ are $\alpha-$ differentiable for all positive t. Then,*

1. $D_t^a(c_1 u + c_2 v) = c_1 D_t^a(u) + c_2 D_t^a(v)$, *for* $\forall\, c_1, c_2 \in \mathbb{R}$.

2. $D_t^a(t^k) = kt^{k-a}, \forall\, k \in \mathbb{R}$.

3. $D_t^a(\lambda) = 0$, *for all constant function* $u(t) = \lambda$.

4. $D_t^a\,(uv) = uD_t^a(v) + vD_t^a(u)$.

5. $D_t^a\left(\frac{u}{v}\right) = \frac{vD_t^a(u) - uD_t^a(v)}{v^2}$.

6. $D_t^a\,(u)\,(t) = t^{1-\alpha}\frac{du}{dt}$.

Conformable fractional differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion and Laplace transform.

2 Description of the IBSEFM

In this section, we give the fundamental properties of the IBSEFM [17–20]. We have five main steps of the IBSEFM below the following:

Step 1: Let us take account of the following conformable time-fractional partial differantial equation of the style

$$P(v, D_t^{(\mu)}v, D_x^{(\mu)}v, D_{xt}^{(2\mu)}v, ...) = 0, \tag{2}$$

where $D_t^{(\mu)}$ is the conformable fractional derivate operator, $v(x,t)$ is an unknown function, P is a polynomial and its partial derivatives contain fractional derivatives. The aim is to convert conformable time fractional nonlinear partial differential equation with a suitable fractional transformation into the ordinary differantial equation. The wave transformation as

$$v(x,t) = V\left(\xi\right), \;\; \xi = \xi\left(x, t^{\alpha}\right). \tag{3}$$

Using the properties of conformable fractional derivate, it enables us to convert (2) into an ODE in the form

$$N(V, V', V'', ...) = 0. \tag{4}$$

If we integrate (4) term to term, we acquire integration constant(s) which may be determined then.

Step 2: We hypothesize that the solution of (4) may be presented below:

$$V(\xi) = \frac{\sum\limits_{i=0}^{n} a_i F^i(\xi)}{\sum\limits_{j=0}^{m} b_j F^j(\xi)} = \frac{a_0 + a_1 F(\xi) + a_2 F^2(\xi) + ...a_n F^n(\xi)}{b_0 + b_1 F(\xi) + b_2 F^2(\xi) + ...b_m F^m(\xi)}, \tag{5}$$

where $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$ are coefficients which will be determined later. $m \neq 0, n \neq 0$ are chosen arbitrary constants to balance principle and considering the form of Bernoulli differential equation below the following;

$$F'(\xi) = \sigma F(\xi) + dF^M(\xi), \; d \neq 0, \sigma \neq 0, \; M \in \mathbb{R}/\left\{0, 1, 2\right\}, \tag{6}$$

where $F(\xi)$ is polynomial.

Step 3: The positive integer m, n, M (are not equal to zero) which is found by balance principle that is both nonlinear term and the highest order derivative term of (4).

Substituting (5), (6) in (2) it gives us an equation of polynomial $\Theta(F)$ of F as follows;

$$\Theta(F(\xi)) = \rho_s F(\xi)^s + ... + \rho_1 F(\xi) + \rho_0 = 0,$$

where $\rho_i, i = 0, ..., s$ are coefficients and will be determined later.

Step 4: The coefficients of $\Theta(F(\xi))$ which will give us a system of algebraic equations, whole be zero.

$$\rho_i = 0, i = 0, .., s.$$

Step 5: When we solve (4), we get the following two cases with respect to σ and d,

$$F(\xi) = \left[\frac{-de^{\sigma(\epsilon-1)} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon-1)\xi}}\right]^{\frac{1}{1-\epsilon}}, \;\; d \neq \sigma, \tag{7}$$

$$F(\xi) = \left[\frac{(\epsilon-1) + (\epsilon+1)\tanh(\sigma(1-\epsilon))\frac{\xi}{2}}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}\right], d = \sigma, \; \epsilon \in \mathbb{R}. \tag{8}$$

Using a complete discrimination system for polynomial of $F(\xi)$, we obtain the analytical solutions of (4) via mathematics software and categorize the exact solutions of (4). To achieve better results, we can plot two and three dimensional figures of analytical solutions by considering proper values of parameters.

3 Implementation of the IBSFM

By taking the traveling wave transformation as:

$$U\left(z,t\right)=u\left(\eta\right),\ \eta=k(z-lt^{\gamma}/\gamma), \tag{9}$$

where k, l are arbitrary constants (are not zero) to be determined later. Using the (9) into (1), we get

$$-2klu+kru^2+2pk^3u''=0. \tag{10}$$

When we apply the balance for the terms u^2 and u'', we obtain the relationship for m, n and M as below:

$$M+m=n+1.$$

This relationship of $m, n,$ and M give us different types of the solutions of (10). Using homogeneous balance principle, for $M=3, n=5$ and $m=1$ in (1), then we get as follows:

$$u(\eta)=\frac{\sum\limits_{i=0}^{5}a_iF^i(\eta)}{\sum\limits_{j=0}^{1}b_jF^j(\eta)}=\frac{a_0+a_1F(\eta)+a_2F^2(\eta)+a_3F^3+a_4F^4(\eta)+a_5F^5(\eta)}{b_0+b_1F(\eta)}=\frac{\psi}{\phi}, \tag{11}$$

$$u'(\eta)=\frac{\psi'\phi-\phi'\psi}{\phi^2},$$

$$u''(\eta)=\frac{\Upsilon'(\eta)\Psi\left(\eta\right)-\Upsilon(\eta)\Psi'(\eta)}{\Psi^2(\eta)}-\frac{\left[\Upsilon(\eta)\Psi'\left(\eta\right)\right]'\Psi^2(\eta)-2\Upsilon(\eta)[\Psi'(\eta)]^2\Psi(\eta)}{\Psi^4(\eta)}. \tag{12}$$

Substituting (11) and (12) in (10), we get an algebraic equation system according to F.

When we solve the system of the equations of F and substitute in each case the obtained result of the coefficients to get the new solution(s) $U(z,t)$. By solving the system with the aid of software, the coefficients are obtained as:

Case 1. For $\sigma\neq d$,

$$a_0=-\frac{8k^2p\sigma^2b_0}{r};\quad a_1=-\frac{8k^2p\sigma^2a_5b_0}{ra_4};\quad a_2=-\frac{-4i\sqrt{3}k\sqrt{p}ra_5\sqrt{b_0}}{\sqrt{r}}; \tag{13}$$

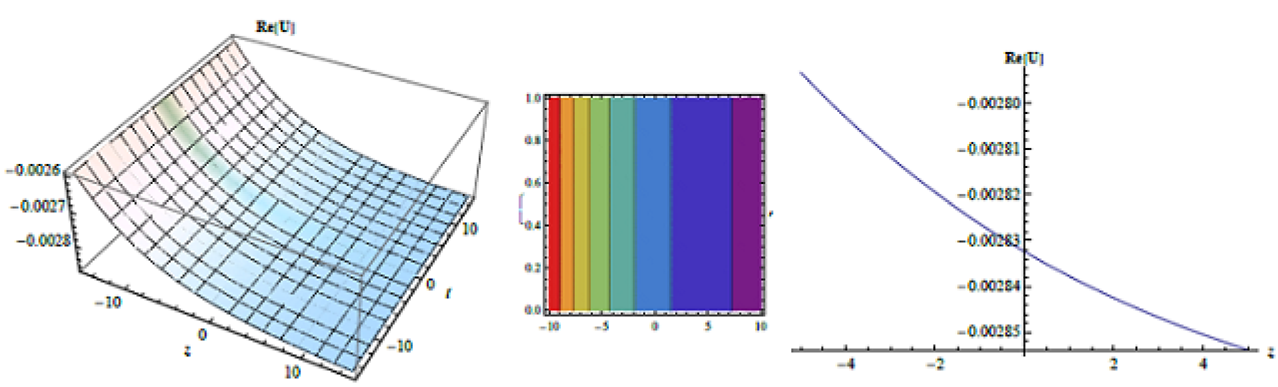
$$a_3=-\frac{-4i\sqrt{3}k\sqrt{p}ra_5\sqrt{b_0}}{\sqrt{ra_4}};\quad b_1=\frac{a_5b_0}{a_4};\quad l=-4k^2p\sigma^2;\quad d=\frac{i\sqrt{r}\sqrt{a_4}}{4\sqrt{3}k\sqrt{p}\sqrt{b_0}},$$

where $k, r, p, a_4, b_0\neq 0$.

Putting (13) along with (7) in (11), we acquire the exponential function solution of the KdV equation as follow:

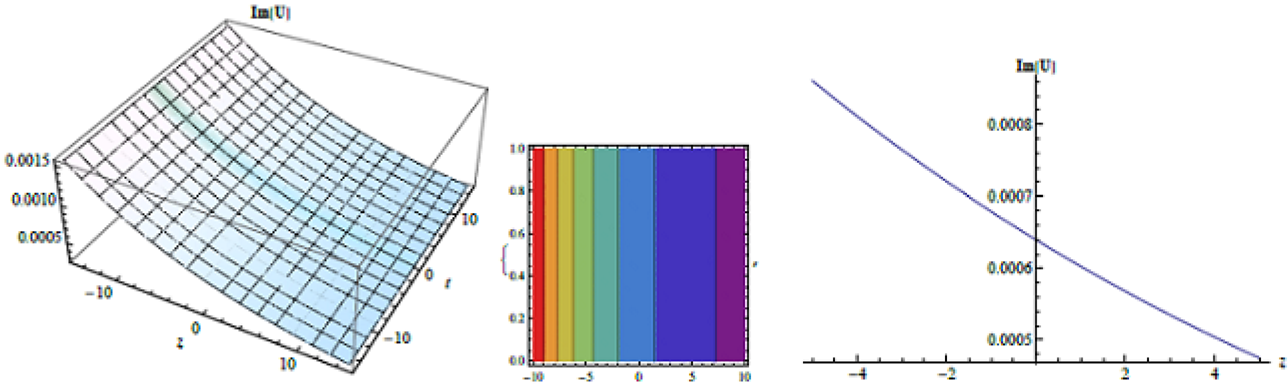
$$U_1(z,\ t)=\frac{24k^2p\sigma^2\left(-e^{4k\sigma\left(z+\frac{4k^2pt^{\alpha}\sigma^2}{\alpha}\right)}ra_4+k\sqrt{p}\sqrt{r}\epsilon\sqrt{a_4}\sqrt{b_0}+i48k^2p\epsilon^2\sigma^2b_016\sqrt{3}e^{2kz\sigma+\frac{8k^3pt^{\alpha}\sigma^3}{\alpha}}\right)^2}{r\left(\sqrt{3}e^{2kz\sigma+\frac{8k^3pt^{\alpha}\sigma^3}{\alpha}}\sqrt{r}\sqrt{a_4}+12ik\sqrt{p}\epsilon\sigma\sqrt{b_0}\right)}.$$

Figure 1.



h The 3D surfaces , contourplot and 2D surfaces of $U_1(z,t)$ for $\alpha=0.6$. By considering the values $\alpha=0.6$; $\epsilon=0.2$; $\sigma=0.1$; $k=0.3$; $p=0.2$; $b_0=1$; $a_4=0.5$; $r=0.5$; $-15<z<15$, $-15<t<15$ for 3D surface, $-10<z<10$, $-10<t<10$ for contourplot and $-15<z<15$; $t=0.1$ for 2D surface.

Figure 2.



The 3D surfaces , contourplot and 2D surfaces of $U_1(z,t)$ for $\alpha = 0.6$. By considering the values $\alpha = 0.6$; $\epsilon = 0.2$; $\sigma = 0.1$; $k = 0.3$; $p = 0.2$; $b_0 = 1$; $a_4 = 0.5$; $r = 0.5$; $-15 < z < 15$, $-15 < t < 15$ for 3D surface, $-10 < z < 10$, $-10 < t < 10$ for contourplot and $-15 < z < 15$; $t = 0.1$ for 2D surface.

4 Conclusion

In this work, we have successfully applied the IBSEFM to the conformable time fractional forms of the KdV equation to investigate new exact solution.

It has been observed that the analytical solution examined in this paper verify the nonlinear ordinary differential equation (10) which is obtained from conformable time fractional forms of the KdV equation under the terms of wave transformation. All necessary computational calculations and graphs have been acquired by using mathematics software. According to the figures, one can see that the formats of travelling wave solutions in two and three dimensional surfaces are similar to the physical meaning of results. If we take more values of coefficients, we can obtain more travelling wave solutions for this model. This method is very reliable, efficient and submits new travelling wave solutions. Therefore, the IBSEFM can be applied to the other nonlinear fractional differential models.

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