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Solving Bigeometric Volterra Integral Equations by Using Successive Approximations Method

Nihan GÜNGÖR*[1]

Abstract

In this study, the successive approximations method has been applied to investigate the solution for the linear bigeometric Volterra integral equations of the second kind in the sense of bigeometric calculus. The conditions to be taken into consideration for the bigeometric continuity and the uniqueness of the solution of linear bigeometric Volterra integral equations of the second kind are researched. Finally, some numerical examples are presented to illustrate successive approximations method.

Keywords: bigeometric calculus, bigeometric Volterra integral equations, successive approximations method.

1. INTRODUCTION

The non-Newtonian calculus comprising of the branches of geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus introduced and studied by Grossman and Katz [13]. Bigeometric calculus which is one of the most popular non-Newtonian calculus is worked by many researchers. Boruah and Hazarika [2,3] named Bigeometric calculus as G -calculus and investigated basic properties of derivative and integral in the sense of bigeometric calculus and also applications in numerical analysis. Boruah et al. [4] researched solvability of bigeometric differential equations by using numerical methods. Güngör [12] defined Volterra integral equations in the bigeometric calculus and investigated the relationship between bigeometric Volterra integral equations and bigeometric differential equations. For understanding non-Newtonian calculus and especially bigeometric calculus, the reader can find more details in [1-16, 21, 22].

Integral equations have used for the solution of several problems in engineering, pure and applied mathematics and mathematical physics. Volterra integral equations which is solved by using analytical and numerical methods, have an important role in the theory of integral equations.

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One can find relevant terminology related to integral equations in [17-20, 23, 24] and also the details of successive approximations method can be found in [17, 19, 20, 24].

A generator is a one-to-one function α whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. The range of generator α is indicated by $\mathbb{R}_\alpha = \{\alpha(x) : x \in \mathbb{R}\}$. α-arithmetic operations are described as indicated, below:

$\alpha -$addition	$x \dot{+} y = \alpha\left[\alpha^{-1}(x) + \alpha^{-1}(y)\right]$
$\alpha -$subtraction	$x \dot{-} y = \alpha\left[\alpha^{-1}(x) - \alpha^{-1}(y)\right]$
$\alpha -$multiplication	$x \dot{\times} y = \alpha\left[\alpha^{-1}(x) \times \alpha^{-1}(y)\right]$
$\alpha -$division	$x \dot{/} y = \alpha\left[\alpha^{-1}(x) / \alpha^{-1}(y)\right]$
$\alpha -$order	$x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$

for $x, y \in \mathbb{R}_\alpha$. $(\mathbb{R}_\alpha, \dot{+}, \dot{\times})$ is complete field. In particular, the identity function I generates classical arithmetic and the exponential function generates geometric arithmetic. The numbers $x \dot{>} \dot{0}$ are α-positive numbers and the numbers $x \dot{<} \dot{0}$ are α-negative numbers in $\mathbb{R}_\alpha$. α-zero and α-one numbers are denoted by $\alpha(0) = \dot{0}$ and $\alpha(1) = \dot{1}$, respectively. α-integers are obtained by successive α-addition of $\dot{1}$ to $\dot{0}$ and successive α-subtraction of $\dot{1}$ from $\dot{0}$. Hence the α-integers are as follows:

$$\ldots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \ldots$$

For each integer n, we set $\dot{n} = \alpha(n)$. If $\dot{n}$ is an α-positive integer, then it is n times sum of $\dot{1}$ [13].

Grosmann and Katz described the *-calculus with the help of two arbitrary selected generators. Let α and β are arbitrarily chosen generators and * is the ordered pair of arithmetic (α-arithmetic, β-arithmetic). The following notions are used:

	$\alpha -$arithmetic	$\beta -$arithmetic
Realm	$A(\subseteq \mathbb{R}_\alpha)$	$B(\subseteq \mathbb{R}_\beta)$
Summation	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}$ (or $-_\alpha$)	$\ddot{/}$ (or $-_\beta$)
Order	$\dot{<}$	$\ddot{<}$

If the generators α and β are chosen as one of I and $\exp$, the following special calculus are obtained:

Calculus	α	β
Classic	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

The ι (iota) which is an isomorphism from α-arithmetic to β-arithmetic uniquely satisfying the following three properties:

(1) ι is one to one,
(2) ι is on A and onto B,
(3) For any numbers x and y in A,

$$\iota(x \dot{+} y) = \iota(x) \ddot{+} \iota(y),$$
$$\iota(x \dot{-} y) = \iota(x) \ddot{-} \iota(y),$$
$$\iota(x \dot{\times} y) = \iota(x) \ddot{\times} \iota(y),$$
$$\iota(x \dot{/} y) = \iota(x) \ddot{/} \iota(y),\ y \neq \dot{0},$$
$$x \dot{<} y \Leftrightarrow \iota(x) \ddot{<} \iota(y).$$

It turns out that $\iota(x) = \beta\{\alpha^{-1}(x)\}$ for every x in A [13].

2. BIGEOMETRIC CALCULUS

Throughout this study, we will deal with Bigeometric calculus which is the *-calculus for which $\alpha = \beta = \exp$ as specified above. In other words, one uses geometric arithmetic on function arguments and values in the bigeometric calculus.

Thereby, we will start by giving the geometric arithmetic and its necessary properties.

If the function exp from $\mathbb{R}$ to $\mathbb{R}^+$ which gives $\alpha^{-1}(x)=\ln x$ is selected as a generator, that is to say that α-arithmetic turns into geometric arithmetic. The range of generator exp is denoted by $\mathbb{R}_{\exp}=\{e^x : x\in\mathbb{R}\}$. The following notions are used:

geometric addition	$x\oplus y=\alpha\left[\alpha^{-1}(x)+\alpha^{-1}(y)\right]=e^{(\ln x+\ln y)}=x.y$
geometric subtraction	$x\ominus y=\alpha\left[\alpha^{-1}(x)-\alpha^{-1}(y)\right]=e^{(\ln x-\ln y)}=x/y,\ y\neq 0$
geometric multiplication	$x\odot y=\alpha\left[\alpha^{-1}(x)\times\alpha^{-1}(y)\right]=e^{(\ln x\times\ln y)}=x^{\ln y}$
geometric division	$x\oslash y=\alpha\left[\alpha^{-1}(x)/\alpha^{-1}(y)\right]=e^{(\ln x\div\ln y)}=x^{\frac{1}{\ln y}},\ y\neq 1$
geometric order	$x<_{\exp} y\Leftrightarrow\alpha^{-1}(x)=\ln x<\alpha^{-1}(y)=\ln y$

$(\mathbb{R}_{\exp},\oplus,\odot)$ is a field with geometric zero 1 and geometric identity e. The geometric positive real numbers and geometric negative real numbers are denoted by $\mathbb{R}^+_{\exp}=\{x\in\mathbb{R}_{\exp} : x>1\}$ and $\mathbb{R}^-_{\exp}=\{x\in\mathbb{R}_{\exp} : x<1\}$, respectively. Now, we will give some useful and necessary relations between geometric and classical arithmetic operations. The geometric absolute valued of $x\in\mathbb{R}_{\exp}$ defined by

$$|x|_{\exp}=\begin{cases} x & ,\ x>1 \\ 1 & ,\ x=1 \\ 1/x & ,\ x<1. \end{cases}$$

Thus $|x|_{\exp}\geq 1$. For all $x,y\in\mathbb{R}_{\exp}$, the following relations hold:

$$x^{2_{\exp}}=x\odot x=x^{\ln x} \qquad \sqrt{x^{2_{\exp}}}^{\exp}=|x|_{\exp}$$

$$\sqrt{x}^{\exp}=e^{(\ln x)^{1/2}} \qquad x^{p_{\exp}}=x^{\ln^{p-1}x}$$

$$x^{-1_{\exp}}=e^{\frac{1}{\ln x}} \qquad e^n\odot x=x^n$$

$$x\odot e=x,\ x\oplus 1=x \qquad 1\ominus e\odot(x\ominus y)=y\ominus x$$

$$\left|e^x\right|_{\exp}=e^{|x|}$$

$$|x\oplus y|_{\exp}\leq_{\exp}|x|_{\exp}\oplus|y|_{\exp} \qquad |x\odot y|_{\exp}=|x|_{\exp}\odot|y|_{\exp}$$

$$|x\ominus y|_{\exp}\geq_{\exp}|x|_{\exp}\ominus|y|_{\exp} \qquad |x\oslash y|_{\exp}=|x|_{\exp}\oslash|y|_{\exp}$$

[2-4, 13, 15]. The geometric factorial notation $!_{\exp}$ denoted by

$$n!_{\exp}=e^n\odot e^{n-1}\odot\ldots\odot e^2\odot e=e^{n!} \text{ [2].}$$

Definition 1. Let (x_n) be sequence and x be a point in metric space $\left(\mathbb{R}_{\exp},|.|_{\exp}\right)$. If for every $\varepsilon>_{\exp}1$, there exits $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that $|x_n\ominus x|_{\exp}<_{\exp}\varepsilon$ for all $n\geq n_0$, then it is said that the sequence (x_n) exp-convergent and denoted by ${}_{\exp}\lim_{n\to\infty}x_n=x$ [22].

Definition 2. Let $f:A\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be a function and $a\in A'^{\exp}$, $b\in\mathbb{R}_{\exp}$. If for every $\varepsilon>_{\exp}1$ there is a number $\delta=\delta(\varepsilon)>_{\exp}1$ such that $|f(x)\ominus b|_{\exp}<_{\exp}\varepsilon$ for all $x\in A$ whenever $1<_{\exp}|x\ominus a|_{\exp}<_{\exp}\delta$, then it is said that the BG-limit function f at the point a is b and it is indicate by ${}_{BG}\lim_{x\to a}f(x)=b$ or $f(x)\xrightarrow{BG}b$. Here $1<_{\exp}|x\ominus a|_{\exp}<_{\exp}\delta\Rightarrow\frac{a}{\delta}<x<a\delta$ and $|f(x)\ominus b|_{\exp}<_{\exp}\varepsilon\Rightarrow\frac{b}{\varepsilon}<f(x)<b\varepsilon$ [2,13,15].

Definition 3. Let $a\in A$ and $f:A\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be a function. If for every $\varepsilon>_{\exp}1$ there is a number $\delta=\delta(\varepsilon)>_{\exp}1$ such that $|f(x)\ominus f(a)|_{\exp}<_{\exp}\varepsilon$ for all $x\in A$ whenever $1<_{\exp}|x\ominus a|_{\exp}<_{\exp}\delta$, then it is said that f is BG-continuous at point $a\in A$. The function f is BG-continuous at the point $a\in A$ iff this point a is an element of domain of the function f and ${}_{BG}\lim_{x\to a}f(x)=f(a)$ [2, 13, 15].

Theorem 1. If the function $f:[r,s]\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be BG-continuous, then f is exp-bounded on $[r,s]\subset\mathbb{R}_{\exp}$ [10].

Definition 4. Let $f:(r,s)\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be a function and $a\in(r,s)$. If the following limit

$$_{BG}\lim_{x\to a}\frac{f(x)\ominus f(a)}{x\ominus a}\,{}_{\exp}=\lim_{x\to a}\left[\frac{f(x)}{f(a)}\right]^{\frac{1}{\ln x-\ln a}}$$

exists, it is indicated by $f^{BG}(a)$ and called the BG-derivative of f at a and say that f is BG-differentiable. If the function f is BG-differentiable at all points of the $\exp$-open interval (r,s), then f is BG-differentiable on (r,s) and BG-derivative of f identified as

$$_{BG}\lim_{h\to 1}\frac{f(x\oplus h)\ominus f(x)}{h}\,{}_{\exp}=\lim_{h\to 1}\left[\frac{f(hx)}{f(x)}\right]^{\frac{1}{\ln h}}$$

for $h\in\mathbb{R}_{\exp}$ and denoted by f^{BG} or $\frac{d^{BG}f}{dx^{BG}}$ [2, 13, 15].

Definition 5. The BG-average of a BG-continuous positive function f on $[r,s]\subset\mathbb{R}_{\exp}$ is defined as the $\exp$-limit of the $\exp$-convergent sequence whose n-th term is geometric average of $f(a_1),f(a_2),\ldots,f(a_n)$ where $a_1,a_2,\ldots,a_n$ is the n-fold $\exp$-partition of $[r,s]$ and denoted by $\overset{BG}{M}{}_r^s f$. The BG-integral of a BG-continuous function f on $[r,s]$ is the positive number $\left[\overset{BG}{M}{}_r^s f\right]^{[\ln(s)-\ln(r)]}$ and is denoted by ${}_{BG}\int_r^s f(x)dx^{BG}$ [3, 13, 15].

Remark 1. If f is BG-continuous positive function on $[r,s]\subset\mathbb{R}_{\exp}$, then

$${}_{BG}\int_r^s f(x)dx^{BG}=\exp\left(\int_{\ln(r)}^{\ln(s)}\ln f\left(e^t\right)dt\right),$$

i.e., the BG-integral of the function f is defined by

$${}_{BG}\int_r^s f(x)dx^{BG}=e^{\int_r^s\frac{\ln f(x)}{x}dx}$$

[3, 13, 15].

Theorem 2. If f and g are BG-continuous positive functions on $[r,s]\subset\mathbb{R}_{\exp}$ and λ,μ are arbitrary constants, then

(1) ${}_{BG}\int_r^s\left(\lambda\odot f(x)\oplus\mu\odot g(x)\right)dx^{BG}$

$$=\lambda\odot{}_{BG}\int_r^s f(x)dx^{BG}\oplus\mu\odot{}_{BG}\int_r^s g(x)dx^{BG}$$

(2) ${}_{BG}\int_r^s\left(f(x)\right)^{\lambda}dx^{BG}=\left({}_{BG}\int_r^s f(x)dx^{BG}\right)^{\lambda}$

(3) ${}_{BG}\int_r^s f(x)dx^{BG}={}_{BG}\int_r^t f(x)dx^{BG}\oplus{}_{BG}\int_t^s f(x)dx^{BG}$

where $r<_{\exp}t<_{\exp}s$

(4) $\left|{}_{BG}\int_r^s f(x)dx^{BG}\right|_{\exp}\leq_{\exp}{}_{BG}\int_r^s\left|f(x)\right|_{\exp}dx^{BG}$

[10, 13, 15].

Theorem 3. *(First fundamental theorem of BG-calculus)* If f is BG-continuous on $[r,s]\subset\mathbb{R}_{\exp}$ and $g(x)={}_{BG}\int_r^x f(x)dx^{BG}$ for every $x\in[r,s]$, then $g^{BG}=f$ on $[r,s]$ [13, 15].

Theorem 4. *(Second fundamental theorem of BG-calculus)* If f^{BG} is BG-continuous on $[r,s]\subset\mathbb{R}_{\exp}$, then

$${}_{BG}\int_r^s f^{BG}(x)dx^{BG}=f(s)\ominus f(r)\ [13, 15].$$

Definition 6. Let A be a nonempty subset of $\mathbb{R}_{\exp}$ and let $n\in\mathbb{N}$. The sequence $(f_n)=(f_1,f_2,\ldots,f_n,\ldots)$ is called BG-function sequence for functions $f_n:A\subseteq\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$. Here all functions defined on same set. The sequence $f_n(x_0)$ is $\exp$-sequence in $\mathbb{R}_{\exp}$ for each $x_0 \in A$ [21].

Definition 7. Let the BG-function sequence (f_n) where $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$. The BG-function sequence (f_n) BG-uniform converges to the function f on the set A, if for any given $\varepsilon >_{\exp} 1$, there exists a naturel number n_0 depends on number ε but not depend on variable x such that $\left| f_n(x) \ominus f(x) \right|_{\exp} <_{\exp} \varepsilon$ for all $n > n_0$ and each $x \in A$. We denote BG-uniform convergence by ${}_{BG}\lim\limits_{n\to\infty} f_n = f$ $(BG-\text{uniform})$ or $f_n \overset{BG}{\to} f$ $(BG-\text{uniform})$ [21].

Definition 8. Let the BG-function sequence (f_n) with $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$. The infinite $\exp$-sum ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f_1 \oplus f_2 \oplus \cdots \oplus f_n \oplus \cdots$ is called BG-function series. The $\exp$-sum $S_n = {}_{\exp}\sum\limits_{k=1}^{n} f_k$ is called n-th partial $\exp$-sum of the series ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ for $n \in \mathbb{N}$ [21].

Definition 9. Let the BG-function series ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ with $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and the function $f : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be given. If the partial $\exp$-sums sequence (S_n) where $S_n = {}_{\exp}\sum\limits_{k=1}^{n} f_k$ is BG-uniform convergent to the function f, then ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ is called BG-uniform convergent to the function f on the set A and ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f$ $(BG\text{-uniform})$ is written [21].

Theorem 5. *(BG-Weierstrass M-criterion)* If there exist $\exp$-numbers M_n such that $\left| f_n(x) \right|_{\exp} <_{\exp} M_n$ for all $x \in A$ where $f : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and the series ${}_{\exp}\sum\limits_{n=1}^{\infty} M_n$ is $\exp$-convergent, then the series ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n$ is BG-uniform convergent and $\exp$-absolutely convergent [21].

Theorem 6. The functions $f_n : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be BG-continuous and the function $f : A \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be given. If ${}_{\exp}\sum\limits_{n=1}^{\infty} f_n = f$ $(BG\text{-uniform})$, then the function f is BG-continuous on the set A [21].

Theorem 7. The functions $f_n : [a,b] \subseteq \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ be BG-continuous on $[a,b] \subseteq \mathbb{R}_{\exp}$ for all $n \in \mathbb{N}$ and $f_n \overset{BG}{\to} f$ $(BG-\text{uniform})$ on $[a,b] \subseteq \mathbb{R}_{\exp}$. Then the function f is BG-continuous on $[a,b] \subseteq \mathbb{R}_{\exp}$ and ${}_{BG}\lim\limits_{n\to\infty} {}_{BG}\int\limits_a^b f_n(x)\, dx^{BG} = {}_{BG}\int\limits_a^b f(x)\, dx^{BG}$ [21].

3. SOLVING BY SUCCESSIVE APPROXIMATIONS METHOD

From [12], we know that the equation of an unknown $\mathbb{R}_{\exp}$-valued function $v(x)$ is occurred form as

$$v(x) = f(x) \oplus \left(\lambda \odot {}_{BG}\int\limits_1^x K(x,s) \odot v(s)\, ds^{BG} \right) \quad (1)$$

where $f(x)$ and $K(x,s)$ are specified $\mathbb{R}_{\exp}$-valued functions and $\lambda \in \mathbb{R}_{\exp}$, is called linear BG-Volterra integral equation of the second kind. The function $K(x,s)$ is the kernel of BG - Volterra equation.

In this method, the zeroth approximation $v_0(x)$ is identified as

$$v_0(x)=f(x).$$

If we substitute $v_0(x)$ instead of the unknown function $v(x)$ on the right side of the equation (1), then the first approximation $v_1(x)$ is found by

$$v_1(x)=f(x)\oplus\left(\lambda\odot{}_{BG}\int_1^x K(x,s)\odot v_0(s)\,ds^{BG}\right).$$

The second approximation is obtained as

$$v_2(x)=f(x)\oplus\left(\lambda\odot{}_{BG}\int_1^x K(x,s)\odot v_1(s)\,ds^{BG}\right)$$

by replacing $v_1(x)$ instead of $v(x)$ on right side of the equation (1). By proceeding similarly, the n-th approximation is obtained in the following form:

$$v_n(x)=f(x)\oplus\left(\lambda\odot{}_{BG}\int_1^x K(x,s)\odot v_{n-1}(s)\,ds^{BG}\right).$$

That is to say, the approximations can be put in a repeated scheme given by

$$v_0(x)=f(x)$$

$$v_n(x)=f(x)\oplus\left(\lambda\odot{}_{BG}\int_1^x K(x,s)\odot v_{n-1}(s)\,ds^{BG}\right),$$
$$n\geq 1. \qquad (2)$$

The successive approximations method gives the exact solution, if it exists, by

$${}_{BG}\lim_{n\to\infty} v_n(x)=v(x).$$

Now, we will give the necessary proposition for using the proof of the theorem that answers the question the BG-convergence of $v_n(x)$.

Proposition 1. Let taken

$$\psi_0(x)=f(x)$$

$$\psi_n(x)={}_{BG}\int_1^x K(x,s)\odot\psi_{n-1}(s)\,ds^{BG},\ n\geq 1. \qquad (3)$$

If $f(x)$ is BG-continuous for $1\leq_{\exp} x\leq_{\exp} a$ and $K(x,s)$ is BG-continuous for $1\leq_{\exp} x\leq_{\exp} a$ and $1\leq_{\exp} s\leq_{\exp} x$, then the series

$${}_{\exp}\sum_{n=0}^{\infty}\lambda^{n_{\exp}}\odot\psi_n(x) \qquad (4)$$

is BG-uniform convergent and $\exp$-absolutely convergent.

Proof. Since $f(x)$ is BG-continuous for $1\leq_{\exp} x\leq_{\exp} a$, there is $m\geq_{\exp} 1$ such that

$$\left|f(x)\right|_{exp}\leq_{\exp} m \qquad (5)$$

on $1\leq_{\exp} x\leq_{\exp} a$. Because of $K(x,s)$ is BG-continuous for $1\leq_{\exp} x\leq_{\exp} a$ and $1\leq_{\exp} s\leq_{\exp} x$, there is $M\geq_{\exp} 1$ such that

$$\left|K(x,s)\right|_{exp}\leq_{\exp} M \qquad (6)$$

on $1\leq_{\exp} s\leq_{\exp} x\leq_{\exp} a$. From (5) and (6), we find

$$\left|\psi_1(x)\right|_{exp}=\left|{}_{BG}\int_1^x K(x,s)\odot\psi_0(s)\,ds^{BG}\right|_{\exp}$$

$$=\left|{}_{BG}\int_1^x K(x,s)\odot f(s)\,ds^{BG}\right|_{\exp}$$

$$\leq_{\exp}{}_{BG}\int_1^x\left|K(x,s)\right|_{\exp}\odot\left|f(s)\right|_{\exp}ds^{BG}$$

$$\leq_{\exp} {}_{BG}\int_1^x M \odot m ds^{BG}$$
$$= m \odot M \odot x$$

by replacing $\psi_0(x)$ right side of the equation in (3). Therefore, we obtain

$$\left|\psi_2(x)\right|_{exp} = \left| {}_{BG}\int_1^x K(x,s) \odot \psi_1(s) ds^{BG} \right|_{\exp}$$
$$\leq_{\exp} {}_{BG}\int_1^x \left|K(x,s)\right|_{\exp} \odot \left|\psi_1(s)\right|_{\exp} ds^{BG}$$
$$\leq_{\exp} {}_{BG}\int_1^x m \odot M^{2_{\exp}} \odot s ds^{BG}$$
$$= m \odot M^{2_{\exp}} \odot \frac{x^{2_{\exp}}}{2!_{\exp}}{}_{\exp}$$

by replacing $\psi_1(x)$ in (3). In a similar manner, we get

$$\left|\psi_n(x)\right|_{exp} \leq_{\exp} m \odot M^{n_{\exp}} \odot \frac{x^{n_{\exp}}}{n!_{\exp}}{}_{\exp}$$

for $1 \leq_{\exp} x \leq_{\exp} a$ and $n \in \mathbb{N}$. Consequently,

$$\left| {}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot \psi_n(x) \right|_{\exp} \leq_{\exp} {}_{\exp}\sum_{n=0}^{\infty} \left|\lambda\right|_{\exp}^{n_{exp}} \odot \left|\psi_n(x)\right|_{\exp}$$
$$\leq_{\exp} {}_{\exp}\sum_{n=0}^{\infty} m \odot \left|\lambda\right|_{\exp}^{n_{exp}} \odot M^{n_{\exp}} \odot \frac{x^{n_{\exp}}}{n!_{\exp}}{}_{\exp}$$
$$= {}_{\exp}\sum_{n=0}^{\infty} \frac{m \odot \left|\lambda\right|_{\exp}^{n_{exp}} \odot M^{n_{\exp}}}{n!_{\exp}}{}_{\exp} \odot x^{n_{\exp}}$$

holds. Let consider the exp-series

$${}_{\exp}\sum_{n=0}^{\infty} \frac{m}{n!_{\exp}}{}_{\exp} \odot \left(\left|\lambda\right|_{\exp} \odot M \right)^{n_{\exp}} .$$

If the rate test is applied as follows

$${}_{\exp}\lim_{n\to\infty} \left| \frac{\frac{m}{(n+1)!_{\exp}}{}_{\exp} \odot \left(\left|\lambda\right|_{\exp} \odot M \right)^{(n+1)_{\exp}}}{\frac{m}{n!_{\exp}}{}_{\exp} \odot \left(\left|\lambda\right|_{\exp} \odot M \right)^{n_{\exp}}}{}_{\exp} \right|$$
$$= {}_{\exp}\lim_{n\to\infty} \left| \frac{\left|\lambda\right|_{\exp} \odot M}{e^{n+1}}{}_{\exp} \right|$$
$$= \lim_{n\to\infty} e^{\left|\frac{M^{\left|\ln\lambda\right|}}{n+1}\right|} = 1 <_{\exp} e$$

then we can see that the series is exp-convergent. This implies via BG-Weierstrass M-criterion, the exp-series ${}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{\exp}} \odot \psi_n(x)$

is BG-uniform convergent and exp-absolutely convergent.

The convergence of $v_n(x)$ will be verified by the following theorem:

Theorem 8. Suppose that the following conditions are satisfied:
(i) $f(x)$ is BG-continuous for $1 \leq_{\exp} x \leq_{\exp} a$,
(ii) $K(x,s)$ is BG-continuous for $1 \leq_{\exp} x \leq_{\exp} a$ and $1 \leq_{\exp} s \leq_{\exp} x$,
then the sequence $v_n(x)$ in (2), converges to solution $v(x)$ of the equation (1), and also the solution $v(x)$ is BG-continuous function on $1 \leq_{\exp} x \leq_{\exp} a$.

Proof. Let taken $v_0(x) = f(x) = \psi_0(x)$. Hence we find

$$v_1(x) = f(x) \oplus \left(\lambda \odot {}_{BG}\int_1^x K(x,s) \odot v_0(s) ds^{BG} \right)$$
$$= f(x) \oplus \left(\lambda \odot {}_{BG}\int_1^x K(x,s) \odot f(s) ds^{BG} \right)$$
$$= f(x) \oplus \lambda \odot \psi_1(x)$$

where $\psi_1(x) = {}_{BG}\int_1^x K(x,s) \odot \psi_0(s)\, ds^{BG}$. The second approximation $v_2(x)$ is obtained as

$$v_2(x) = f(x) \oplus \left(\lambda \odot {}_{BG}\int_1^x K(x,s) \odot v_1(s)\, ds^{BG} \right)$$

$$= f(x) \oplus \left(\lambda \odot {}_{BG}\int_1^x K(x,s) \odot \left[f(s) \oplus \lambda \odot \psi_1(s) \right] ds^{BG} \right)$$

$$= f(x) \oplus \lambda \odot {}_{BG}\int_1^x K(x,s) \odot f(s)\, ds^{BG} \oplus$$

$$\oplus \lambda^{2_{exp}} \odot {}_{BG}\int_1^x K(x,s) \odot \psi_1(s)\, ds^{BG}$$

$$= f(x) \oplus \lambda \odot \psi_1(x) \oplus \lambda^{2_{exp}} \odot \psi_2(x)$$

where $\psi_2(x) = {}_{BG}\int_1^x K(x,s) \odot \psi_1(s)\, ds^{BG}$.

Proceeding this manner, we get

$$v_n(x) = {}_{\exp}\sum_{k=0}^{n} \lambda^{k_{exp}} \odot \psi_k(x)$$

$$= f(x) \oplus_{\exp} \sum_{k=1}^{n} \lambda^{k_{exp}} \odot \psi_k(x).$$

Under the hypothesis, $v_n(x)$ is BG-continuous on $1 \leq_{\exp} x \leq_{\exp} a$ and also the series ${}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{exp}} \odot \psi_n(x)$ is BG-uniform convergent by Proposition 1. Therefore there exists a BG-continuous function $v(x)$ such that ${}_{BG}\lim_{n\to\infty} v_n(x) = v(x)$, i.e,

$${}_{\exp}\sum_{n=0}^{\infty} \lambda^{n_{exp}} \odot \psi_n(x) = v(x)$$

by Theorem 6. Now, we need to show $v(x)$ is solution of the equation (1). If the expressions

$$\psi_0(x) = f(x)$$

$$\lambda^{k_{exp}} \odot \psi_k(x) = \lambda^{k_{exp}} \odot {}_{BG}\int_1^x K(x,s) \odot \psi_{k-1}(s)\, ds^{BG}$$

for $k \in \mathbb{N}$, is added side to side, we find

$$\psi_0(x) \oplus_{\exp} \sum_{k=1}^{n} \lambda^{k_{exp}} \odot \psi_k(x) =$$

$$f(x) \oplus \lambda \odot {}_{BG}\int_1^x K(x,s) \odot_{\exp} \sum_{k=1}^{n} \lambda^{(k-1)_{exp}} \odot \psi_{k-1}(s) ds^{BG}.$$

As a result of this, we can write

$${}_{\exp}\sum_{k=0}^{n} \lambda^{k_{exp}} \odot \psi_k(x) =$$

$$f(x) \oplus \lambda \odot {}_{BG}\int_1^x K(x,s) \odot_{\exp} \sum_{k=0}^{n-1} \lambda^{k_{exp}} \odot \psi_k(s) ds^{BG}.$$

If we take BG-limit as $n \to \infty$ on both sides of the equation, we obtain

$$v(x) = f(x) \oplus \lambda \odot$$

$${}_{BG}\int_1^x K(x,s) \odot \left({}_{BG}\lim_{n\to\infty \exp} \sum_{k=0}^{n-1} \lambda^{k_{exp}} \odot \psi_k(s) \right) ds^{BG}$$

$$= f(x) \oplus \lambda \odot {}_{BG}\int_1^x K(x,s) \odot v(s)\, ds^{BG}$$

by Theorem 7. This completes the proof.

Remark 2. Under the hypothesis of Theorem 8, the series in (4) BG-convergences and equals to solution $v(x)$ of the equation (1). For this reason, the solution of (1) also can be determined by aid of the system (3)-(4).

Theorem 9. Under the hypothesis of Theorem 8, the BG-Volterra integral equation (1) has an unique solution on $[1, a] \subset \mathbb{R}_{\exp}$.

Proof. Assume that $v(x)$ and $u(x)$ are different solutions of the equation (1). Then, it is written

$$v(x) = f(x) \oplus \left(\lambda \odot {}_{BG}\int_1^x K(x,s) \odot v(s)\, ds^{BG} \right)$$

$$u(x)=f(x)\oplus\left(\lambda\odot{}_{BG}\int_1^x K(x,s)\odot u(s)\,ds^{BG}\right).$$

If we set $\phi(x)=v(x)\ominus u(x)$, then

$$\begin{aligned}|\phi(x)|_{\exp}&=\left|\lambda\odot{}_{BG}\int_1^x K(x,s)\odot\phi(s)\,ds^{BG}\right|_{\exp}\\&\le_{\exp}|\lambda|_{\exp}\odot{}_{BG}\int_1^x |K(x,s)|_{exp}\odot|\phi(s)|_{\exp}\,ds^{BG}\\&\le_{\exp}M\odot|\lambda|_{\exp}\odot{}_{BG}\int_1^x|\phi(s)|_{\exp}\,ds^{BG}\end{aligned}$$

from (6). If it is taken

$$h(x)={}_{BG}\int_1^x|\phi(s)|_{\exp}\,ds^{BG}\ge_{\exp}1, \tag{7}$$

then we find

$$|\phi(x)|_{\exp}\le_{\exp}|\lambda|_{\exp}\odot M\odot h(x)$$
$$|\phi(x)|_{\exp}\ominus\left(|\lambda|_{\exp}\odot M\odot h(x)\right)\le_{\exp}1$$
$$h^{BG}(x)\ominus\left(|\lambda|_{\exp}\odot M\odot h(x)\right)\le_{\exp}1$$

from first fundamental theorem of BG-calculus. By multiplication with $e^{x^{-|\ln\lambda|\ln M}}$ both sides of this inequality

$$\left[e^{x^{-|\ln\lambda|\ln M}}\odot h(x)\right]^{BG}\le_{\exp}1. \tag{8}$$

We find

$$e^{x^{-|\ln\lambda|\ln M}}\odot h(x)\le_{\exp}1$$

by BG-integration both sides of the inequality (8) according from 1 to x. Thereby, we write

$$h(x)\le_{\exp}1 \tag{9}$$

From (7) and (9) we obtain

$$h(x)={}_{BG}\int_1^x|\phi(s)|_{\exp}\,ds^{BG}=1.$$

Hence it must be ${}_{BG}\int_1^x|\phi(s)|_{\exp}\,ds^{BG}=1$. Therefore

$$e^{\int_1^x\frac{\ln|\phi(s)|_{\exp}}{s}ds}=1$$
$$e^{\int_1^x\frac{|\ln\phi(s)|}{s}ds}=1$$
$$\int_1^x\frac{|\ln\phi(s)|}{s}ds=0.$$

Thus, we find $\frac{\ln|\phi(s)|}{s}=0$ for all $s\in[1,x]\subset\mathbb{R}_{\exp}$, i.e., $\phi(s)=1$ for all $1\le_{\exp}s\le_{\exp}x$. As a result of this $v(x)=u(x)$ for all $x\in[1,a]\subset\mathbb{R}_{\exp}$.

3.1. Numerical Examples

Example 1. Find the solution of BG-Volterra integral equation

$$v(x)=e\oplus{}_{BG}\int_1^x(s\ominus x)\odot v(s)\,ds^{BG}$$

by aid of the series (4).

Solution. Proceeding with the recurrence relation in (3), that gives

$$\psi_0(x)=e$$

$$\psi_1(x)={}_{BG}\int_1^x(s\ominus x)\odot\psi_0(s)\,ds^{BG}$$

$$={}_{BG}\int_1^x(s\ominus x)\odot e\,ds^{BG}=e^{\int_1^x\frac{\ln s-\ln x}{s}ds}$$

$$= e^{-\frac{(\ln x)^2}{2!}} = \left[\left(\left(e^{\ln x}\right)^{\ln x}\right)^{\frac{1}{2!}}\right]^{-1}$$

$$= \left[\left((x)^{\ln x}\right)^{\frac{1}{\ln e^{2!}}}\right]^{-1} = (1 \ominus e) \odot \frac{x^{2_{\exp}}}{2!_{\exp}}\exp$$

$$\psi_2(x) = {}_{BG}\int_1^x (s \ominus x) \odot \psi_1(x)\, ds^{BG}$$

$$= {}_{BG}\int_1^x (s \ominus x) \odot \left((1 \ominus e) \odot \frac{s^{2_{\exp}}}{2!_{\exp}}\exp\right) ds^{BG}$$

$$= {}_{BG}\int_1^x (s \ominus x) \odot e^{-\frac{(\ln s)^2}{2}} ds^{BG}$$

$$= e^{-\frac{1}{2}\int_1^x \frac{(\ln s)^3 - (\ln s)^2 \ln x}{s} ds}$$

$$= e^{\frac{(\ln x)^4}{4!}} = \frac{x^{4_{\exp}}}{4!_{\exp}}\exp$$

and so on. The solution of the integral equation is obtained as a series form is given by

$$v(x) = e \oplus (1 \ominus e) \odot \frac{x^{2_{\exp}}}{2!_{\exp}}\exp \oplus \frac{x^{4_{\exp}}}{4!_{\exp}}\exp \oplus$$

$$(1 \ominus e) \odot \frac{x^{6_{\exp}}}{6!_{\exp}} \oplus \frac{x^{8_{\exp}}}{8!_{\exp}} \oplus \cdots$$

$$= e \oplus_{\exp} \sum_{n=1}^{\infty} (1 \ominus e)^{n_{\exp}} \odot \frac{x^{(2n)_{\exp}}}{(2n)!_{\exp}}\exp$$

$$= e \oplus_{\exp} \sum_{n=1}^{\infty} \psi_n(x)$$

and the closed form by

$$v(x) = e^{\cos \ln x}$$

obtained upon using the geometric Taylor expansion for $e^{\cos \ln x}$.

Example 2. Solve the BG-Volterra integral equation

$$v(x) = x \oplus {}_{BG}\int_1^x (s \ominus x) \odot v(s)\, ds^{BG}$$

by using successive approximations method.

Solution. Taking the zero approximation as

$v_0(x) = x$.

Substituting this equality into $v(x)$ under the BG-integral in the iteration formula (2), we find the first approximation as

$$v_1(x) = x \oplus {}_{BG}\int_1^x (s \ominus x) \odot s\, ds^{BG}$$

$$= x \oplus e^{\int_1^x \left(\frac{\ln s \ln \frac{s}{x}}{s}\right) ds} = x \oplus e^{\int_1^x \left(\frac{\ln^2 s - \ln s \ln x}{s}\right) ds}$$

$$= x \oplus e^{-\frac{\ln^3 x}{6}} = x \ominus \frac{x^{3_{\exp}}}{3!_{\exp}}\exp.$$

Then, we find the second approximation as

$$v_2(x) = x \oplus {}_{BG}\int_1^x (s \ominus x) \odot \left(s \ominus \frac{s^{3_{\exp}}}{3!_{\exp}}\exp\right) ds^{BG}$$

$$= x \oplus e^{\int_1^x \left(\frac{(\ln s - \ln x)\left(\ln s - \frac{\ln^3 s}{6}\right)}{s}\right) ds}$$

$$= x \oplus e^{\int_1^x \left(\frac{\ln^2 s - \frac{\ln^4 s}{6} - \ln x \ln s + \ln x \frac{\ln^3 s}{6}}{s}\right) ds}$$

$$= x \ominus e^{-\frac{\ln^3 x}{6} + \frac{\ln^5 x}{60}} = x \ominus e^{\frac{\ln^3 x}{3!}} \oplus e^{\frac{\ln^5 x}{5!}}$$

$$= x \ominus \frac{x^{3_{\exp}}}{3!_{\exp}}\exp \oplus \frac{x^{5_{\exp}}}{5!_{\exp}}\exp.$$

Proceeding the same manner we get the n-th approximation by

$$v_n(x) = x \ominus \frac{x^{3_{\exp}}}{3!_{\exp}}\exp \oplus \frac{x^{5_{\exp}}}{5!_{\exp}}\exp \oplus \cdots$$

$$\oplus (1 \ominus e)^{n_{\exp}} \odot \frac{x^{(2n+1)_{\exp}}}{(2n+1)!_{\exp}}\exp$$

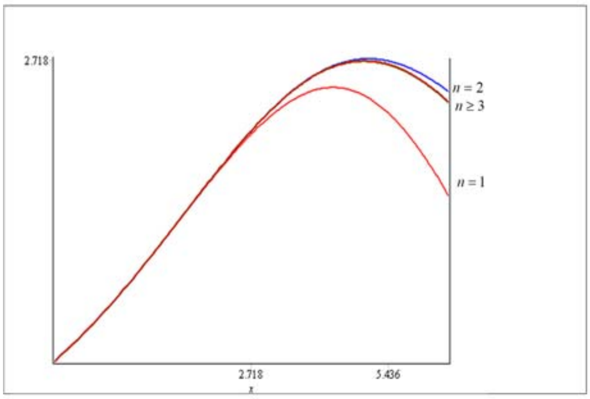
for $n \geq 1$. Since the function $v_n(x)$ is n-th partial $\exp$-sum of the series

$$\exp\sum_{n=0}^{\infty}(1\ominus e)^{n_{\exp}}\odot\frac{x^{(2n+1)_{\exp}}}{(2n+1)!_{\exp}}\exp=e^{\sin\ln x}$$

this implies that ${}_{BG}\lim_{n\to\infty} v_n(x)=e^{\sin\ln x}$. As a result, the solution is obtained by $v(x)=e^{\sin\ln x}$.

By using the relation bigeometric and classic calculus, we can find the iterations with Maple as follows:

```
> k:=9; lambda:=exp(1); u0:=x-> x;
a:=1; f:=x->x;K:=(x,t)->t/x;
> for n from 1 to k do
  u||n:=f(x)*(exp(1)^(ln(lambda)*in
t((1/t)*ln(K(x,t))*ln(u||(n-
1)(t)),t=a..x)))assuming x>1;
  u||n:=unapply(f(x)*(exp(1)^(ln(la
mbda)*int((1/t)*ln(K(x,t))*ln(u||(n
-1)(t)),t=a..x))),x)assuming   x>1;
od;
```



Here, we plot the successive approximations. In this figure, n is the number of iterations performed.

4. CONCLUSION

In this paper, the successive approximations method is constructed in the bigeometric calculus. The exact solution of linear bigeometric Volterra integral equations of the second kind is presented by using this analytical method. Also, we get the result that the solution of the linear bigeometric Volterra equation of the second kind can be found by aid of the series which is obtained in Proposition 1. Moreover, the necessary conditions for the bigeometric continuity and uniqueness of the solution of these equations are given. Finally, some applications are presented to explain the procedure of solutions of these equations by using these methods and how to find the approximations by using Maple are expressed.

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The author of the paper declares that she complies with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that she does not make any falsification on the data collected. In addition, she declares that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication

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