Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (4) (2022), 970 – 980
DOI : 10.15672/hujms.779452

Research Article

Oscillation criteria of second order differential equations with positive and negative coefficients

Yutaka Shoukaku

Faculty of Engineering, Kanazawa University, Kanazawa 920-1192, Japan

Abstract

In this paper we obtain oscillation criteria for solutions of homogeneous and nonhomogeneous cases of second order neutral differential equations with positive and negative coefficients. Our results improve and extend the results of [Oscillation criteria for a class of second order neutral delay differential equations, Appl. Math. Comput. **210**, 303–312, 2009].

Mathematics Subject Classification (2020). 34C10, 34K11, 34K40

Keywords. second order neutral differential equations, oscillation, positive and negative coefficients

1. Introduction

In this paper we consider the oscillation of the second order neutral delay differential equations

$$\text{(E)}\qquad \left[x(t) \pm \sum_{i=1}^{l} h_i(t)x(\alpha_i(t))\right]'' + \sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = f(t), \ \ t > 0.$$

It is assumed throughout this paper that:

(H1) $h_i(t) \in C([0,\infty);[0,\infty))$ $(i = 1,2,\dots,l)$,
$p_i(t)$ $(i = 1,2,\dots,m), q_i(t)$ $(i = 1,2,\dots,n) \in C([0,\infty);[0,\infty))$,
$f(t) \in C([0,\infty);\mathbb{R})$;

(H2) $\alpha_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \alpha_i(t) = \infty$, $\alpha_i(t) \le t$ $(i = 1,2,\dots,l)$,
$\beta_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \beta_i(t) = \infty$ $(i = 1,2,\dots,m)$,
$\gamma_i(t) \in C([0,\infty);\mathbb{R})$, $\lim_{t\to\infty} \gamma_i(t) = \infty$, $\gamma_i(t) \le t$ $(i = 1,2,\dots,n)$;

(H3) $h_i(t) \le h_i$ $(i = 1,2,\dots,l)$, where h_i are nonnegative constants;

(H4) $G_i(\xi) \in C(\mathbb{R};\mathbb{R})$, $uG_i(u) > 0$ $(i = 1,2)$ for $u \ne 0$, $G_1(\xi)$ is nondecreasing and there exists a positive constant M such that

$$\liminf_{|u|\to\infty} \frac{G_2(u)}{u} \le M;$$

Email address: shoukaku@se.kanazawa-u.ac.jp
Received: 12.08.2020; Accepted: 25.08.2021

(H5) there exist two bounded functions $F(t) \in C^2([0,\infty);\mathbb{R})$ and $F'(t) \in C^1([0,\infty);\mathbb{R})$ such that $\lim_{t\to\infty} F'(t) = \lim_{t\to\infty} F(t) = 0$, where

$$F(t) = \int_t^\infty \int_s^\infty f(\xi)d\xi ds.$$

Definition 1.1. By a *solution* of (E) we mean a continuous function $x(t)$ which is defined for $t \geq T$, and satisfies $\sup\{|x| : t \geq t_0\} > 0$ for all $t_0 \geq T$, where $T = \min\{\alpha, \beta, \gamma\}$ and

$$\alpha = \inf_{t>0}\left\{\min_{1\leq i\leq l} \alpha_i(t)\right\}, \ \beta = \inf_{t>0}\left\{\min_{1\leq i\leq m} \beta_i(t)\right\}, \ \gamma = \inf_{t>0}\left\{\min_{1\leq i\leq n} \gamma_i(t)\right\}.$$

Definition 1.2. A solution of (E) is called *oscillatory* if it has arbitrary large zeros, otherwise, it is called nonoscillatory.

There is much current interest in studying the oscillatory behavior of solutions of neutral differential equations. Various models of neutral differential equations have been studied recently. We refer the reader to [3, 5]. The study of oscillatory behavior of solutions of neutral differential equations is of both theoretical and practical importance. As a matter of fact, neutral differential equations have numerous applications in engineering and natural sciences (e.g., neutral differential equations arise in a variety of real world problems such as in the study of non-Newtonian fluid theory and porous medium problems [4]). In particular, some new developments in the oscillation and asymptotic behavior of solutions of second order neutral differential equations with positive and negative coefficients have been reported by authors [1, 2, 6–11]. Many of these approaches employ linearized oscillation theory [2, 7, 8] to obtain criteria which guarantee that all solutions are oscillatory or tend to zero as $t \to \infty$.

However, as far as the author knows, there are no results for every solution of (E) to be oscillatory. Here, our interest is to establish the criteria which ensure the oscillation of every solution of (E).

2. Oscillation of solutions of homogenous equations

In this section we will present the following oscillation criteria for the equations

$$(\mathrm{E}_\pm) \qquad \left[x(t) \pm \sum_{i=1}^{l} h_i(t)x(\alpha_i(t))\right]'' + \sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = 0, \ t > 0.$$

Theorem 2.1. *If for some* $j \in \{1, 2, \dots, m\}$

$$\int_0^\infty p_j(t)dt = \infty \tag{2.1}$$

and

$$\sum_{i=1}^{n} \int_0^\infty \int_t^\infty q_i(\xi)d\xi dt < \frac{1}{M}, \tag{2.2}$$

then every solution of (E_+) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (E_+). Without any loss of generality, we assume that $x(t) > 0$, $t \ge t_0$ for some $t_0 > 0$. We set

$$\begin{aligned} z(t) &= x(t) + \sum_{i=1}^{l} h_i(t)x(\alpha_i(t)) + \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds \quad (2.3) \\ &= X(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds, \ t \ge t_0. \end{aligned}$$

Differentiating the above equation twice, we see from (E_+) that

$$z''(t) = X''(t) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = -\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))), \ t \ge t_0.$$

In the sequel, we obtain

$$z''(t) \le -p_j(t)G_1(x(\beta_j(t))) \le 0, \ t \ge t_0 \quad (2.4)$$

for some $j \in \{1, 2, \dots, m\}$. Hence $z'(t)$ is nonincreasing. Then we see that $z'(t) \ge 0$ or $z'(t) < 0$, $t \ge t_1$ for some $t_1 \ge t_0$.

Case 1. $z'(t) < 0$ for $t \ge t_1$. Integrating (2.4) twice over $[t_1, t]$ yields

$$z(t) \le z(t_1) + z'(t_1)(t - t_1),$$

which implies $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t) > 0$.

Case 2. $z'(t) \ge 0$ for $t \ge t_1$. Since (H4) holds, we see that

$$\begin{aligned} z(t) &\le X(t) + M\sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)x(\gamma_i(\xi))d\xi ds \\ &= X(t) + Mx(\gamma_i(\xi_0))\sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)d\xi ds \end{aligned}$$

for some $\xi_0 \in [t_0, \infty)$. Substituting $z(t) \ge x(t)$ into the above inequality, we obtain

$$z(t) \le X(t) + Mz(t)\sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)d\xi ds.$$

On the other hand, since $z'(t) \ge 0$ and $z(t) > 0$, there exists a positive constant k_0 such that

$$z(t) \ge k_0, \ t \ge t_2$$

for some $t_2 \ge t_1$. Hence we observe that

$$K_0 \equiv k_0 \left(1 - M\sum_{i=1}^{n} \int_{0}^{\infty} \int_{t}^{\infty} q_i(\xi)d\xi dt\right) \le X(t).$$

Since

$$X(t) \le x(t) + \sum_{i=1}^{l} h_i x(\alpha_i(t)),$$

we obtain by taking inferior limit that

$$K_0 \le \left(1 + \sum_{i=1}^{l} h_i\right) \liminf_{t\to\infty} x(t).$$

This means that

$$\liminf_{t\to\infty} x(t) \ge \frac{K_0}{\left(1 + \sum_{i=1}^{l} h_i\right)} \equiv K_1,$$

that is,

$$x(\beta_j(t)) \geq \frac{K_1}{2}, \ t \geq t_3 \tag{2.5}$$

for some $t_3 \geq t_2$. Integrating (2.4) over $[t_3, t]$ yields

$$G_1\left(\frac{K_1}{2}\right)\int_{t_3}^{t} p_j(s)ds \leq -z'(t) + z'(t_3) < \infty.$$

This is a contradiction and completes the proof. □

Example 2.2. We consider the equation

$$\begin{aligned}[x(t) + 2x(t-\pi)]'' + \left(1 + \frac{1}{2}e^{-t}\right)x(t-\pi) \\ -\frac{1}{2}e^{-t}x(t-3\pi) = 0, \ t > 0,\end{aligned} \tag{2.6}$$

which satisfies the all conditions of Theorem 2.1. Hence every solutions of (2.6) oscillates. For example, $x(t) = \sin t$ is such a solution.

Theorem 2.3. *Assume that*

(H6) $\sum_{i=1}^{l} h_i \leq 1.$

If the condition (2.1) *holds, moreover,*

$$M\sum_{i=1}^{n}\int_0^\infty\int_t^\infty q_i(\xi)d\xi dt \leq \sum_{i=1}^{l} h_i \tag{2.7}$$

holds, then every solution of (E$_-$) *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (E$_-$). Without any loss of generality, we assume that $x(t) > 0$, $t \geq t_0$ for some $t_0 > 0$. We define

$$\begin{aligned} w(t) &= x(t) - \sum_{i=1}^{l} h_i(t)x(\alpha_i(t)) + \sum_{i=1}^{n}\int_{t_0}^{t}\int_s^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds \\ &= Y(t) + \sum_{i=1}^{n}\int_{t_0}^{t}\int_s^\infty q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds, \ t \geq t_0. \end{aligned} \tag{2.8}$$

Differentiating the above equation, we have

$$w'(t) = Y'(t) + \sum_{i=1}^{n}\int_t^\infty q_i(s)G_2(x(\gamma_i(s)))ds, \ t \geq t_0. \tag{2.9}$$

By differentiating again and noting (E$_-$), we see that

$$w''(t) = Y''(t) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = -\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))), \ t \geq t_0. \tag{2.10}$$

We can rewrite (2.10) as follows

$$w''(t) \leq -p_j(t)G_1(x(\beta_j(t))) \leq 0, \ t \geq t_0 \tag{2.11}$$

for some $j \in \{1, 2, \ldots, m\}$. Hence $w'(t)$ is nonincreasing. Then we see that $w'(t) \geq 0$ or $w'(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0$.

Case 1. $w'(t) < 0$ for $t \geq t_1$. As in the proof of Theorem 2.1, we obtain $\lim_{t\to\infty} w(t) = -\infty$. If $x(t)$ is not bounded from above, then there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ such that

$$\lim_{\bar{n}\to\infty} t_{\bar{n}} = \infty \quad \text{and} \quad \max_{t_1\leq t\leq t_{\bar{n}}} x(t) = x(t_{\bar{n}}). \tag{2.12}$$

Hence we have

$$w(t_{\bar{n}}) \geq \left(1 - \sum_{i=1}^{l} h_i\right) x(t_{\bar{n}}).$$

Taking limit as $\bar{n} \to \infty$, we get the following contradiction

$$\lim_{\bar{n}\to\infty} w(t_{\bar{n}}) \geq \left(1 - \sum_{i=1}^{l} h_i\right) \lim_{\bar{n}\to\infty} x(t_{\bar{n}}) = \infty. \tag{2.13}$$

Next we assume that $x(t)$ is bounded from above, then there exists a constant L such that

$$x(t) < L \quad \text{and} \quad \limsup_{t\to\infty} x(t) = L. \tag{2.14}$$

Thus we obtain

$$w(t) \geq x(t) - L \sum_{i=1}^{l} h_i.$$

Taking superior limit as $t \to \infty$, we have

$$\lim_{t\to\infty} w(t) \geq \left(1 - \sum_{i=1}^{l} h_i\right) L \geq 0. \tag{2.15}$$

This is a contradiction.

Case 2. $w'(t) \geq 0$ for $t \geq t_1$.

Subcase 2.1. $w(t) < 0$ for $t \geq t_1$, then $\lim_{t\to\infty} w(t) = \mu_0 \in (-\infty, 0]$. So, because of (2.13) and (2.15), we conclude that $\mu_0 = 0$, which implies that $\lim_{t\to\infty} x(t) = 0$ (cf. [7]). By $\lim_{t\to\infty} x(t) = 0$ and the definition of $Y(t)$, it is not difficult to see that $\lim_{t\to\infty} Y(t) = 0$. Since $w''(t) \leq 0$, $w'(t) \geq 0$ and $\lim_{t\to\infty} w(t) = 0$, there exists a constant k_0 such that $\lim_{t\to\infty} w'(t) = k_0 \geq 0$. If $k_0 > 0$, then

$$w'(t) \geq k_0 - \varepsilon_0$$

for some $k_0 > \varepsilon_0 > 0$. Integrating the above inequality over $[t_1, t]$ yields

$$w(t) \geq w(t_1) + (k_0 - \varepsilon_0)(t - t_1),$$

which implies that $\lim_{t\to\infty} w(t) = \infty$. This is a contradiction. Hence we obtain $k_0 = 0$. In view of $\lim_{t\to\infty} x(t) = 0$, there exists a $\varepsilon > 0$ such that

$$0 < x(t) < \varepsilon. \tag{2.16}$$

It follows from (2.9) that

$$-\varepsilon M \sum_{i=1}^{n} \int_t^\infty q_i(s)ds \leq Y'(t) \leq w'(t),$$

which reduces to $\lim_{t\to\infty} Y'(t) = 0$ by using the condition (2.7). Consequently we find that

$$\lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Y'(t) = 0,$$
$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} w'(t) = 0.$$

Integrating (2.10) twice and using the above fact, we derive

$$w(t) + \sum_{i=1}^{m} \int_t^\infty \int_s^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds = 0.$$

Similarly, integrating (E_-) yields

$$Y(t)+\sum_{i=1}^{m}\int_t^{\infty}\int_s^{\infty}p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds$$
$$-\sum_{i=1}^{n}\int_t^{\infty}\int_s^{\infty}q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds=0.$$

Substituting the above into $w(t)\geq Y(t)$ yields

$$\begin{aligned}&-\sum_{i=1}^{m}\int_t^{\infty}\int_s^{\infty}p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds\\ \geq\ &-\sum_{i=1}^{m}\int_t^{\infty}\int_s^{\infty}p_i(\xi)G_1(x(\beta_i(\xi)))d\xi ds\\ &\qquad+\sum_{i=1}^{n}\int_t^{\infty}\int_s^{\infty}q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds,\end{aligned}$$

which leads the following contradiction

$$0\geq\sum_{i=1}^{n}\int_t^{\infty}\int_s^{\infty}q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds.$$

Subcase 2.2. $w(t)\geq 0$ for $t\geq t_1$. Since $w'(t)\geq 0$ and $w(t)\geq 0$, we can show that

$$w(t)\geq k_1,\ \ t\geq t_2 \tag{2.17}$$

for some constant $k_1>0$ and some $t_2\geq t_1$. If $x(t)$ is not bounded from above, there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ satisfies (2.12). It follows from (2.8) and (2.17) that

$$k_1\leq\left(1+M\sum_{i=1}^{n}\int_{t_0}^{t_{\bar{n}}}\int_s^{\infty}q_i(\xi)d\xi ds\right)x(t_{\bar{n}})\leq 2x(t_{\bar{n}}).$$

Then we see that

$$\lim_{\bar{n}\to\infty}x(t_{\bar{n}})\geq\frac{k_1}{2}\equiv K_1,$$

which implies that (2.5) holds for some $t_3\geq t_2$. Integrating (2.11) over $[t_3,t]$ yields

$$G_1\left(\frac{K_1}{2}\right)\int_{t_3}^{t}p_j(s)ds\leq -w'(t)+w'(t_3)<\infty. \tag{2.18}$$

This contradicts the conditon (2.1). Next we assume that $x(t)$ is bounded from above. There exists a constant $L>0$ such that (2.14) holds. It follows from (2.8) and (2.17) that

$$k_1\leq x(t)-\sum_{i=1}^{l}h_ix(\alpha_i(t))+ML\sum_{i=1}^{n}\int_{t_0}^{t}\int_s^{\infty}q_i(\xi)d\xi ds.$$

Taking inferior limit as $t\to\infty$, we observe that

$$k_1=\liminf_{t\to\infty}x(t)+\left(M\sum_{i=1}^{n}\int_0^{\infty}\int_t^{\infty}q_i(\xi)d\xi dt-\sum_{i=1}^{l}h_i\right)L\leq\liminf_{t\to\infty}x(t),$$

which implies that

$$x(t)\geq\frac{k_1}{2},\ \ t\geq t_4$$

for some $t_4\geq t_2$. This contradicts the condition (2.1) by obtaining (2.18). Therefore, we complete the proof of the theorem. □

Example 2.4. Consider the equation

$$[x(t) - x(t-\pi)]'' + \left(2 - \frac{1}{3}e^{-t}\right)x(t-2\pi) \tag{2.19}$$
$$-\frac{1}{3}e^{-t}x(t-\pi) = 0, \ t > 0.$$

It is easy to see that all conditions of Theorem 2.3 hold. Therefore, every solution of (2.19) oscillates. In fact, $x(t) = \cos t$ is such a solution.

3. Oscillation of solutions of nonhomogenous equations

In this section we consider the following oscillation criteria for the equations

$$(\tilde{\mathrm{E}}_\pm) \qquad \left[x(t) \pm \sum_{i=1}^{l} h_i(t)x(\alpha_i(t))\right]''$$
$$+\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) = f(t), \ t > 0.$$

Theorem 3.1. *If* (2.1) *and* (2.2) *hold, then every solution of* $(\tilde{\mathrm{E}}_+)$ *oscillates.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of $(\tilde{\mathrm{E}}_+)$. Without any loss of generality, we assume that $x(t) > 0$, $t \geq t_0$ for some $t_0 > 0$. In view of (H5) there exists a $\varepsilon_F > 0$ such that $F(t) \leq \varepsilon_F$. We define

$$Z(t) = X(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \int_{s}^{\infty} q_i(\xi)G_2(x(\gamma_i(\xi)))d\xi ds \tag{3.1}$$
$$-F(t) + \varepsilon_F, \ t \geq t_1$$

for sufficiently large $t_1 > t_0$. Differentiating the above equation, we have

$$Z'(t) = X'(t) + \sum_{i=1}^{n} \int_{t}^{\infty} q_i(s)G_2(x(\gamma_i(s)))ds - F'(t), \ t \geq t_1. \tag{3.2}$$

After differentiating, this together with $(\tilde{\mathrm{E}}_+)$ implies that

$$Z''(t) = X''(t) - \sum_{i=1}^{n} q_i(t)G_2(x(\gamma_i(t))) - f(t) \tag{3.3}$$
$$= -\sum_{i=1}^{m} p_i(t)G_1(x(\beta_i(t))), \ t \geq t_1,$$

which derives that

$$Z''(t) \leq -p_j(t)G_1(x(\beta_j(t))) \leq 0, \ t \geq t_1 \tag{3.4}$$

for some $j \in \{1, 2, \ldots, m\}$. Hence $Z'(t)$ is nonincreasing. Then we see that $Z'(t) \geq 0$ or $Z'(t) < 0$, $t \geq t_2$ for some $t_2 \geq t_1$.

Case 1. $Z'(t) < 0$ for $t \geq t_2$. As in the proof of Theorem 2.1, we obtain $\lim_{t\to\infty} Z(t) = -\infty$. Otherwise, it follows from (3.1) that $Z(t) \geq 0$. This is a contradiction.

Case 2. $Z'(t) \geq 0$ for $t \geq t_2$. Since $Z'(t) \geq 0$ and $Z(t) \geq 0$ hold, there exists a constant $k_0 > 0$ such that

$$Z(t) \geq k_0, \ t \geq t_3$$

for some $t_3 \geq t_2$. From (3.1) we conclude that $Z(t) \geq x(t)$. Proceeding as in the proof of Theorem 2.1, we obtain

$$K_0 \equiv \left(1 - M \sum_{i=1}^{n} \int_0^{\infty} \int_t^{\infty} q_i(\xi) d\xi dt\right) k_0 \leq \liminf_{t\to\infty} X(t) + \varepsilon_F.$$

Choosing $K_0 > \varepsilon_F > 0$, we see that

$$x(t) + \sum_{i=1}^{l} h_i x(\alpha_i(t)) \geq X(t) \geq K_0 - \varepsilon_F, \ t \geq t_3,$$

which leads to the inequality (2.5) by using the same method of Case 2 of Theorem 2.1. Hence, integrating (3.4) over $[t_3, t]$ and noting (2.5) yields

$$G_1\left(\frac{K_1}{2}\right) \int_{t_3}^{t} p_j(s) ds \leq -Z'(t) + Z'(t_3) < \infty.$$

This is a contradiction. We complete the proof of the theorem. □

Example 3.2. Consider the equation

$$[x(t) + 3x(t - 2\pi)]'' + \left(4 + 2e^{-t}\right) x(t - \pi) \qquad (3.5)$$
$$-e^{-t} x(t - 2\pi) = e^{-t} \sin t, \ t > 0.$$

It is not difficult to see that all conditions of Theorem 3.1 are satisfied. Hence every solution of (3.5) oscillates. Indeed, $x(t) = \sin t$ is such a solution.

Theorem 3.3. *Assume that* (H6). *If* (2.1) *and* (2.7) *hold, then every solution of* ($\tilde{\text{E}}_-$) *oscillates*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of ($\tilde{\text{E}}_-$). Without any loss of generality, we assume that $x(t) > 0$, $t \geq t_0$ for some $t_0 > 0$. We define

$$\begin{aligned} W(t) &= Y(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \int_s^{\infty} q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds \\ &\quad - F(t) + \varepsilon_F, \ t \geq t_1 \end{aligned} \qquad (3.6)$$

for sufficiently large $t_1 > t_0$. Differentiating the above equation, we have

$$W'(t) = Y'(t) + \sum_{i=1}^{n} \int_t^{\infty} q_i(s) G_2(x(\gamma_i(s))) ds - F'(t), \ t \geq t_1. \qquad (3.7)$$

After differentiating, it follows from ($\tilde{\text{E}}_-$) that

$$\begin{aligned} W''(t) &= Y''(t) - \sum_{i=1}^{n} q_i(t) G_2(x(\gamma_i(t))) - f(t) \\ &= -\sum_{i=1}^{m} p_i(t) G_1(x(\beta_i(t))), \ t \geq t_1, \end{aligned} \qquad (3.8)$$

which rewritten as

$$W''(t) = -p_j(t) G_1(x(\beta_j(t))) \leq 0, \ t \geq t_1 \qquad (3.9)$$

for some $j \in \{1, 2, \ldots, m\}$. Hence $W'(t)$ is nonincreasing. Then we see that $W'(t) \geq 0$ or $W'(t) < 0$, $t \geq t_2$ for some $t_2 \geq t_1$.

Case 1. $W'(t) < 0$ for $t \geq t_2$. As in the proof of Theorem 2.1, we obtain $\lim_{t\to\infty} W(t) = -\infty$. If $x(t)$ is not bounded from above, then there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ such that (2.12) holds. Hence we see that

$$W(t_{\bar{n}}) \geq \left(1 - \sum_{i=1}^{l} h_i\right) x(t_{\bar{n}}),$$

and letting $\bar{n} \to \infty$,

$$\lim_{\bar{n}\to\infty} W(t_{\bar{n}}) \geq \left(1 - \sum_{i=1}^{l} h_i\right) \lim_{\bar{n}\to\infty} x(t_{\bar{n}}) = \infty. \tag{3.10}$$

Next we assume that $x(t)$ is bounded from above, then there exists a constant $L > 0$ satisfies (2.14). It is obvious that

$$W(t) \geq x(t) - L\sum_{i=1}^{l} h_i.$$

Taking the superior limit as $t \to \infty$ yields

$$\varlimsup_{t\to\infty} W(t) \geq \left(1 - \sum_{i=1}^{l} h_i\right) L \geq 0. \tag{3.11}$$

This is a contradiction.

Case 2. $W'(t) \geq 0$ for $t \geq t_2$.

Subcase 2.1. $W(t) < 0$ for $t \geq t_2$, then $\lim_{t\to\infty} W(t) = \mu_0 \in (-\infty, 0]$. Taking account into (3.10) and (3.11), we see that $\mu_0 = 0$ and $\lim_{t\to\infty} x(t) = 0$. Then we obtain $\lim_{t\to\infty} Y(t) = 0$. Since $W''(t) \leq 0$, $W'(t) \geq 0$ and $\lim_{t\to\infty} W(t) = 0$, we can prove that $\lim_{t\to\infty} W'(t) = 0$. From (3.7) there exists an $\varepsilon > 0$ such that

$$-\varepsilon M \sum_{i=1}^{n} \int_t^{\infty} q_i(s)ds + F'(t) \leq Y'(t) \leq W'(t) + F'(t).$$

We observe that $\lim_{t\to\infty} Y'(t) = 0$ as $t \to \infty$. From the above discussion, we show that

$$\lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Y'(t) = 0,$$
$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty} W'(t) = 0.$$

Integrating (3.8) two times yields

$$W(t) + \sum_{i=1}^{m} \int_t^{\infty} \int_s^{\infty} p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds = 0.$$

Similarly, integrating ($\tilde{\text{E}}_-$) two times we have

$$Y(t) + \sum_{i=1}^{m} \int_t^{\infty} \int_s^{\infty} p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds - \sum_{i=1}^{n} \int_t^{\infty} \int_s^{\infty} q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds = F(t).$$

Substituting the above facts into $W(t) \geq Y(t) - F(t) + \varepsilon_F$, we see that

$$\begin{aligned} & -\sum_{i=1}^{m} \int_t^\infty \int_s^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds \\ \geq\ & \left\{ -\sum_{i=1}^{m} \int_t^\infty \int_s^\infty p_i(\xi) G_1(x(\beta_i(\xi))) d\xi ds \right. \\ & \left. + \sum_{i=1}^{n} \int_t^\infty \int_s^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds + F(t) \right\} - F(t) + \varepsilon_F, \end{aligned}$$

which leads the following contradiction

$$-\varepsilon_F \geq \sum_{i=1}^{n} \int_t^\infty \int_s^\infty q_i(\xi) G_2(x(\gamma_i(\xi))) d\xi ds.$$

Subcase 2.2. $W(t) \geq 0$ for $t \geq t_1$. Since $W'(t) \geq 0$ and $W(t) \geq 0$, we can show that $W(t) \geq k_1,\ t \geq t_2$ for some constant $k_1 > 0$ and some $t_2 \geq t_1$. If $x(t)$ is not bounded from above, then there exists a sequence $\{t_{\bar{n}}\}_{\bar{n}=1}^{\infty}$ satisfies (2.12). Then we obtain

$$\begin{aligned} k_1 \leq W(t_{\bar{n}}) & \leq \left(1 + M \sum_{i=1}^{n} \int_{t_0}^{t_{\bar{n}}} \int_s^\infty q_i(\xi) d\xi ds \right) x(t_{\bar{n}}) - F(t_{\bar{n}}) + \varepsilon_F \\ & \leq 2x(t_{\bar{n}}) - F(t_{\bar{n}}) + \varepsilon_F, \end{aligned}$$

which implies that

$$\lim_{\bar{n}\to\infty} x(t_{\bar{n}}) \geq \frac{(k_1 - \varepsilon_F)}{2}$$

for $k_1 > \varepsilon_F > 0$. If we choose $\varepsilon_F = k_1/2$, then we see that (2.5) holds. Integrating (3.9) over $[t_3, t]$ yields

$$G_1\left(\frac{K_1}{2}\right) \int_{t_3}^{t} p_j(s) ds \leq -W''(t) + W''(t_3) < \infty. \tag{3.12}$$

This is a contradiction. Hence $x(t)$ is bounded from above. Then there exists a constant $L > 0$ such that (2.14) holds. By choosing $k_1 > \varepsilon_F > 0$, it follows from (2.8) that

$$k_1 \leq x(t) - \sum_{i=1}^{l} h_i x(\alpha_i(t)) + ML \int_{t_0}^{t} \int_s^\infty q_i(\xi) d\xi ds - F(t) + \varepsilon_F.$$

Taking inferior limit as $t \to \infty$, we observe that

$$\liminf_{t\to\infty} x(t) \geq k_1 - \varepsilon_F.$$

Letting $\varepsilon_F = k_1/2$, we obtain (2.5), therefore it is easy to show that contradiction (3.12) holds. We complete the proof of the theorem. □

Example 3.4. Consider the equation

$$\begin{aligned} \left[x(t) - \frac{1}{2}x(t-2\pi)\right]'' + \left(\frac{1}{2} + e^{-t}\right) x(t-2\pi) & \\ -\frac{3}{4}e^{-t}x(t) = \frac{3}{4}e^{-t}\cos t,\ t > 0 & \end{aligned} \tag{3.13}$$

satisfying all conditions of Theorem 3.3. Therefore, every solution of (3.13) oscillates. For example, $x(t) = \cos t$ is such a solution.

References

[1] M.R. Kulenović, S. Hadžiomersphaić, *Existence of nonoscillatory solution of second order linear neutral delay equation*, J. Math. Anal. Appl. **228**, 436–448, 1998.

[2] B. Karpuz, J. Manojlovic, Ö. Öcalan, and Y. Shoukaku, *Oscillation criteria for a class of second order neutral delay differential equations*, Appl. Math. Comput. **210**, 303–312, 2009.

[3] T. Li, Z. Han, S. Sun, and D. Yang, *Existence of nonoscillatory solutions to second-order neutral delay dynamic equations on time scales*, Adv. Difference Equ. **2009**, 1–10, 2009.

[4] T. Li, N. Pintus, and G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys. **70** (3), 1–18, 2019.

[5] T. Li, and Y.V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett. **105**, 1–7, 2020.

[6] J. Manojlović, Y. Shoukaku, T. Tanigawa, and N. Yoshida, *Oscillation criteria for second order differential equations with positive and negative coefficients*, Appl. Math. Comput. **181**, 853–863, 2006.

[7] S. Padhi, *Oscillation and asymptotic behavior of solutions of second order neutral differential equations with positive and negative coefficients*, Fasc. Math. **38**, 105–114, 2007.

[8] S. Padhi, *Oscillation and asymptotic behavior of solutions of second order homogeneous neutral differential equations with positive and negative coefficients*, Funct. Differ. Equ. **14**, 363–371, 2007.

[9] N. Parhi, and S. Chand, *Oscillation of second order neutral delay differential equations with positive and negative coefficients*, J. Ind. Math. Soc. **66**, 227–235, 1999.

[10] N. Parhi, and S. Chand, *On second order neutral delay differential equations with positive and negative coefficients*, Bull. Cal. Math. Soc. **94**, 7–16, 2002.

[11] A. Weng, and J. Sun, *Oscillation of second order delay differential equations*, Appl. Math. Comput. **198**, 930–935, 2007.