Heine Diferansiyel Denkleminin Komütativite Koşulları

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Öz

Çoğu mühendislik sistemlerinin gerçekleştirilmesi, daha basit sistemlerin ardışık bağlantıları ile yapılmaktadır. Bu durum elektrik ve elektronik sistemlerinin tasarımında çok önemlidir. Her ne kadar bu alt sistemlerin bağlantı sırası, kullanılan özel tasarım yöntemlerine, mühendislik tecrübesine, alışılagelmiş sentez yöntemlerine bağlı olmakla beraber, hassasiyet, kararlılık, doğrusallık, gürültüden etkilenme ve dayanıklılık hususları göz önüne alındığında toplam sistemin ana fonksiyonunu değiştirmeden alt sistemlerin bağlantı sırasının değiştirilmesi (komütativite) pozitif sonuçlara yol açabilmektedir. Bu nedenle pratik uygulamalar açısından komütativite çok önemlidir. Bu çalışmada, Heine diferansiyel denkleminin komütativitesi ele alınmıştır. Bir Heine diferansiyel denklemi ile modellenen sistemin, denklemin parametrelerine bağlı olarak komütatif bir çifti olduğu ya da olmadığı gösterilmiştir. Komütatif olabilmesi için gerekli ve yeterli koşullar belirtilmiştir. Uygulama için bir örnek ele alınmıştır. Bu çalışmada başlangıç koşullarının sıfır olduğu varsayılmıştır. Sıfır olmayan başlangıç koşulları (veya geri beslemeli sistemler) için, bu çalışmada türetilen Heine diferansiyel sistemleri için komütativite koşullarına ek olarak bazı ilave koşullar gereklidir. Sıfır olmayan başlangıç koşulları durumu, genel ikinci dereceden sürekli zamanla değişen doğrusal sistemler için literatürde (Koksal, 2019) daha önce çalışılmış olduklarından bu makalede bunun üzerinde orijinal sonuçlar elde etmemektedir. Komütativite varlığının koşullarını sağlayan bazı parametreler için komütatif Heine diferansiyel denklem çiftleri elde edilmiştir ve bunların Heine tipi olmadığı görülmüştür. Bu nedenle, Heine diferansiyel sistemlerinin komütatif çiftleri, komütatif çiftlerle ilgili keyfi sabitlerin genel seçimi için Heine tipi değildir.

Anahtar Kelimeler: Komütativite, Analog Sistemler, Heine Diferansiyel Denklemi, Zamanla Değişen Sistemler.

Commutativity Conditions of Heine's Differential Equation

Abstract

The realization of many engineering systems consists of cascade connection of systems of simple orders, which is very important in design of electrical and electronic systems. Although the order of connection of the systems mainly depends on the special design approach, engineering ingenuity, traditional synthetic methods, when the sensitivity, stability, linearity, noise disturbance, robustness effects are considered the change of the order of connection without changing the main function of the total system (commutativity) may lead positive results. Therefore, the commutativity is very important from the practical point of view. In this paper, commutativity of Heine differential equation is considered. It is shown that the system modeled by a Heine differential equation has a commutative pair or not depending on the parameters of the equation. The conditions that must be satisfied for the commutativity are set in this contribution. An example is considered for application. The initial conditions are assumed to be zero in this study. For the nonzero initial conditions (or for unrelaxed systems), some further relations should hold in addition to the commutativity conditions for relaxed Heine differential systems derived in this paper. The case of nonzero initial conditions do not gain original results on the paper since they have already been studied in the literature (Koksal, 2019) for general second-order continuous-time linear time-varying systems. The commutative pairs of Heine differential equation are obtained for some parameters satisfying the existence conditions of commutativity and it is seen that they are not of Heine type. So, the commutative pairs of Heine differential systems are not Heine type for the general choice of arbitrary constants relating the commutative pairs.

Keywords: Commutativity, Analog System, Heine's Differential Equations, Time-varying Systems.

1. Introduction

The commutativity is a very important subject concerning engineers and mathematicians. Especially, it places an important role to improve different system performances in system and control design as a main branch of electrical and electronics engineering. For example, commutative subsystems are used in cascade connected and feedback structures to design higher order composite systems for achieving several beneficial properties such as controllability, sensitivity, robustness, and design flexibility.

Firstly, this subject was studied in the literature by E. Marshall in 1982 (Marshall, 1977) for the first-order linear time-varying differential systems. After that, commutativity conditions of second-order (Koksal, 1982), third-order (Koksal, 1985a), and fourth-order (Koksal, 1985b), systems were obtained by M. Koksal in 1982, 1985a and 1985b respectively. After a long time, commutativity of fifth-order systems (Koksal et al., 2011) was studied by M. Koksal and M.E. Koksal in 2011.

Although most of the famous second-order linear time-varying differential systems were subjected from the commutativity point of view in (Koksal, 2018a, 2018b), Heine's differential equation (Zwillinger, 1997) is not among them. So the purpose of this presentation is to study the commutativity conditions of Heine differential equation; namely

$$\ddot{y} + \frac{1}{2}\left(\frac{1}{x-e} + \frac{2}{x-h}\right)\dot{y} + \frac{a + bx + cx^2 + dx^3}{4(x-e)(x-f)^2(x-h)^2}y = 0. \tag{1}$$

Here a, b, c, d, e, f, h are constants. Note that $\ddot{y}_A(t) = \frac{d^2}{dt^2}y_A(t)$, $\dot{y}_A(t) = \frac{d}{dt}y_A(t)$, and the initial conditions are assumed to be zero.

2. Commutativity Conditions

Let A be linear time-varying second-order differential system described by

$$a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t); t \geq 0, \tag{2}$$

where $x_A(t)$ is the independent excitation and $y_A(t)$ is the resulting response. For the unique solution of Eq. (2) for $t \geq t_0$, it is sufficient that the excitation and the time-varying coefficients $a_2(t)$, $a_1(t)$, $a_0(t)$ be piece-wise continuous functions of time with $a_2(t) \not\equiv 0$.

It is shown in (Koksal, 1982) that all the commutative pairs of (2) are obtained by

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & a_2^{-0.5}(2a_1 - \dot{a}_2)/4 & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} \tag{3}$$

if the coefficients of A satisfy

$$-a_2^{0.5}\frac{d}{dt}\left[a_0 - \frac{1}{16a_2}(4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1a_2 - 4a_2\ddot{a}_2)\right]k_1 = 0. \tag{4}$$

In Eqs. (2) and (3), k_2, k_1, k_0 are some constants. Then any commutative pair B of A is described by

$$b_2(t)\ddot{y}_B(t) + b_1(t)\dot{y}_B(t) + b_0(t)y_B(t) = x_B(t). \tag{5}$$

If $k_1 = 0$, Eq. (3) is automatically satisfied for any second-order linear time-varying differential system A. The case $k_1 = 0$ correspods to a system B through Eq. (3) which can be obtained from A by constant feedback and constant feedforword gains (constant feedback conjugate of A) (Koksal, et al., 2011). Therefore , every second-order linear time-varying system A has a commutative pair which is its constant feedback conjugate. In this presentation, we look for the commutativity not of this type, so $k_1 \neq 0$, and the coefficients of A must satisfy

$$a_0 - \frac{1}{16a_2}(4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1a_2 - 4a_2\ddot{a}_2) = C \tag{6}$$

for all $t \geq t_0$ where C is some constant.

3. Commutativity of Heine Differential Equation

Under the light of the above given theoretical bases, we investigate the commutativity conditions for the Heine differential equations given in (1).

With the coefficients of Heine differential equation in (1)

$$a_2(x) = 1,$$

$$a_1(x) = \frac{1}{2}\left(\frac{1}{x-e} + \frac{2}{x-h}\right),$$

$$a_0(x) = \frac{a + bx + cx^2 + dx^3}{4(x-e)(x-f)^2(x-h)^2}.$$

Because of the fact that derivative of $a_2(x)$ is equal to zero, the commutativity condition in Eq. (6) yields

$$4a_0 - a_1^2 - 2\dot{a}_1 = K, \tag{7}$$

where $K = 4C$ is a constant. Using the formulas of $a_0(x)$, $a_1(x)$ and $\dot{a}_1(x) = -\frac{1}{2}\left[\frac{1}{(x-e)^2} + \frac{2}{(x-h)^2}\right]$ in Eq. (7), we obtain

$$\frac{dx^4 + (c-ed)x^3 + (b-ec)x^2 + (a-eb)x - ea}{(x-e)^2(x-f)^2(x-h)^2}$$

$$-\frac{1}{4}\left[\frac{1}{(x-e)^2} + \frac{4}{(x-e)(x-h)} + \frac{4}{(x-h)^2}\right] + \left[\frac{1}{(x-e)^2} + \frac{2}{(x-h)^2}\right] = K.$$

For finding the unknown parameters in the above equation, we equalize the denominators at the left side of the equation. Then, we multiply both sides of the equation by $(x-e)^2(x-f)^2(x-h)^2$ and obtain the following expression:

$$4[dx^4 + (c-ed)x^3 + (b-ec)x^2 + (a-eb)x - ea] - 4(x-f)^2(x-e)(x-h)$$

$$+3(x-f)^2(x-h)^2 + 4(x-f)^2(x-e)^2 = K(x-e)^2(x-f)^2(x-h)^2.$$

So, we obtain a fourth-order and six-order polynomials at the left and right sides of the equation respectively. Because of the fact that the right side of the equation is sixth-order though the left side is fourth-order, K should be equal to zero. We obtain the following equations by equalizing the coefficients:

for x^6,

$$K = 0$$

for x^5,

$$-2(h+f)K = 0$$

for x^4,

$$4d + 3 = 0$$

for x^3,

$$4c = 4ed + 2h + 6f + 4e$$

for x^2,

$$4b - 4ec - 4eh + 3h^2 + 3f^2 + 4fh + 4e^2 + 8fe = 0$$

for x^1,

$$4a - 4eb + 8feh - 6fh^2 - 2f^2h - 8fe^2 - 4f^2e = 0$$

for x^0,

$$-4ea - 4f^2eh + 3f^2h^2 + 4f^2e^2 = 0.$$

Arranging these equations, we get

$$d = -3/4, \tag{8}$$

$$c = \frac{e + 6f + 2h}{4}, \tag{9}$$

$$b = -\frac{3e^2 + 2e(f - 3h) + 3(f^2 + h^2) + 4fh}{4}, \tag{10}$$

$$a = \frac{-3e^3 + 6e^2(f + h) + e(f^2 - 3h^2 - 12fh) + 2fh(f + 3h)}{4}, \tag{11}$$

$$e^4 - 2e^3(f + h) + e^2(f + h)^2 - 2efh(f + h) + f^2h^2 = 0. \tag{12}$$

These equations indicate that e, f and h can be chosen freely as to satisfy Eq. (12) and c, b, a are expressed in terms of these freely chosen parameters by Eqs. (9-11).

The parameter d must be strictly equal to $-\frac{3}{4}$ as given in Eq. (8).

4. Application

As an application, let us consider Eq. (1) such that $f = 1$ and $h = 0$. For these values, from Eq. (12), we find $e = 0$ or 1.

i)   For $e = 0$, from Eqs. (9-11), we get

$$a = 0, b = -\frac{3}{4}, c = \frac{3}{2}.$$

Then, the Heine differential equation reduces to $\ddot{y} + \frac{3}{2x}\dot{y} - \frac{3}{16x^2}y = 0$. For this differential equation, $4a_0 - a_1^2 - 2\dot{a}_1 = 0$. So, Eq. (7) is satisfied. Commutative pair of A is obtained through Eq. (3) as

$$k_2\ddot{y}_B(t) + \left(\frac{3}{2x}k_2 + k_1\right)\dot{y}_B(t) + \left(-\frac{3}{16x^2}k_2 + \frac{3}{4x}k_1 + k_0\right)y_B(t) = x_B(t)$$

which is not an Heine differential equation at all.

ii)   For $e = 1$, from Eqs. (9-11), we get

$$a = 1, b = -2, c = \frac{7}{4}.$$

Then, the Heine differential equation reduces to $\ddot{y} + \frac{1}{2}\left(\frac{1}{x-1} + \frac{2}{x}\right)\dot{y} + \frac{4 - 8x + 7x^2 - 3x^3}{16(x-1)^3x^2}y = 0$. For this differential equation, $4a_0 - a_1^2 - 2\dot{a}_1 = 0$. So, Eq. (7) is satisfied. Commutative pair of A is obtained through Eq. (3) as

$$k_2\ddot{y}_B(t) + \left[\frac{1}{2}\left(\frac{1}{x-1} + \frac{2}{x}\right)k_2 + k_1\right]\dot{y}_B(t) + \left[\frac{4 - 8x + 7x^2 - 3x^3}{16(x-1)^3x^2}k_2 + \frac{1}{4}\left(\frac{1}{x-1} + \frac{2}{x}\right)k_1 + k_0\right]y_B(t) = x_B(t)$$

which is not an Heine differential equation at all.

5. Conclusions

The conditions for the existence of second or lower order commutative pairs of the Heine differential equation with zero initial conditions are derived in this contribution. It is shown that Heine's system has many second or lower order commutative pairs other than its feedback conjugates obtained from itself by constant feed-forward and constant feed-back gains. The case of nonzero initial conditions do not gain original results on the paper since they have already been studied in the literature (Koksal, 2019) for general second-order continuous-time linear time-varying systems. The commutative pairs of Heine differential equation are obtained for some parameters satisfying the existence conditions of commutativity and it is seen that they are not of Heine type.

6. Acknowledgment

7. References

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