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Remarks on F -weak contractions and discontinuity at the fixed point

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Abstract

The aim of this paper is to generalize celebrated results due to Wardowski[8, 9] and also to provide yet new solutions to the once open problem[6, p.242] on the existence of a contractive mapping which possesses a fixed point but is not continuous at the fixed point via generalized F^{**} -weak contractions. Finally, an example is given to illustrate our results.

Keywords: F -contraction; F -weak contraction; F^{**} -weak contraction; contractive mapping; fixed point theorem; discontinuity at the fixed point.

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1. Introduction and Preliminary Notes

The first contractive definition is that of Banach[1], which states that if G satisfies, $d(Gx, Gy) \leq \sigma d(x, y)$ for each $x, y \in X$, where $\sigma \in [0, 1)$, then G has a unique fixed point in X . This hypothesis has numerous applications but endures from one disadvantage, the definition requires G be continuous throughout X . Kannan[3] gave an example of a contractive concept in 1968, which did not require that G be continuous. There after several authors made several extensions of this result. Rhoades[5] compared 250 contractive definitions, and he found that while most of the contractive definitions do not require the mapping to be continuous over the whole domain, they all require continuous mapping over the entire domain, they all require continuous mapping at the fixed point. Rhoades[6] motivated by his observations formulated a

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fascinating open problem, “whether there exists a contractive definition which is strong enough to ensure the existence and uniqueness of a fixed point which does not force the mapping to be continuous at the underlying fixed point”(p.242).

Recently, some solutions to the open question has been proposed and investigated. For example, Pant[4], Wardowski[8], Wardowski and Van Dung[9] and Alfaqih[7]. Motivated by the studies above, we are investigating new contractive conditions to get another solution to the open question. We recall the following definitions, which are necessary in the next section, before stating our main results.

Definition 1.1. [8] Let \mathfrak{S} be the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:
 (F_1) : F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty), \alpha < \beta \Rightarrow F(\alpha) < F(\beta)$,
 (F_2) : for every sequence $\beta_n \subseteq (0, \infty), \lim_{n \rightarrow \infty} F(\beta_n) = -\infty \Rightarrow \lim_{n \rightarrow \infty} \beta_n = 0$,
 (F_3) : there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Example 1.1. [8] The following functions belong to F :

- (i) $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$,
- (ii) $F(\alpha) = \log \alpha$,
- (iii) $F(\alpha) = \alpha + \log \alpha$.

Definition 1.2. [8] Let (X, d) be a metric space. A self-mapping G on X is said to be an F -contraction if there exist $\sigma > 0$ and $F \in \mathfrak{S}$ such that (for all $x, y \in X$)

$$d(Gx, Gy) > 0 \Rightarrow \sigma + F(d(Gx, Gy)) \leq F(d(x, y))$$

Wardowski[8] proved the finding as follows:

Theorem 1.1. Every F -contraction mapping G defined on a complete metric space (X, d) has a unique fixed point. Moreover, for any $x \in X$, the sequence $\{G_n x\}$ converges to the fixed point of G .

In 2014, Wardowski and Dung[9] utilized the same lesson of auxiliary functions to present the idea of F -weak contractions as follows:

Definition 1.3. [9] Let (X, d) be a metric space. A self-mapping G on X is said to be an F -weak contraction if there exist $\sigma > 0$ and $F \in \mathfrak{S}$ such that (for all $x, y \in X$)

$$d(Gx, Gy) > 0 \Rightarrow \sigma + F(d(Gx, Gy)) \leq F(M(x, y)),$$

where $M(x, y) = \max \left\{ d(x, y), d(x, Gx), d(y, Gy), \frac{d(x, Gy) + d(y, Gx)}{2} \right\}$.

Wardowski and Dung[9] demonstrated the following hypothesis:

Theorem 1.2. Let (X, d) be a complete metric space and $G : X \rightarrow X$ is a F -weak contraction. If F or G is continuous then,

- (a) G has unique fixed point (say $w \in X$)
- (b) $\lim_{n \rightarrow \infty} G^n(x) = w$ for all $x \in X$

Alfaqih, et al.[7] interpret that, theorem(1.2) will survive without F_1 and F_3 assumptions besides eliminating one way inference of F_2 assumption and present the following Auxiliary class functions. Let \mathfrak{S}' be the set of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following condition:

F_2^* : for every sequence $\beta_n \subseteq (0, \infty), \lim_{n \rightarrow \infty} F(\beta_n) = -\infty \Rightarrow \lim_{n \rightarrow \infty} \beta_n = 0$.

Next, Alfaqih, et al.[7] introduce the notion of F^* -weak contraction mappings, which is as follows

Definition 1.4. Let (X, d) be a metric space. A self-mapping G on X is said to be an F -weak contraction if there exist $\sigma > 0$ and $F \in \mathfrak{S}$ such that

$$d(Gx, Gy) > 0 \Rightarrow \sigma + F(d(Gx, Gy)) \leq F(m(x, y)),$$

where $m(x, y) = \max \{d(x, y), d(x, Gx), d(y, Gy)\}$.

Alfaqih, et al.[7] proved the subsequent theorem:

Theorem 1.3. Let (X, d) be a complete metric space and $G : X \rightarrow X$ is a F^* -weak contraction. If F is continuous then,

(a) G has unique fixed point (say $w \in X$)

(b) $\lim_{n \rightarrow \infty} G^n(x) = w$ for all $x \in X$

Moreover, G is continuous at w if and only if $\lim_{n \rightarrow \infty} m(x, z) = 0$.

Now, we give the following lemma which will be used effectively in proving this study's key theorem.

Lemma 1.1. [2] Let $\{x_n\}$ be a sequence in a metric space (X, d) . If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$k \leq m(k) < n(k), \quad d(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad \text{and} \quad d(x_{m(k)}, x_{n(k)-1}) < \epsilon,$$

for all $k \in \mathbb{N}$. Furthermore, if $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, then

$$\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon.$$

The aim of this article is to present the idea of F^{**} -weak mapping and obtain new solutions to the open question on the existence of F^{**} -weak contractive conditions that are strong enough to generate a fixed point but which do not require the mapping to be continuous at the point.

2. Main Results

First, we present the idea of F^{**} -weak mapping of the contraction, which operates as follows.

Definition 2.1. Let (X, d) be a metric space. A map $G : X \rightarrow X$ is said to be an F^{**} -weak contraction on (X, d) if there exist $F \in \mathfrak{S}$ and $\sigma > 0$ such that, for all $x, y \in X$ satisfying $d(Gx, Gy) > 0$, the following holds:

$$\begin{aligned} & \sigma + F(d(Gx, Gy)) \\ & \leq F(\max \{d(x, y), d(x, Gx), d(y, Gy), d(x, Gy), d(y, Gx)\}) \end{aligned}$$

Remark 2.1. Every F -contraction is an F^{**} -weak contraction but inverse implication does not hold.

Now, we are able to construct this section's main theorem.

Theorem 2.1. Let (X, d) be a complete metric space and $G : X \rightarrow X$ is a F^{**} -weak contraction. If F is continuous and $\sigma > 0$ then,

(a) G has unique fixed point $w \in X$

(b) $\lim_{n \rightarrow \infty} G^n(x) = w$ for all $x \in X$

Additionally, G is continuous at w if and only if

$$\lim_{n \rightarrow \infty} (\max \{d(x, y), d(x, Gx), d(y, Gy), d(x, Gy), d(y, Gx)\}) = 0$$

Proof. Let $x_0 \in X$ be an arbitrary and fixed. We define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Gx_n$. If there exist $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$ then $Gx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of G .

Now we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. It follows from definition(2.1) that for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
 F(d(x_{n+1}, x_n)) &= F(d(Gx_n, Gx_{n-1})) \\
 &\leq F(\max d(x_n, x_{n-1}), d(x_n, Gx_n), d(x_{n-1}, Gx_{n-1}), d(x_n, Gx_{n-1}), d(x_{n-1}, Gx_n)) - \sigma \\
 &= F(\max d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), d(x_{n-1}, x_{n+1})) - \sigma \\
 &= F(\max d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})) - \sigma \\
 &\leq F(\max d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n) + d(x_n, x_{n+1})) - \sigma \\
 &= F(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) - \sigma
 \end{aligned}
 \tag{1}$$

For all $n \in \mathbb{N}$. Thus we have

$$F(d(x_{n+1}, x_n)) = F(d(Gx_n, Gx_{n-1})) \leq F(d(x_0, x_1) + d(x_1, x_2)) - n\sigma
 \tag{2}$$

For all $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ in eq.(2) we get

$$\lim_{n \rightarrow \infty} F(d(x_{n+1}, x_n)) = -\infty$$

which together with F'_2 gives

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0
 \tag{3}$$

Now, we prove $\{x_n\}$ is a Cauchy sequence. Contradictory, suppose that $\{x_n\}$ isn't a Cauchy sequence. As a result of lemma (1.1) and eq.(3), there exist $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that

$$k \leq m(k) < n(k), \quad d(x_{n(k)-1}, x_{m(k)}) < \epsilon d(x_{n(k)}, x_{m(k)}), \quad \forall k \in \mathbb{N} \cup \{0\}$$

and

$$\lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon.
 \tag{4}$$

Now by eq.(4), there exists $N \in \mathbb{N} \cup \{0\}$ such that $d(x_{n(k)}, x_{m(k)}) > 0$, for all $k > N$. Now from Definition (2.1) with $x = x_{n(k)-1}$ and $y = x_{m(k)-1}$, we have

$$\begin{aligned}
 F(d(x_{n(k)}, x_{m(k)})) &= F(d(Gx_{n(k)-1}, Gx_{m(k)-1})) \\
 &\leq F(\max\{d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), \\
 &\quad d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})\}) - \sigma
 \end{aligned}
 \tag{5}$$

Assuming $k \rightarrow \infty$ and using eq.(4) and eq.(5), we get

$$\begin{aligned}
 \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)-1}) &= \epsilon, \\
 \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{n(k)}) &= \epsilon, \\
 \lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{m(k)}) &= \epsilon, \\
 \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)}) &= \epsilon, \\
 \lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)}) &= \epsilon,
 \end{aligned}
 \tag{6}$$

As F is continuous so by using eq.(4), eq.(5) and eq.(6), we obtain $F(\epsilon) \leq F(\epsilon) - \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a cauchy sequence. Since X is complete, there exist $w \in X$ such that $\lim_{n \rightarrow \infty} x_n = w$.

Now we prove w is an unique fixed point of G . Let $Q = \{n \in \mathbb{N} \cup \{0\} : x_n = Gw\}$. Here we consider two cases:

(a) **Q is infinite:**

In the event that Q is infinite at that point there exists a subsequence $\{x_{n(k)}\} \subseteq \{x_n\}$ which converges to

Gw . Since limit is unique we get $Gw = w$.

(b) Q is finite:

If Q is finite then $d(x_n, Gw) > 0$ for infinitesimal $n \in \mathbb{N} \cup \{0\}$. Therefore there exist a subsequence $\{x_{n(k)}\} \subseteq \{x_n\}$ in such a manner that $d(x_{n(k)}, Gw) > 0$ for every $k \in \mathbb{N} \cup \{0\}$. Now by utilizing eq.(1), we find

$$\begin{aligned} F(d(x_{n(k)}, Gw)) &= F(d(Gx_{n(k)-1}, Gw)) \\ &\leq F(\max\{d(x_{n(k)-1}, w), d(x_{n(k)-1}, x_{n(k)}), \\ &\quad d(w, Gw), d(x_{n(k)-1}, Gw), d(w, x_{n(k)})\}) - \sigma \end{aligned} \tag{7}$$

If $d(Gw, w) > 0$, then by letting $k \rightarrow \infty$ in eq.(7), we obtain $F(d(w, Gw)) \leq F(d(w, Gw)) - \sigma$. Which is a contradiction, therefore $d(Gw, w) = 0$. Thus we get $Gw = w$. That means w is a fixed point of G . To prove the uniqueness, let u be some other fixed point of G . Then definition(2.1) gives $F(d(w, u)) \leq F(d(w, u)) - \sigma$ a contradiction. Hence G has unique point.

Lastly, let us assume that G is continuous at w and $\{z_n\} \rightarrow w$. Then we have $\{Gz_n\} \rightarrow Gw = w$ and $\lim_{n \rightarrow \infty} d(z_n, Gz_n) = 0$.

Therefore

$$\lim_{n \rightarrow \infty} \max\{d(z_n, w), d(z_n, Gz_n), d(w, Gw), d(z_n, Gw), d(w, Gz_n)\} = 0.$$

Conversely, let $\{z_n\} \rightarrow w$, if we assume that

$$\begin{aligned} \lim_{n \rightarrow \infty} \max\{d(z_n, w), d(z_n, Gz_n), d(w, Gw), d(z_n, Gw), d(w, Gz_n)\} &= 0 \\ \Rightarrow \lim_{n \rightarrow \infty} d(z_n, Gz_n) &= 0 \\ \Rightarrow \lim_{n \rightarrow \infty} Gz_n = \lim_{n \rightarrow \infty} z_n = w = Gw \end{aligned}$$

Therefore G is continuous at w . □

Example 2.1. Let $X = [0, 1]$ and d be the usual metric on X . Define $G : X \rightarrow X$ as

$$Gx = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < 1 \\ \frac{1}{6} & \text{if } x = 1 \end{cases}$$

Now, for $0 \leq x < 1$ and $y = 1$, we have

$$\begin{aligned} d(Gx, 1) &= d\left(\frac{1}{2}, \frac{1}{6}\right) \\ &= \left|\frac{1}{2} - \frac{1}{6}\right| \\ &= \left|\frac{1}{3}\right| \\ &= \frac{1}{3} > 0 \end{aligned}$$

and

$$\begin{aligned} d(1, G1) &= d\left(1, \frac{1}{6}\right) \\ &= \left|1 - \frac{1}{6}\right| \\ &= \frac{5}{6} \end{aligned}$$

Therefore

$$\begin{aligned} & \max\{d(x, 1), d(x, Gx), d(1, G1), d(x, G1), d(1, Gx)\} \\ & \geq d(1, G1) \\ & = \frac{5}{6} \end{aligned}$$

Now, Let us consider $F : (0, \infty) \rightarrow \mathbb{R}$ be inclined by $F(\alpha) = \cos\alpha - \frac{1}{\alpha}$, $\sigma = 2$, G is a F^{**} -weak contraction then by theorem (2.1), G has a unique fixed point $\frac{1}{2}$.

$$\lim_{x \rightarrow \frac{1}{2}} \max \left\{ d \left(x, \frac{1}{2} \right), d \left(x, G \left(\frac{1}{2} \right) \right), d \left(\frac{1}{2}, G \left(\frac{1}{2} \right) \right), d \left(x, G \left(\frac{1}{2} \right) \right), d \left(\frac{1}{2}, Gx \right) \right\} = 0$$

and G is continuous at its fixed point nonetheless it's discontinuous at its domain.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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