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DUAL-COMPLEX GENERALIZED k-HORADAM NUMBERS

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Abstract. The purpose of this paper is to provide a broad overview of the generalization of the various dual−complex number sequences, especially in the disciplines of mathematics and physics. By the help of dual numbers and dual−complex numbers, in this paper, we define the dual−complex generalized $k-$Horadam numbers. Furthermore, we investigate the Binet formula, generating function, some conjugation identities, summation formula and a theorem which is generalization of the Catalan's identity, Cassini's identity and d'Ocagne's identity.

1. Introduction

Dual numbers and dual−complex numbers arise in many areas in physics and mathematics such as coordinate transformation, matrix modeling, displacement analysis, rigid body dynamics, velocity analysis, static analysis, dynamic analysis, 2D rigid transformation, mechanics, kinematics and applications of geometry. In the 19 th century, Clifford described the dual numbers with in the form $A = a+\varepsilon a^*$, where $a, a^* \in \mathbb{R}$, $\varepsilon^2 = 0$ and $\varepsilon \neq 0$ [4]. Up to this time, there are number of studies in the literature that concern about the dual numbers and dual−complex numbers [1,3,5,8–10,17–20]. For instance, Fjelstad and Gal examined the extensions of the hyperbolic complex numbers to n dimensions and they presented $n-$dimensional dual complex numbers [9]. Matsuda et al. examine the ring of anti−commutative dual complex numbers, which parametrizes two dimensional rotation and translation all together, by modifying the ordinary dual number construction for the complex numbers [18]. Majernik presented three types of the four−component number

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TABLE 1. Multiplication scheme of dual−complex numbers

x	1	$\mathbf{i}$	ε	$\mathbf{i}\varepsilon$
1	1	$\mathbf{i}$	ε	$\mathbf{i}\varepsilon$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{i}\varepsilon$	$-\varepsilon$
ε	ε	$\mathbf{i}\varepsilon$	0	0
$\mathbf{i}\varepsilon$	$\mathbf{i}\varepsilon$	$-\varepsilon$	0	0

systems which are built by using the complex, binary and dual two-component number [17]. Messelmi, in [19], expressed the dual−complex numbers as

$$\mathbb{DC} = \{w = z_1 + \varepsilon z_2 | z_1, z_2 \in \mathbb{C} \quad where \quad \varepsilon^2 = 0, \varepsilon \neq 0\}. \tag{1}$$

He also generalized the concept of holomorphicity to dual−complex functions inspired by topics in complex analysis. In Eq. (1), if $z_1 = x_1 + \mathbf{i}x_2$ and $z_2 = y_1 + \mathbf{i}y_2$, then any dual−complex number can be formulated by

$$w = x_1 + \mathbf{i}x_2 + \varepsilon y_1 + \mathbf{i}\varepsilon y_2. \tag{2}$$

Moreover, multiplication and subtraction of dual−complex numbers w_1 and w_2 are defined by

$$w_1 \pm w_2 = (z_1 + \varepsilon z_2) \pm (z_3 + \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon (z_2 \pm z_4) \tag{3}$$

and

$$w_1 \times w_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon (z_1 z_4 + z_2 z_3). \tag{4}$$

Also, the division of dual−complex numbers can be given by

$$\frac{w_1}{w_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{(z_1 + \varepsilon z_2)(z_3 - \varepsilon z_4)}{(z_3 + \varepsilon z_4)(z_3 - \varepsilon z_4)} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{z_3^2}. \tag{5}$$

Another important topic is the number sequences that have been studied over many years. For $n \in \mathbb{N}_0$, the Fibonacci and Lucas numbers are defined by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1 \tag{6}$$

and

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1, \tag{7}$$

respectively. Recently, many researchers have studied several applications and generalizations of the number sequences(see [6,7,11–15,21,22,24]). Yazlık and Taşkara introduced the generalized $k-$Horadam sequence, which is generalization of many number sequences in the literature [24]. For $n \in \mathbb{N}_0$ and $f(k)^2 + 4g(k) > 0$, the generalized $k-$Horadam sequence is defined by

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \quad H_{k,0} = a, \quad H_{k,1} = b. \tag{8}$$

Note that, the Binet formula of the generalized $k-$Horadam sequence is given by, for $n \in \mathbb{N}_0$,

$$H_{k,n} = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta}, \tag{9}$$

where $X = b - a\beta$ and $Y = b - a\alpha$. Up until now, some researchers have studied dual$-$complex numbers with Fibonacci and Lucas numbers. For example, Güngör and Azak defined dual$-$complex Fibonacci and Lucas numbers and investigate some properties of these numbers such as Binet formulas, Cassini and Catalan identities [10]. Aydın, presented dual$-$complex $k-$Fibonacci numbers and then examine some algebraic properties of dual$-$complex $k-$Fibonacci numbers which are related to dual$-$complex numbers and $k-$Fibonacci numbers [1]. She also described some algebraic properties of dual$-$complex $k-$Pell numbers and quaternions which are related to dual$-$complex numbers and $k-$Pell numbers [2].

Inspiring by these studies and by combining dual$-$complex numbers and generalized $k-$Horadam numbers, in this paper, we define the dual$-$complex generalized $k-$Horadam numbers which is a generalization of the studies [10], [1] as presented in Table 2. The next section describes the dual$-$complex generalized $k-$Horadam numbers and contains various theorems and corollaries regarding the dual$-$complex generalized $k-$Horadam numbers.

2. Dual$-$Complex Generalized $k-$Horadam Numbers

Definition 1. *For $n \in \mathbb{N}_0$, the dual$-$complex generalized $k-$Horadam numbers are defined by*

$$\mathbb{DC}H_{k,n} = H_{k,n} + \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3}, \tag{10}$$

where $H_{k,n}$ is the generalized $k-$Horadam numbers which is defined in (8).

It can seen easily from the Table 2 that the dual$-$complex generalized $k-$Horadam numbers can be reduced into several dual$-$complex numbers for the special cases of $f(k)$, $g(k)$, a and b. Let $\mathbb{DC}H_{k,n}$ and $\mathbb{DC}H_{k,m}$ be two dual$-$complex generalized $k-$Horadam numbers. Then, the addition and subtraction of the dual$-$complex generalized $k-$Horadam numbers are defined by

$$\begin{aligned}\mathbb{DC}H_{k,n} \pm \mathbb{DC}H_{k,m} &= (H_{k,n} + \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3}) \\ &\qquad \pm (H_{k,m} + \mathbf{i}H_{k,m+1} + \varepsilon H_{k,m+2} + \mathbf{i}\varepsilon H_{k,m+3}) \\ &= (H_{k,n} \pm H_{k,m}) + \mathbf{i}\,(H_{k,n+1} \pm H_{k,m+1}) \\ &\qquad + \varepsilon\,(H_{k,n+2} \pm H_{k,m+2}) + \mathbf{i}\varepsilon\,(H_{k,n+3} \pm H_{k,m+3})\,.\end{aligned} \tag{11}$$

The multiplication of a dual$-$complex generalized $k-$Horadam number by the real scalar λ is defined as:

$$\lambda\mathbb{DC}H_{k,n} = \lambda H_{k,n} + \mathbf{i}\lambda H_{k,n+1} + \varepsilon\lambda H_{k,n+2} + \mathbf{i}\varepsilon\lambda H_{k,n+3}.$$

Furthermore, the multiplication of two dual−complex generalized $k-$Horadam numbers is defined by:

$$\begin{aligned}
&\mathbb{DC}H_{k,n} \times \mathbb{DC}H_{k,m} \\
&= \left(H_{k,n} + \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3}\right) \\
&\quad \times \left(H_{k,m} + \mathbf{i}H_{k,m+1} + \varepsilon H_{k,m+2} + \mathbf{i}\varepsilon H_{k,m+3}\right) \\
&= \left(H_{k,n}H_{k,m} - H_{k,n+1}H_{k,m+1}\right) \\
&\quad + \mathbf{i}\left(H_{k,n}H_{k,m+1} + H_{k,n+1}H_{k,m}\right) \\
&\quad + \varepsilon\left(H_{k,n}H_{k,m+2} + H_{k,n+2}H_{k,m} - H_{k,n+1}H_{k,m+3} - H_{k,n+3}H_{k,m+1}\right) \\
&\quad + \mathbf{i}\varepsilon\left(H_{k,n}H_{k,m+3} + H_{k,n+3}H_{k,m} + H_{k,n+1}H_{k,m+2} + H_{k,n+2}H_{k,m+1}\right).
\end{aligned} \tag{12}$$

In addition, complex, dual, coupled and anti−dual conjugations of the dual−complex generalized $k-$Horadam numbers are defined by

$$\left(\mathbb{DC}H_{k,n}\right)^{\star_1} = H_{k,n} - \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} - \mathbf{i}\varepsilon H_{k,n+3} \tag{13}$$
$$\left(\mathbb{DC}H_{k,n}\right)^{\star_2} = H_{k,n} + \mathbf{i}H_{k,n+1} - \varepsilon H_{k,n+2} - \mathbf{i}\varepsilon H_{k,n+3} \tag{14}$$
$$\left(\mathbb{DC}H_{k,n}\right)^{\star_3} = H_{k,n} - \mathbf{i}H_{k,n+1} - \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3} \tag{15}$$
$$\left(\mathbb{DC}H_{k,n}\right)^{\star_4} = H_{k,n+2} + \mathbf{i}H_{k,n+3} - \varepsilon H_{k,n} - \mathbf{i}\varepsilon H_{k,n+1}, \tag{16}$$

respectively. Some properties about the conjugations of the dual−complex generalized $k-$Horadam numbers are given in the following theorem.

Theorem 2. *The four types of conjugation of the dual−complex generalized $k-$Horadam numbers $\left(\mathbb{DC}H_{k,n}\right)^{\star_1}$, $\left(\mathbb{DC}H_{k,n}\right)^{\star_2}$, $\left(\mathbb{DC}H_{k,n}\right)^{\star_3}$ and $\left(\mathbb{DC}H_{k,n}\right)^{\star_4}$ satisfies the following multiplication and summation identities.*

$$\begin{aligned}
\left(\mathbb{DC}H_{k,n}\right)\left(\mathbb{DC}H_{k,n}\right)^{\star_1} &= \left(H_{k,n}^2 + H_{k,n+1}^2\right) + 2\varepsilon\left(H_{k,n}H_{k,n+2} + H_{k,n+1}H_{k,n+3}\right), \\
\left(\mathbb{DC}H_{k,n}\right)\left(\mathbb{DC}H_{k,n}\right)^{\star_2} &= \left(H_{k,n}^2 - H_{k,n+1}^2\right) + 2\mathbf{i}\left(H_{k,n}H_{k,n+1}\right), \\
\left(\mathbb{DC}H_{k,n}\right)\left(\mathbb{DC}H_{k,n}\right)^{\star_3} &= \left(H_{k,n}^2 + H_{k,n+1}^2\right) + 2\mathbf{i}\varepsilon\left(H_{k,n}H_{k,n+3} - H_{k,n+1}H_{k,n+2}\right), \\
\left(\mathbb{DC}H_{k,n}\right) + \left(\mathbb{DC}H_{k,n}\right)^{\star_1} &= 2\left(H_{k,n} + \varepsilon H_{k,n+2}\right), \\
\left(\mathbb{DC}H_{k,n}\right) + \left(\mathbb{DC}H_{k,n}\right)^{\star_2} &= 2\left(H_{k,n} + \mathbf{i}H_{k,n+1}\right), \\
\left(\mathbb{DC}H_{k,n}\right) + \left(\mathbb{DC}H_{k,n}\right)^{\star_3} &= 2\left(H_{k,n} + \mathbf{i}\varepsilon H_{k,n+3}\right), \\
\varepsilon\left(\mathbb{DC}H_{k,n}\right) + \left(\mathbb{DC}H_{k,n}\right)^{\star_4} &= \left(H_{k,n+2} + \mathbf{i}H_{k,n+3}\right), \\
\left(\mathbb{DC}H_{k,n}\right) - \varepsilon\left(\mathbb{DC}H_{k,n}\right)^{\star_4} &= \left(H_{k,n} + \mathbf{i}H_{k,n+1}\right).
\end{aligned} \tag{17}$$

Proof. By considering the Eqs. (10), (13), (14), (15) and (16), the theorem can be proved easily. □

Theorem 3. *Dual−complex generalized $k-$Horadam number, $\mathbb{DC}H_{k,n}$, satisfies the following relation:*

$$\mathbb{DC}H_{k,n+2} = f(k)\mathbb{DC}H_{k,n+1} + g(k)\mathbb{DC}H_{k,n}. \tag{18}$$

Proof. By using Eq. (10) we get,

$f(k)\mathbb{DC}H_{k,n+1}+g(k)\mathbb{DC}H_{k,n}$

$$\begin{aligned}
&= f(k)\left(H_{k,n+1}+\mathbf{i}H_{k,n+2}+\varepsilon H_{k,n+3}+\mathbf{i}\varepsilon H_{k,n+4}\right)\\
&\quad + g(k)\left(H_{k,n}+\mathbf{i}H_{k,n+1}+\varepsilon H_{k,n+2}+\mathbf{i}\varepsilon H_{k,n+3}\right)\\
&= \left(f(k)H_{k,n+1}+g(k)H_{k,n}\right)+\mathbf{i}\left(f(k)H_{k,n+2}+g(k)H_{k,n+1}\right)\\
&\quad + \varepsilon\left(f(k)H_{k,n+3}+g(k)H_{k,n+2}\right)+\mathbf{i}\varepsilon\left(f(k)H_{k,n+4}+g(k)H_{k,n+3}\right)\\
&= H_{k,n+2}+\mathbf{i}H_{k,n+3}+\varepsilon H_{k,n+4}+\mathbf{i}\varepsilon H_{k,n+5}\\
&= \mathbb{DC}H_{k,n+2}.
\end{aligned}$$

□

Binet's formula is a formula which is used to find the $n-$th term of the Fibonacci sequence. Now, we give the Binet formula for the dual$-$complex generalized $k-$Horadam numbers.

Theorem 4. *The Binet formula for the dual$-$complex generalized $k-$Horadam numbers is as follows:*

$$\mathbb{DC}H_{k,n}=\frac{X\overline{\alpha}\alpha^n-Y\overline{\beta}\beta^n}{\alpha-\beta}, \tag{19}$$

where $\overline{\alpha}=1+\mathbf{i}\alpha+\varepsilon\alpha^2+\mathbf{i}\varepsilon\alpha^3$, $\overline{\beta}=1+\mathbf{i}\beta+\varepsilon\beta^2+\mathbf{i}\varepsilon\beta^3$, $X=b-a\beta$, $Y=b-a\alpha$, $\alpha=\frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ *and* $\beta=\frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$.

Proof. By considering the Binet formula of the generalized $k-$Horadam numbers in (9), we have

$$\begin{aligned}
\mathbb{DC}H_{k,n} &= H_{k,n}+\mathbf{i}H_{k,n+1}+\varepsilon H_{k,n+2}+\mathbf{i}\varepsilon H_{k,n+3}\\
&= \frac{X\alpha^n-Y\beta^n}{\alpha-\beta}+\mathbf{i}\frac{X\alpha^{n+1}-Y\beta^{n+1}}{\alpha-\beta}+\varepsilon\frac{X\alpha^{n+2}-Y\beta^{n+2}}{\alpha-\beta}\\
&\quad +\mathbf{i}\varepsilon\frac{X\alpha^{n+3}-Y\beta^{n+3}}{\alpha-\beta}\\
&= \frac{X\alpha^n}{\alpha-\beta}\left(1+\mathbf{i}\alpha+\varepsilon\alpha^2+\mathbf{i}\varepsilon\alpha^3\right)-\frac{Y\beta^n}{\alpha-\beta}\left(1+\mathbf{i}\beta+\varepsilon\beta^2+\mathbf{i}\varepsilon\beta^3\right)\\
&= \frac{1}{\alpha-\beta}\left(X\overline{\alpha}\alpha^n-Y\overline{\beta}\beta^n\right),
\end{aligned}$$

where $\overline{\alpha}=1+\mathbf{i}\alpha+\varepsilon\alpha^2+\mathbf{i}\varepsilon\alpha^3$, $\overline{\beta}=1+\mathbf{i}\beta+\varepsilon\beta^2+\mathbf{i}\varepsilon\beta^3$, $X=b-a\beta$, $Y=b-a\alpha$, $\alpha=\frac{f(k)+\sqrt{f(k)^2+4g(k)}}{2}$ and $\beta=\frac{f(k)-\sqrt{f(k)^2+4g(k)}}{2}$. Therefore the proof is completed. □

TABLE 2. The dual−complex generalized k−Horadam numbers

$f(k)$	$g(k)$	a	b	Dual−complex generalized k−Horadam numbers $\mathbb{DC}H_{k,n} = H_{k,n} + \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3}$ $H_{k,n} = f(k)H_{k,n-1} + g(k)H_{k,n-2}$, $H_{k,0} = a$ and $H_{k,1} = b$
1	1	0	1	Dual−complex Fibonacci numbers [10] $\mathbb{DC}F_n = F_n + \mathbf{i}F_{n+1} + \varepsilon F_{n+2} + \mathbf{i}\varepsilon F_{n+3}$ $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$
1	1	2	1	Dual−complex Lucas numbers [10] $\mathbb{DC}L_n = L_n + \mathbf{i}L_{n+1} + \varepsilon L_{n+2} + \mathbf{i}\varepsilon L_{n+3}$ $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$ and $L_1 = 1$
2	1	0	1	Dual−complex Pell numbers $\mathbb{DC}P_n = P_n + \mathbf{i}P_{n+1} + \varepsilon P_{n+2} + \mathbf{i}\varepsilon P_{n+3}$ $P_n = 2P_{n-1} + P_{n-2}$, $P_0 = 0$ and $P_1 = 1$
2	1	2	2	Dual−complex Pell-Lucas numbers $\mathbb{DC}Q_n = Q_n + \mathbf{i}Q_{n+1} + \varepsilon Q_{n+2} + \mathbf{i}\varepsilon Q_{n+3}$ $Q_n = 2Q_{n-1} + Q_{n-2}$, $Q_0 = 2$ and $Q_1 = 2$
1	2	0	1	Dual−complex Jacobsthal numbers $\mathbb{DC}J_n = J_n + \mathbf{i}J_{n+1} + \varepsilon J_{n+2} + \mathbf{i}\varepsilon J_{n+3}$ $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$ and $J_1 = 1$
1	2	2	1	Dual−complex Jacobsthal−Lucas numbers $\mathbb{DC}j_n = j_n + \mathbf{i}j_{n+1} + \varepsilon j_{n+2} + \mathbf{i}\varepsilon j_{n+3}$ $j_n = j_{n-1} + 2j_{n-2}$, $j_0 = 2$ and $j_1 = 1$
k	1	0	1	Dual−complex k−Fibonacci numbers [1] $\mathbb{DC}F_{k,n} = F_{k,n} + \mathbf{i}F_{k,n+1} + \varepsilon F_{k,n+2} + \mathbf{i}\varepsilon F_{k,n+3}$ $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, $F_{k,0} = 0$ and $F_{k,1} = 1$
k	1	2	k	Dual−complex k−Lucas numbers $\mathbb{DC}L_{k,n} = L_{k,n} + \mathbf{i}L_{k,n+1} + \varepsilon L_{k,n+2} + \mathbf{i}\varepsilon L_{k,n+3}$ $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$, $L_{k,0} = 2$ and $L_{k,1} = k$
p	q	a	b	Dual−complex Horadam numbers $\mathbb{DC}H_n = H_n + \mathbf{i}H_{n+1} + \varepsilon H_{n+2} + \mathbf{i}\varepsilon H_{n+3}$ $H_n = pH_{n-1} + qH_{n-2}$, $H_0 = a$ and $H_1 = b$

Theorem 5. *Let $\mathbb{DC}H_{k,n}$ be the dual−complex generalized k−Horadam number. Then the summation formula for $\mathbb{DC}H_{k,n}$ is as follows:*

$$\sum_{s=1}^{n} \mathbb{DC}H_{k,s} = \begin{cases} \frac{\mathbb{DC}H_{k,n+1} + g(k)\mathbb{DC}H_{k,n} - \mathbb{DC}H_{k,1} - g(k)\mathbb{DC}H_{k,0}}{f(k)+g(k)-1}, & \text{if } f(k)+g(k) \neq 1 \\ \frac{g(k)\mathbb{DC}H_{k,n} + \mathbb{DC}H_{k,1} + (n-1)[g(k)a+b](1+\mathbf{i}+\varepsilon+\mathbf{i}\varepsilon)}{1+g(k)}, & \text{if } f(k)+g(k) = 1 \end{cases} \tag{20}$$

Proof. We prove the theorem in two different cases. First we assume that $f(k)+g(k)\neq 1$. By using the Eq. (10), we have

$$\sum_{s=1}^{n}\mathbb{DC}H_{k,s}=\sum_{s=1}^{n}H_{k,s}+\mathbf{i}\sum_{s=1}^{n}H_{k,s+1}+\varepsilon\sum_{s=1}^{n}H_{k,s+2}+\mathbf{i}\varepsilon\sum_{s=1}^{n}H_{k,s+3}. \tag{21}$$

Summation of the generalized $k-$Horadam numbers are defined in Eq. (12) in [16] as:

$$\sum_{s=1}^{n}H_{k,s}=\frac{H_{k,n+1}+g(k)H_{k,n}-H_{k,1}-g(k)H_{k,0}}{f(k)+g(k)-1}. \tag{22}$$

Furthermore, the following equations are defined in Eq. (2.22), Eq. (2.23) and Eq. (2.24) in [23].

$$\sum_{s=1}^{n}H_{k,s+1}=\frac{H_{k,n+2}+g(k)H_{k,n+1}-H_{k,2}-g(k)H_{k,1}}{f(k)+g(k)-1}. \tag{23}$$

$$\sum_{s=1}^{n}H_{k,s+2}=\frac{H_{k,n+3}+g(k)H_{k,n+2}-H_{k,3}-g(k)H_{k,2}}{f(k)+g(k)-1} \tag{24}$$

$$\sum_{s=1}^{n}H_{k,s+3}=\frac{H_{k,n+4}+g(k)H_{k,n+3}-H_{k,4}-g(k)H_{k,3}}{f(k)+g(k)-1}. \tag{25}$$

By substituting the Equations (22), (23), (24) and (25) in the Eq. (21), we can obtain the equation as follows:

$$\sum_{s=1}^{n}\mathbb{DC}H_{k,s}=\frac{\mathbb{DC}H_{k,n+1}+g(k)\mathbb{DC}H_{k,n}-\mathbb{DC}H_{k,1}-g(k)\mathbb{DC}H_{k,0}}{f(k)+g(k)-1}. \tag{26}$$

Next, we assume that $f(k)+g(k)=1$. By using the Eq. (13) in [16], we have

$$\sum_{s=1}^{n}H_{k,s}=\frac{g(k)H_{k,n}+H_{k,1}+(n-1)[g(k)a+b]}{1+g(k)}. \tag{27}$$

Moreover, the following equations are defined in Eq. (2.27), Eq. (2.28) and Eq. (2.29) in [23].

$$\sum_{s=1}^{n}H_{k,s+1}=\frac{g(k)H_{k,n+1}+H_{k,2}+(n-1)[g(k)a+b]}{1+g(k)} \tag{28}$$

$$\sum_{s=1}^{n}H_{k,s+2}=\frac{g(k)H_{k,n+2}+H_{k,3}+(n-1)[g(k)a+b]}{1+g(k)} \tag{29}$$

$$\sum_{s=1}^{n}H_{k,s+3}=\frac{g(k)H_{k,n+3}+H_{k,4}+(n-1)[g(k)a+b]}{1+g(k)}. \tag{30}$$

Hence, by substituting the Equations (27), (28), (29) and (30) in Eq. (21), we can obtain the equation as follows:

$$\sum_{s=1}^{n} \mathbb{DC}H_{k,s} = \frac{g(k)\mathbb{DC}H_{k,n} + \mathbb{DC}H_{k,1} + (n-1)[g(k)a+b](1+\mathbf{i}+\varepsilon+\mathbf{i}\varepsilon)}{1+g(k)}. \quad (31)$$

□

In the following theorem, the general form of the some identities are given such as Catalan's identity, Cassini's identity and d'Ocagne's identity, for the dual−complex generalized $k-$Horadam numbers for the special cases of s and r.

Theorem 6. *The dual−complex generalized $k-$Horadam numbers satisfies the following identity:*

$$\begin{aligned}
&\mathbb{DC}H_{k,n}\mathbb{DC}H_{k,n-r+s} - \mathbb{DC}H_{k,n+s}\mathbb{DC}H_{k,n-r} \\
&= \frac{(-g(k))^{n-r}(bH_{k,r} - aH_{k,r+1})(bH_{k,s} - aH_{k,s+1})}{b^2 - a^2g(k) - abf(k)} \\
&\quad \times \Bigg[1 + g(k) + \mathbf{i}f(k) + \varepsilon(1+g(k))\left(f(k)^2 + 2g(k)\right) \\
&\quad +\mathbf{i}\varepsilon f(k)\left(f(k)^2 + 2g(k)\right)\Bigg]. \quad (32)
\end{aligned}$$

Proof. By using Theorem 7 in [24] and definition of the generalized $k-$Horadam numbers, we get

$$\begin{aligned}
&\mathbb{DC}H_{k,n}\mathbb{DC}H_{k,n-r+s} - \mathbb{DC}H_{k,n+s}\mathbb{DC}H_{k,n-r} \\
&= (H_{k,n} + \mathbf{i}H_{k,n+1} + \varepsilon H_{k,n+2} + \mathbf{i}\varepsilon H_{k,n+3}) \\
&\quad \times (H_{k,n-r+s} + \mathbf{i}H_{k,n-r+s+1} + \varepsilon H_{k,n-r+s+2} + \mathbf{i}\varepsilon H_{k,n-r+s+3}) \\
&\quad - (H_{k,n+s} + \mathbf{i}H_{k,n+s+1} + \varepsilon H_{k,n+s+2} + \mathbf{i}\varepsilon H_{k,n+s+3}) \\
&\quad \times (H_{k,n-r} + \mathbf{i}H_{k,n-r+1} + \varepsilon H_{k,n-r+2} + \mathbf{i}\varepsilon H_{k,n-r+3}) \\
&= H_{k,n}H_{k,n-r+s} - H_{k,n+1}H_{k,n-r+s+1} - H_{k,n+s}H_{k,n-r} + H_{k,n+s+1}H_{k,n-r+1} \\
&\quad + \mathbf{i}\Big(H_{k,n+1}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+1} - H_{k,n+s+1}H_{k,n-r} - H_{k,n+s}H_{k,n-r+1}\Big) \\
&\quad + \varepsilon\Big(H_{k,n+2}H_{k,n-r+s} + H_{k,n}H_{k,n-r+s+2} - H_{k,n+1}H_{k,n-r+s+3} \\
&\quad - H_{k,n+3}H_{k,n-r+s+1} - H_{k,n+s}H_{k,n-r+2} + H_{k,n+s+1}H_{k,n-r+3} \\
&\quad - H_{k,n-r}H_{k,n+s+2} + H_{k,n-r+1}H_{k,n+s+3}\Big) \\
&+ \mathbf{i}\varepsilon\Big(H_{k,n}H_{k,n-r+s+3} + H_{k,n+3}H_{k,n-r+s} + H_{k,n+1}H_{k,n-r+s+2} \\
&\quad + H_{k,n+2}H_{k,n-r+s+1} - H_{k,n+s}H_{k,n-r+3} - H_{k,n+s+3}H_{k,n-r}
\end{aligned}$$

$$
\begin{aligned}
&\quad - H_{k,n+s+1}H_{k,n-r+2} - H_{k,n+s+2}H_{k,n-r+1}\Big)\\
&= \frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}\Bigg[1+g(k)\Bigg]\\
&+\mathbf{i}\frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}\Bigg[f(k)\Bigg]\\
&+\varepsilon\frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}\\
&\quad\times\Bigg[(1+g(k))\left(f(k)^2+2g(k)\right)\Bigg]\\
&+\mathbf{i}\varepsilon\frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}\Bigg[f(k)\left(f(k)^2+2g(k)\right)\Bigg]\\
&= \frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,s}-aH_{k,s+1})}{b^2-a^2g(k)-abf(k)}\\
&\quad\times\Bigg[1+g(k)+\mathbf{i}f(k)+\varepsilon(1+g(k))\left(f(k)^2+2g(k)\right)+\mathbf{i}\varepsilon f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned}
$$

□

Corollary 7. *By taking $s=m-n+r$ in the Theorem 6, we obtain the following identity:*

$$
\begin{aligned}
&\mathbb{DC}H_{k,n}\mathbb{DC}H_{k,m} - \mathbb{DC}H_{k,m+r}\mathbb{DC}H_{k,n-r}\\
&\quad= \frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})(bH_{k,m-n+r}-aH_{k,m-n+r+1})}{b^2-a^2g(k)-abf(k)}\\
&\qquad\times\Bigg[1+g(k)+\mathbf{i}f(k)+\varepsilon(1+g(k))\left(f(k)^2+2g(k)\right)\\
&\qquad\quad+\mathbf{i}\varepsilon f(k)\left(f(k)^2+2g(k)\right)\Bigg].
\end{aligned}
$$

Corollary 8. *(Catalan's identity) By taking $m=n$ in the Corollary 7, we obtain the Catalan's identity for the dual−complex generalized k−Horadam numbers as:*

$$
\begin{aligned}
&(\mathbb{DC}H_{k,n})^2 - \mathbb{DC}H_{k,n+r}\mathbb{DC}H_{k,n-r}\\
&\quad= \frac{(-g(k))^{n-r}(bH_{k,r}-aH_{k,r+1})^2}{b^2-a^2g(k)-abf(k)}
\end{aligned}
$$

$$\times \left[1 + g(k) + \mathbf{i}f(k) + \varepsilon\left(1 + g(k)\right)\left(f(k)^2 + 2g(k)\right)\right.$$
$$\left. + \mathbf{i}\varepsilon f(k)\left(f(k)^2 + 2g(k)\right)\right].$$

Corollary 9. *(Cassini's identity) By taking $m = n$ and $r = 1$ in the Corollary 7, we obtain the Cassini's identity for the dual−complex generalized $k-$Horadam numbers as:*

$$\begin{aligned}&\left(\mathbb{D}\mathbb{C}H_{k,n}\right)^2 - \mathbb{D}\mathbb{C}H_{k,n+1}\mathbb{D}\mathbb{C}H_{k,n-1}\\ &\quad= \left(-g(k)\right)^{n-1}\left(b^2 - a^2 g(k) - abf(k)\right)\\ &\qquad\times\left[1 + g(k) + \mathbf{i}f(k) + \varepsilon\left(1 + g(k)\right)\left(f(k)^2 + 2g(k)\right)\right.\\ &\qquad\quad\left. + \mathbf{i}\varepsilon f(k)\left(f(k)^2 + 2g(k)\right)\right].\end{aligned}$$

Corollary 10. *(d'Ocagne's identity) By taking $n = n+1$ and $r = 1$ in the Corollary 7, we obtain the d'Ocagne's identity for the dual−complex generalized $k-$Horadam numbers as:*

$$\begin{aligned}&\mathbb{D}\mathbb{C}H_{k,n+1}\mathbb{D}\mathbb{C}H_{k,m} - \mathbb{D}\mathbb{C}H_{k,m+1}\mathbb{D}\mathbb{C}H_{k,n}\\ &\quad= \left(-g(k)\right)^{n}\left(bH_{k,m-n} - aH_{k,m-n+1}\right)\\ &\qquad\times\left[1 + g(k) + \mathbf{i}f(k) + \varepsilon\left(1 + g(k)\right)\left(f(k)^2 + 2g(k)\right)\right.\\ &\qquad\quad\left. + \mathbf{i}\varepsilon f(k)\left(f(k)^2 + 2g(k)\right)\right].\end{aligned}$$

Generating functions are useful tools with many applications to number sequences. The following theorem explains the generating function of the dual−complex generalized $k-$Horadam numbers.

Theorem 11. *For $n \in \mathbb{N}_0$, the generating function for the dual−complex generalized $k-$Horadam numbers is*

$$H(t) = \frac{\mathbb{D}\mathbb{C}H_{k,0} + \left(\mathbb{D}\mathbb{C}H_{k,1} - f(k)\mathbb{D}\mathbb{C}H_{k,0}\right)t}{1 - f(k)t - g(k)t^2}. \tag{33}$$

Proof. We use the formal power series to obtain the generating function of $\mathbb{D}\mathbb{C}H_{k,n}$. Now, we define

$$H(t) = \sum_{n=0}^{\infty}\mathbb{D}\mathbb{C}H_{k,n}t^n = \mathbb{D}\mathbb{C}H_{k,0} + \mathbb{D}\mathbb{C}H_{k,1}t + \sum_{n=2}^{\infty}\mathbb{D}\mathbb{C}H_{k,n}t^n. \tag{34}$$

Multiplying the Eq. (34) both $f(k)t$ and $g(k)t^2$, we get

$$f(k)tH(t) = \sum_{n=0}^{\infty} f(k)\mathbb{DC}H_{k,n}t^{n+1} = f(k)\mathbb{DC}H_{k,0}t + \sum_{n=2}^{\infty} f(k)\mathbb{DC}H_{k,n-1}t^n \quad (35)$$

and

$$g(k)t^2H(t) = \sum_{n=2}^{\infty} g(k)\mathbb{DC}H_{k,n-2}t^n. \quad (36)$$

By considering the above equations and doing some basic operations, we get

$$\begin{aligned}&\left(1 - f(k)t - g(k)t^2\right) H(t)\\ &= \mathbb{DC}H_{k,0} + \mathbb{DC}H_{k,1}t - f(k)\mathbb{DC}H_{k,0}t\\ &+ \sum_{n=2}^{\infty}\left(\underbrace{\mathbb{DC}H_{k,n} - f(k)\mathbb{DC}H_{k,n-1} - g(k)\mathbb{DC}H_{k,n-2}}_{0}\right)t^n.\end{aligned}$$

Therefore, we obtain

$$H(t) = \frac{\mathbb{DC}H_{k,0} + (\mathbb{DC}H_{k,1} - f(k)\mathbb{DC}H_{k,0})\,t}{1 - f(k)t - g(k)t^2}, \quad (37)$$

which is the desired result. □

The matrix representation of the dual−complex generalized $k-$Horadam numbers can be given in the following theorem.

Theorem 12. *Let $n \geq 1$ be integer. Then*

$$\begin{pmatrix} \mathbb{DC}H_{k,2n+l} & \mathbb{DC}H_{k,2(n-1)+l} \\ \mathbb{DC}H_{k,2(n+1)+l} & \mathbb{DC}H_{k,2n+l} \end{pmatrix} = \begin{pmatrix} \mathbb{DC}H_{k,2+l} & \mathbb{DC}H_{k,l} \\ \mathbb{DC}H_{k,4+l} & \mathbb{DC}H_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{n-1}, \quad (38)$$

where $l \in \{0, 1\}$.

Proof. We prove the theorem by induction on n. If $n = 1$ then the result is clear. Now we assume that, for any integer m such as $1 \leq m \leq n$,

$$\begin{pmatrix} \mathbb{DC}H_{k,2m+l} & \mathbb{DC}H_{k,2(m-1)+l} \\ \mathbb{DC}H_{k,2(m+1)+l} & \mathbb{DC}H_{k,2m+l} \end{pmatrix} = \begin{pmatrix} \mathbb{DC}H_{k,2+l} & \mathbb{DC}H_{k,l} \\ \mathbb{DC}H_{k,4+l} & \mathbb{DC}H_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1}.$$

Then, for $n = m + 1$, we get

$$\begin{aligned}&\begin{pmatrix} \mathbb{DC}H_{k,2+l} & \mathbb{DC}H_{k,l} \\ \mathbb{DC}H_{k,4+l} & \mathbb{DC}H_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m}\\ &= \begin{pmatrix} \mathbb{DC}H_{k,2+l} & \mathbb{DC}H_{k,l} \\ \mathbb{DC}H_{k,4+l} & \mathbb{DC}H_{k,2+l} \end{pmatrix} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}^{m-1} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \mathbb{DC}H_{k,2m+l} & \mathbb{DC}H_{k,2(m-1)+l} \\ \mathbb{DC}H_{k,2(m+1)+l} & \mathbb{DC}H_{k,2m+l} \end{pmatrix} \begin{pmatrix} f(k)^2 + 2g(k) & 1 \\ -g(k)^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{DC}H_{k,2(m+1)+l} & \mathbb{DC}H_{k,2m+l} \\ \mathbb{DC}H_{k,2(m+2)+l} & \mathbb{DC}H_{k,2(m+1)+l} \end{pmatrix},$$

where $l \in \{0,1\}$. Therefore, the proof is completed. □

3. Conclusion

This study aims to generalize several dual−complex numbers in the literature such as [1, 10]. Also, we obtain numerous new dual−complex numbers for the special cases of $f(k), g(k), a$ and b (see Table 2). Further, we present the Binet formula, generating function, matrix representation and the summation formula for the dual−complex generalized k−Horadam numbers. We also give Theorem (6) which is the generalization of the Catalan's identity, Cassini's identity and d'Ocagne's identity. Hence this study involves some new generalized results for dual−complex numbers and it contributes to the literature in dual−complex numbers.

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