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Research Article

Grüss and Grüss-Voronovskaya-type estimates for complex convolution polynomial operators

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ABSTRACT. The aim of this paper is to obtain Grüss and Grüss-Voronovskaya inequalities with exact quantitative estimates (with respect to the degree) for the complex convolution polynomial operators of de la Vallée Poussin, of Zygmund-Riesz and of Jackson, acting on analytic functions.

Keywords: Complex convolution polynomials, de la Vallée-Poussin kernel, Riesz-Zygmund kernel, Jackson kernel, Grüss-type estimate, Grüss-Voronovskaya-type estimate, analytic functions.

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Dedicated to Professor Francesco Altomare, on occasion of his 70th birthday, with esteem and friendship.

1. INTRODUCTION

A classical well-known result in approximation theory is the Grüss inequality for positive linear functionals $L : C[0,1] \to \mathbb{R}$, which gives an upper bound for the Chebyshev-type functional

$$T(f,g) := L(f \cdot g) - L(f) \cdot L(g), \quad f, g \in C[0,1].$$

Starting also from a problem posed in [3], this inequality was investigated in terms of the least concave majorants of the moduli of continuity and for positive linear operators $H : C[0,1] \to C[0,1]$, for the first time in [1] and in the note [5], where the cases of classical Hermite-Fejér and Fejér-Korovkin convolution operators were considered.

Refined versions of the Grüss-type inequality in the spirit of Voronovskaya's theorem were obtained in [4] for Bernstein and Păltaănea operators of real variable and for complex Bernstein, genuine Bernstein-Durrmeyer and Bernstein-Faber operators attached to analytic functions of complex variable.

After the appearance of these results, several papers by other authors have developed these directions of research.

For example, let $C_{2\pi} = \{f : \mathbb{R} \to \mathbb{R}; f$ continuous and 2π periodic on $\mathbb{R}\}$. A classical method to construct trigonometric approximating polynomials for $f \in C_{2\pi}$ is that of convolution of f with various trigonometric even polynomials $K_n(t)$ (called kernels), under the form

$$L_n(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)K_n(x-u)du, \quad x \in \mathbb{R}, n \in \mathbb{N}. \tag{1.1}$$

Upper estimate in the Grüss-type inequality for convolution trigonometric polynomials with respect to general form of the kernel $K_n(t)$, was obtained in [1].

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Now, by analogy, for f analytic in a disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and continuous in the closure of the disk, one can attach the convolution complex (algebraic) polynomials by

$$\mathcal{L}_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it}) \cdot K_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{-it}) \cdot K_n(t)dt, z \in \mathbb{D}_R, \quad n \in \mathbb{N}. \tag{1.2}$$

The goal of this paper is to continue the above mentioned directions of research, obtaining Grüss and Grüss-Voronovskaya exact estimates (with respect to the degree) for the de la Vallée-Poussin complex polynomials in Section 2, for Zygmund-Riesz complex polynomials in Section 3 and for Jackson complex polynomials in Section 4.

2. De la Vallée-Poussin complex convolution

In this section, we extend the Grüss and the Grüss-Voronovskaya estimates for the de la Vallée-Poussin complex polynomials given by the general formula (1.2) and based on the convolution with the de la Vallée-Poussin kernel

$$K_n(t) = \frac{1}{2} \cdot \frac{(n!)^2}{(2n)!} \cdot (2\cos(t/2))^{2n},$$

defined by

$$\mathcal{V}_n(f)(z) = \frac{1}{\binom{2n}{n}} \sum_{j=0}^{n} c_j \binom{2n}{n+j} z^j = \sum_{j=0}^{n} c_j \frac{(n!)^2}{(n-j)!(n+j)!} z^j, \tag{2.3}$$

attached to analytic functions in compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

Let us denote $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and $\|f\|_r = \max\{|f(z)|; |z| \le r\}$.

Firstly, we prove a theorem for the general complex convolutions given by (1.2).

Theorem 2.1. *Suppose that $R > 1$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

Let $1 \le r < R$ and consider the operators $\mathcal{L}_n$ given by (1.2). For all $n \in \mathbb{N}$, it follows

$$\|\mathcal{L}_n(fg) - \mathcal{L}_n(f)\mathcal{L}_n(g)\|_r \le \sum_{m=0}^{\infty} \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}| \cdot \|A_{n,m,j}\|_r\right],$$

where denoting $A_{n,m,j}(z) = \mathcal{L}_n(e_m)(z) - \mathcal{L}_n(e_j)(z) \cdot \mathcal{L}_n(e_{m-j})(z)$ and $e_m(z) = z^m$, $m \in \mathbb{N} \cup \{0\}$, we have

$$\|A_{n,m,j}\|_r \le \|\mathcal{L}_n(e_m) - e_m\|_r + \|e_j\|_r \cdot \|e_{m-j} - \mathcal{L}_n(e_{m-j})\|_r + \|\mathcal{L}_n(e_{m-j})\|_r \cdot \|e_j - \mathcal{L}_n(e_j)\|_r.$$

Proof. Since $f(z)g(z) = \sum_{m=0}^{\infty} c_m z^m$, where $c_m = \sum_{j=0}^{m} a_j b_{m-j}$, it follows

$$\mathcal{L}_n(fg)(z) = \sum_{m=0}^{\infty} \left[\sum_{j=0}^{m} a_j b_{m-j}\right] \mathcal{L}_n(e_m)(z).$$

Also,

$$\mathcal{L}_n(f)(z) = \sum_{k=0}^{\infty} a_k \mathcal{L}_n(e_k)(z), \;\; \mathcal{L}_n(g)(z) = \sum_{k=0}^{\infty} b_k \mathcal{L}_n(e_k)(z)$$

and

$$\mathcal{L}_n(f)(z)\mathcal{L}_n(g)(z) = \sum_{m=0}^{\infty} \left[\sum_{j=0}^{m} a_j b_{m-j} \mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)\right],$$

which immediately implies

$$|\mathcal{L}_n(fg)(z)-\mathcal{L}_n(f)(z)\mathcal{L}_n(g)(z)| = \left|\sum_{m=0}^{\infty}\left[\sum_{j=0}^{m} a_j b_{m-j}\left(\mathcal{L}_n(e_m)(z)-\mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)\right)\right]\right|$$
$$\leq \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\cdot|\mathcal{L}_n(e_m)(z)-\mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|\right].$$

Then, we get

$$\begin{aligned}|A_{n,m,j}(z)| &= |\mathcal{L}_n(e_m)(z)-\mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|\\ &\leq |\mathcal{L}_n(e_m)(z)-e_m(z)|+|e_j(z)\cdot e_{m-j}(z)-\mathcal{L}_n(e_j)(z)\mathcal{L}_n(e_{m-j})(z)|\\ &\leq |\mathcal{L}_n(e_m)(z)-e_m(z)|+|e_j(z)|\cdot|e_{m-j}(z)-\mathcal{L}_n(e_{m-j})(z)|\\ &+|\mathcal{L}_n(e_{m-j})(z)|\cdot|e_j(z)-\mathcal{L}_n(e_j)(z)|,\end{aligned}$$

which immediately proves the lemma. □

The following Grüss-type estimate holds.

Corollary 2.1. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

For all $n \in \mathbb{N}$, we have

$$\|\mathcal{V}_n(fg)-\mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \leq \frac{3}{n}\cdot\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m,$$

where $\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m < \infty$.

Proof. We estimate $\|A_{n,m,j}\|_r$ in the statement of Theorem 2.1 for $\mathcal{L}_n = \mathcal{V}_n$. For that purpose, by [2, p. 182], we easily get $\|\mathcal{V}_n(e_k)\|_r \leq r^k$, for all $n \in \mathbb{N}$, $k \in \mathbb{N}\bigcup\{0\}$, while from [2, p. 183], we have $\|\mathcal{V}_n(e_k)-e_k\|_r \leq \frac{k^2}{n}r^k$, for all k, n. This implies, for all $n, m, j \in \mathbb{N}$ and $j \leq m$

$$\begin{aligned}\|A_{n,m,j}\|_r &\leq \frac{m^2}{n}r^m + r^j\cdot\frac{(m-j)^2}{n}\cdot r^{m-j} + r^{m-j}\cdot\frac{j^2}{n}\cdot r^j\\ &\leq \frac{3}{n}\cdot m^2 r^m,\end{aligned}$$

which by Theorem 2.1, immediately implies the estimate in the statement of the corollary.

It remains to show that $\sum_{m=1}^{\infty} m^2\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m < \infty$. Indeed, since f and g are analytic it follows that the series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ converge uniformly for $|z| \leq r$ and all $1 \leq r < R$, that is the series $\sum_{k=0}^{\infty}|a_k|r^k$ and $\sum_{k=0}^{\infty}|b_k|r^k$ converge for all $1 \leq r < R$. Then, by Mertens' theorem (see e.g. [6, Theorem 3.50, p. 74] their (Cauchy) product is a convergent series and therefore

$$\sum_{m=0}^{\infty}\left[\sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|\right]r^m$$

is a convergent series for all $1 \leq r < R$. Denoting $A_m = \sum_{j=0}^{m}|a_j|\cdot|b_{m-j}|$, this means that the power series $F(z) = \sum_{m=0}^{\infty} A_m z^m$ is uniformly convergent for $|z| \leq r$, for all $1 \leq r < R$, which implies that $F''(z) = \sum_{m=2}^{\infty} m(m-1)A_m z^{m-2}$ also is uniformly convergent for $|z| \leq r$, with $1 \le r < R$ arbitrary, fixed. Indeed, choose an r' with $1 \le r < r' < R$ and consider the uniformly convergent series $F(z) = \sum_{m=0}^{\infty} A_m z^m$ on $|z| \le r'$.

Therefore $\sum_{m=2}^{\infty} m(m-1)A_m r^{m-2} < \infty$, which immediately implies that

$$\sum_{m=1}^{\infty} m^2 \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < \infty.$$

□

In what follows, it is natural to ask for the limit

$$\lim_{n\to\infty} n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)].$$

By simple calculation, we have

$$\begin{aligned}
& n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)] \\
=& n\Bigg\{\mathcal{V}_n(fg)(z) - f(z)g(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))' \\
&- g(z)\left[\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n}f''(z) + \frac{z}{n}f'(z)\right] \\
&- \mathcal{V}_n(f)(z)\left[\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right] \\
&+ \left(\frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right)[\mathcal{V}_n(f)(z) - f(z)] - \frac{2z^2}{n}f'(z)g'(z)\Bigg\}.
\end{aligned}$$

Indeed, the above equality easily follows by simple algebraic manipulations, replacing in the right-hand side of the equality $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$, $[f(z)g(z)]'' = f''(z)g(z) + 2f'(z)g'(z) + f(z)g''(z)$ and reducing the corresponding terms.

Taking into account the estimate in [2, Theorem 3.1.2, p. 183] applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n[\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z)] = -2z^2 f'(z)g'(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 2.2. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then, for all $|z| \le r$, there exists a constant $C(r, f, g) > 0$ depending on r, f, g, such that*

$$\left|\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z)\right| \le \frac{C(r,f,g)}{n^2},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
& \mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n}f'(z)g'(z) \\
=& \left[\mathcal{V}_n(fg)(z) - (fg)(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))'\right] \\
&-f(z)\left[\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n}g''(z) + \frac{z}{n}g'(z)\right] \\
&-g(z)\left[\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n}f''(z) + \frac{z}{n}f'(z)\right] \\
&+[g(z) - \mathcal{V}_n(g)(z)] \cdot [\mathcal{V}_n(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in [2, Theorem 3.1.1, (i), p. 182] and [2, Theorem 3.1.2, p. 183], we get

$$\begin{aligned}
&\left|\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n} f'(z)g'(z)\right| \\
\le & \left|\mathcal{V}_n(fg)(z) - (fg)(z) + \frac{z^2}{n}(f(z)g(z))'' + \frac{z}{n}(f(z)g(z))'\right| \\
&+|f(z)| \left|\mathcal{V}_n(g)(z) - g(z) + \frac{z^2}{n} g''(z) + \frac{z}{n} g'(z)\right| \\
&+|g(z)| \left|\mathcal{V}_n(f)(z) - f(z) + \frac{z^2}{n} f''(z) + \frac{z}{n} f'(z)\right| \\
&+|g(z) - \mathcal{V}_n(g)(z)| \cdot |\mathcal{V}_n(f)(z) - f(z)|. \\
\le & \frac{C_1(r,f,g)}{n^2} + \|f\|_r \cdot \frac{C_2(r,g)}{n^2} + \|g\|_r \cdot \frac{C_3(r,f)}{n^2} + \frac{C_4(r,g)}{n} \cdot \frac{C_5(r,f)}{n} \\
\le & \frac{C(r,f,g)}{n^2},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r,f,g) > 0$ independent of n and depending on r, f, g. □

In what follows, the above theorem is used to obtain a lower estimate in the Grüss-type inequality.

Corollary 2.2. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then there exists an $n_0 \in \mathbb{N}$, depending only on r, f and g, such that*

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \ge \frac{1}{n} \cdot \|e_2 f'g'\|_r, \; n \ge n_0.$$

Proof. We can write

$$\begin{aligned}
&\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) \\
=&\frac{1}{n}\left\{-2z^2 f'(z)g'(z) + \frac{1}{n}\left[n^2\left(\mathcal{V}_n(fg)(z) - \mathcal{V}_n(f)(z)\mathcal{V}_n(g)(z) + \frac{2z^2}{n} f'(z)g'(z)\right)\right]\right\}.
\end{aligned}$$

Applying to the above identity, the obvious inequality

$$\|F + G\|_r \ge |\|F\|_r - \|G\|_r| \ge \|F\|_r - \|G\|_r,$$

and denoting $e_2(z) = z^2$, we obtain

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \ge \frac{1}{n}\left\{\|2e_2 f'g'\|_r - \frac{1}{n}\left[n^2\left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n} f'g'\right\|_r\right]\right\}.$$

Since f and g are not constant functions, we easily get $\|2e_2 f'g'\|_r > 0$.

Taking into account that by Theorem 2.2, we get

$$n^2\left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n} f'g'\right\|_r \le C(r,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\|2e_2 f'g'\|_r - \frac{1}{n}\left[n^2\left\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g) + \frac{2e_2}{n}f'g'\right\|_r\right] \geq \frac{\|2e_2 f'g'\|_r}{2} = \|e_2 f'g'\|_r > 0,$$

which for all $n \geq n_0$ implies

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \geq \frac{1}{n} \cdot \|e_2 f'g'\|_r.$$

□

As an immediate consequence of Corollary 2.1 and Corollary 2.2, we obtain the following exact estimate.

Corollary 2.3. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. If f and g are not constant functions, then there exists $n_0 \in \mathbb{N}$ depending only on r, f and g, such that we have*

$$\|\mathcal{V}_n(fg) - \mathcal{V}_n(f)\mathcal{V}_n(g)\|_r \sim \frac{1}{n}, \; n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n but depend on r, f, g.

3. Zygmund-Riesz complex convolution

This section deals with the Grüss and the Grüss-Voronovskaya estimates for the Zygmund-Riesz complex polynomials based on the convolution with the Zygmund-Riesz kernel

$$K_{n,s}(t) = \frac{1}{2} + \sum_{j=1}^{n-1}\left(1 - \frac{j^s}{n^s}\right)\cos(jt), \; s \in \mathbb{N} \text{ fixed},$$

defined by

$$\mathcal{R}_{n,s}(f)(z) = \sum_{j=0}^{n-1} c_j \left[1 - \left(\frac{j}{n}\right)^s\right] z^j, \tag{3.4}$$

attached to analytic functions in compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$.

Firstly, as a consequence of Theorem 2.1, the following Grüss-type estimate holds for Zygmund-Riesz complex polynomial convolution.

Corollary 3.4. *Suppose that $1 \leq r < R$, $s \in \mathbb{N}$ are fixed arbitrary and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

For all $n \in \mathbb{N}$, we have

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \leq \frac{3}{n^s}\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m,$$

where $\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < +\infty$.

Proof. Denote $e_m(z) = z^m$. We will estimate $\|A_{n,m,j}\|_r$ in the case when in Theorem 2.1, we take $\mathcal{L}_n = \mathcal{R}_{n,s}$.

From the formula (3.4), we immediately get that $\mathcal{R}_{n,s}(e_k)(z) = 0$ if $k \geq n$ and that $\mathcal{R}_{n,s}(e_k)(z) = \left[1 - \frac{k^s}{n^s}\right] e_k(z)$ if $k \leq n-1$. This immediately implies $\|\mathcal{R}_{n,s}(e_k)\|_r \leq r^k$ for all n, k. Also,

$\|\mathcal{R}_{n,s}(e_k) - e_k\|_r = r^k \le \frac{k^s}{n^s} \cdot r^k$ if $k \ge n$ and $\|\mathcal{R}_{n,s}(e_k) - e_k\|_r \le \frac{k^s}{n^s} r^k$ if $k \le n-1$, which easily implies

$$\|A_{m,n,j}\|_r \le \frac{m^s}{n^s} r^m + r^j \cdot \frac{(m-j)^s}{n^s} \cdot r^{m-j} + r^{m-j} \cdot \frac{j^s}{n^s} \cdot r^j \le \frac{3}{n^s} m^s r^m.$$

It remains to show that $\sum_{m=1}^{\infty} m^s \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m < +\infty$. This follows immediately by reasoning exactly as in the proof of Corollary 2.1. Indeed, keeping the notation there for the series $F(z) = \sum_{m=0}^{\infty} A_m z^m$, we analogously get that for any $1 \le r < R$, all the series $F'(z)$, ..., $F^{(s)}(z)$ are uniformly convergent for $|z| \le r$.

In conclusion, we obtain the conclusions in the statement. □

In what follows, it is natural to ask for the limit

$$\lim_{n\to\infty} n^s [\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)].$$

For this purpose, for arbitrary $k, s \in \mathbb{N}$, let us denote the coefficients $\alpha_{j,s} \in \mathbb{N}$, independent of k which satisfy (see, e.g., [2, Lemma 3.1.7, p. 190])

(3.5) $$k^s = \sum_{j=1}^{s} \alpha_{j,s} k(k-1) \cdot ... \cdot (k-(j-1)),$$

and the recurrence formula

(3.6) $$\alpha_{j,s+1} = \alpha_{j-1,s} + j\alpha_{j,s}, j = 2, ..., s, s \ge 2, \text{ with } \alpha_{1,s} = \alpha_{s,s} = 1, \text{ for all } s \ge 1.$$

By simple calculation (see the indications for the relation after the proof of Corollary 2.1), we have

$$\begin{aligned}
& n^s [\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)] \\
=& n^s \left\{ \mathcal{R}_{n,s}(fg)(z) - f(z)g(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j (f(z)g(z))^{(j)} \right. \\
& - g(z) \left[\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j f^{(j)}(z) \right] \\
& - \mathcal{R}_{n,s}(f)(z) \left[\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j g^{(j)}(z) \right] \\
& \left. + \left(\frac{1}{n^s} \sum_{j=1}^{s} \alpha_{j,s} z^j g^{(j)}(z) \right) [\mathcal{R}_{n,s}(f)(z) - f(z)] + E_{n,s}(f,g)(z) \right\},
\end{aligned}$$

where $E_{n,s}(f,g)(z) = \frac{1}{n^s} G_s(f,g)(z)$ with

(3.7) $$G_s(f,g)(z) = \sum_{j=1}^{s} \alpha_{j,s} z^j [f(z)g^{(j)}(z) + g(z)f^{(j)}(z) - (f(z)g(z))^{(j)}].$$

Taking into account the estimate in [2, Theorem 3.1.8, p. 190], applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n^s [\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)] = G_s(f,g)(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 3.3. *Suppose that* $1 \le r < R$, $s \in \mathbb{N}$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$. *Then, for all* $|z| \le r$, *there exists a constant* $C(r, s, f, g) > 0$ *depending on* r, s, f, g, *such that*

$$\left|\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right| \le \frac{C(r,s,f,g)}{n^{s+1}},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
&\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\\
&= \left[\mathcal{R}_{n,s}(fg)(z) - (fg)(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j(f(z)g(z))^{(j)}\right]\\
&-f(z)\left[\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j g^{(j)}(z)\right]\\
&-g(z)\left[\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j f^{(j)}(z)\right]\\
&+[g(z) - \mathcal{R}_{n,s}(g)(z)]\cdot[\mathcal{R}_{n,s}(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in the second line of the proof of [2, Theorem 3.1.6, p. 189] and [2, Theorem 3.1.8, p. 190], we get

$$\begin{aligned}
&\left|\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right|\\
&\le \left|\mathcal{R}_{n,s}(fg)(z) - (fg)(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j(f(z)g(z))^{(j)}\right|\\
&+|f(z)|\left|\mathcal{R}_{n,s}(g)(z) - g(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j g^{(j)}(z)\right|\\
&+|g(z)|\left|\mathcal{R}_{n,s}(f)(z) - f(z) + \frac{1}{n^s}\sum_{j=1}^{s}\alpha_{j,s}z^j f^{(j)}(z)\right|\\
&+|g(z) - \mathcal{R}_{n,s}(g)(z)|\cdot|\mathcal{R}_{n,s}(f)(z) - f(z)|\\
&\le \frac{C_1(r,s,f,g)}{n^{s+1}} + \|f\|_r\cdot\frac{C_2(r,s,g)}{n^{s+1}} + \|g\|_r\cdot\frac{C_3(r,s,f)}{n^{s+1}} + \frac{C_4(r,s,g)}{n^s}\cdot\frac{C_5(r,s,f)}{n^s}\\
&\le \frac{C(r,s,f,g)}{n^{s+1}},
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r,s,f,g) > 0$ independent of n and depending on r, s, f, g. □

Remark 3.1. *Taking* $s = 1$ *in Theorem 3.3 and using that* $G_1(f,g)(z) = 0$ *for all* $z \in \mathbb{D}_R$, *in this case we get a better estimate in the Grüss-type inequality than that in Corollary 3.4, namely*

$$\|\mathcal{R}_{n,1}(fg) - \mathcal{R}_{n,1}(f)\mathcal{R}_{n,1}(g)\|_r \le \frac{C(f,g)}{n^2}.$$

In what follows, the above theorem is used to obtain lower estimate in the Grüss-type inequality.

Corollary 3.5. *Suppose that* $1 \leq r < R$, $s \in \mathbb{N}$, $s \geq 2$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$. *Then there exists* $n_0 \in \mathbb{N}$ *depending on* r, s, f *and* g, *such that*

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \geq \frac{1}{n^s} \cdot \frac{\|G_s(f,g)\|_r}{2},\ n \in \mathbb{N}, n \geq n_0,$$

where $G_s(f,g)(z)$ *is given by relation (3.7).*

Proof. We can write

$$\begin{aligned}&\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z)\\ =&\frac{1}{n^s}\left\{G_s(f,g)(z) + \frac{1}{n^s}\left[n^{2s}\left(\mathcal{R}_{n,s}(fg)(z) - \mathcal{R}_{n,s}(f)(z)\mathcal{R}_{n,s}(g)(z) - \frac{1}{n^s}G_s(f,g)(z)\right)\right]\right\}.\end{aligned}$$

Applying to the above identity, the obvious inequality

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r,$$

we obtain

$$\begin{aligned}&\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r\\ \geq&\frac{1}{n^s}\left\{\|G_s(f,g)\|_r - \frac{1}{n^s}\left[n^{2s}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) - \frac{1}{n^s}G_s(f,g)\right\|_r\right]\right\}.\end{aligned}$$

By hypothesis, we easily get $\|G_s(f,g)\|_r > 0$.

Taking into account that by Theorem 3.3, we get

$$n^{s+1}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) + \frac{1}{n^s}G_s(f,g)\right\|_r \leq C(r,s,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\begin{aligned}&\|G_s(f,g)\|_r - \frac{1}{n^s}\left[n^{2s}\left\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g) + \frac{1}{n^s}G_s(f,g)\right\|_r\right]\\ \geq&\|G_s(f,g)\|_r - \frac{K(r,s,f,g)}{n}\\ \geq&\frac{\|G_s(f,g)\|_r}{2}\\ >&0,\end{aligned}$$

which for all $n \geq n_0$ implies

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \geq \frac{1}{n^s} \cdot \frac{\|G_s(f,g)\|_r}{2}.$$

The corollary is proved. □

As an immediate consequence of Corollary 3.4 and Corollary 3.5, we obtain the following exact estimate.

Corollary 3.6. *Suppose that* $1 \leq r < R$, $s \in \mathbb{N}$, $s \geq 2$ *and* $f, g : \mathbb{D}_R \to \mathbb{C}$ *are analytic in* $\mathbb{D}_R$. *If* f *and* g *are such that* $G_s(f,g)(z)$ *is not identical zero in* $\mathbb{D}_r$ *, then there exists* $n_0 \in \mathbb{N}$ *depending only on* r, s, f, g, *such that we have*

$$\|\mathcal{R}_{n,s}(fg) - \mathcal{R}_{n,s}(f)\mathcal{R}_{n,s}(g)\|_r \sim \frac{1}{n^s},\ n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n but depend on r, s, f, g.

Remark 3.2. *The statements of Corollaries 3.5 and 3.6 suggest to be of interest to examine the pair of functions f, g, for which $G_s(f,g)(z) \equiv 0$. For example, in the particular case $s = 2$, taking into account the formula for $G_s(f,g)(z)$ in (3.7), we easily obtain that*

$$f(z)g''(z) + f''(z)g(z) - [f(z)g(z)]'' \equiv 0.$$

This easily one reduces to $f'(z)g'(z) \equiv 0$, which means that f is a constant function and g is an arbitrary analytic function, or f is an arbitrary analytic function and g is a constant function.

The cases $s \geq 3$ are more complicated and remain as open questions.

4. Jackson complex convolution

In this section, we study the Jackson complex polynomials based on the convolution with the Jackson kernel

$$K_n(t) = \frac{3}{2n(2n^2+1)} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^4,$$

defined by

$$\mathcal{J}_n(f)(z) = c_0 + \sum_{j=1}^{2n-2} c_j \cdot \lambda_{j,n} \cdot z^j, \tag{4.8}$$

attached to analytic functions on compact disks, $f(z) = \sum_{j=0}^{\infty} c_j z^j$, where $\lambda_{j,n} = \frac{4n^3-6j^2n+3j^3-3j+2n}{2n(2n^2+1)}$ if $1 \leq j \leq n$, $\lambda_{j,n} = \frac{j-2n-(j-2n)^3}{2n(2n^2+1)}$ if $n \leq j \leq 2n-2$.

As a consequence of Theorem 2.1, the following Grüss-type estimate holds for Jackson complex convolution.

Corollary 4.7. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in \mathbb{D}_R$.*

For all $n \in \mathbb{N}$, we have

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \leq \frac{3C_r}{n^2} \sum_{m=1}^{\infty} m^2 \left[\sum_{j=0}^{m} |a_j| \cdot |b_{m-j}|\right] r^m.$$

Here, $C_r > 0$ is a constant depending only on r.

Proof. Denote $e_m(z) = z^m$. We will estimate $\|A_{n,m,j}\|_r$ in the case when in Theorem 2.1, we take $\mathcal{L}_n = \mathcal{J}_n$.

From the formula for $\mathcal{J}_n$ in (4.8), we get $\mathcal{J}_n(e_k)(z) = 0$, if $k > 2n-2$ and $\mathcal{J}_n(e_k) = \lambda_{k,n} e_k(z)$ if $0 \leq k \leq 2n-2$, which implies that $\|\mathcal{J}_n(e_k)\|_r \leq r^k$, for all k, n (here we take into account that by e.g. [2, Remark 3, p. 195], we have $0 \leq \lambda_{k,n} \leq 1$ for all k, n).

Also, from [2, Theorem 3.1.10, (iv), p. 195], combined with the mean value theorem applied to the divided difference of the complex valued function $g(t) = f(re^{it})$, we immediately get

$$\begin{aligned}|\mathcal{J}_n(f)(z)-f(z)| &\le C_r\omega_2(f;1/n)_{\partial\mathbb{D}_r}\\ &\le \frac{C_r}{n^2}\|g''\|_{[0,2\pi]}\\ &\le \frac{C_r}{n^2}\left[\|f'\|_r+\|f''\|_r\right]\\ &\le \frac{C_r}{n^2}\left[\sum_{k=1}^{\infty}|c_k|kr^{k-1}+\sum_{k=2}^{\infty}|c_k|(k-1)kr^{k-2}\right]\\ &\le \frac{C_r}{n^2}\sum_{k=1}^{\infty}|c_k|\cdot k^2\cdot r^k.\end{aligned}$$

Note that here, the constant C_r depends only on r and is different at each occurrence.

It is worth noting here that the above estimate corrects a little the constant in the estimate in [2, Corollary 3.1.11, (i)] (where instead of $\sum_{k=1}^{\infty}|c_k|\cdot k^2\cdot r^k$ we got the incorrect constant $\sum_{k=1}^{\infty}|c_k|\cdot k(k-1)\cdot r^{k-2}$, which appears because in [2, p. 196] we used the incorrect estimate $\|g''\|_{[0,2\pi]}\le\|f''\|_r$).

Now, if we put above e_k instead of f, we easily arrive at

$$\|\mathcal{J}_n(e_k)-e_k\|_r\le\frac{C_r}{n^2}\cdot k^2r^k$$

for all k,n.

Therefore, for all $j\le m$, it follows

$$\begin{aligned}\|A_{m,n,j}\|_r &\le \frac{C_r}{n^2}m^2r^m+r^j\cdot\frac{C_r}{n^2}(m-j)^2r^{m-j}+r^{m-j}\cdot\frac{C_r}{n^2}j^2r^j\\ &\le\frac{3C_r}{n^2}\cdot m^2r^m,\end{aligned}$$

which combined with Theorem 2.1 proves the corollary. □

In what follows, it is natural to ask for the limit

$$\lim_{n\to\infty}n^2[\mathcal{J}_n(fg)(z)-\mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)].$$

By simple calculation, we have (see the indications for the relation after the proof of Corollary 2.1)

$$\begin{aligned}&n^2[\mathcal{J}_n(fg)(z)-\mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)]\\ =&n^2\left\{\mathcal{J}_n(fg)(z)-f(z)g(z)+\frac{3z^2}{2n^2}(f(z)g(z))''+\frac{3z}{2n^2}(f(z)g(z))'\right.\\ &-g(z)\left[\mathcal{J}_n(f)(z)-f(z)+\frac{3z^2}{2n^2}f''(z)+\frac{3z}{2n^2}f'(z)\right]\\ &-\mathcal{J}_n(f)(z)\left[\mathcal{J}_n(g)(z)-g(z)+\frac{3z^2}{2n^2}g''(z)+\frac{3z}{2n^2}g'(z)\right]\\ &\left.+\left(\frac{3z^2}{2n^2}g''(z)+\frac{3z}{2n^2}g'(z)\right)[\mathcal{J}_n(f)(z)-f(z)]-\frac{3z^2}{n^2}f'(z)g'(z)\right\}.\end{aligned}$$

Taking into account the estimate in [2, Theorem 3.1.12, p. 196], applied successively there for $f \cdot g$, f and g, passing to the limit it easily follows

$$\lim_{n\to\infty} n^2[\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)] = -3z^2 f'(z)g'(z).$$

This suggests us to prove the following Grüss-Voronovskaya-type estimate.

Theorem 4.4. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then, for all $|z| \le r$, there exists a constant $C(r, f, g) > 0$ depending on r, f, g, such that*

$$\left| \mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z) \right| \le \frac{C(r,f,g)}{n^3},\ n \in \mathbb{N}.$$

Proof. Firstly, note that we have the decomposition formula

$$\begin{aligned}
&\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z) \\
&= \left[\mathcal{J}_n(fg)(z) - (fg)(z) + \frac{3z^2}{2n^2}(f(z)g(z))'' + \frac{3z}{2n^2}(f(z)g(z))' \right] \\
&-f(z)\left[\mathcal{J}_n(g)(z) - g(z) + \frac{3z^2}{2n^2} g''(z) + \frac{3z}{2n^2} g'(z) \right] \\
&-g(z)\left[\mathcal{J}_n(f)(z) - f(z) + \frac{3z^2}{2n^2} f''(z) + \frac{3z}{2n^2} f'(z) \right] \\
&+[g(z) - \mathcal{J}_n(g)(z)] \cdot [\mathcal{J}_n(f)(z) - f(z)].
\end{aligned}$$

Passing to modulus with $|z| \le r$ and taking into account the estimates in [2, Theorem 3.1.12, p. 196] and the estimate in the proof of Corollary 4.7, we get

$$\begin{aligned}
&\left| \mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2} f'(z)g'(z) \right| \\
&\le \left| \mathcal{J}_n(fg)(z) - (fg)(z) + \frac{3z^2}{2n^2}(f(z)g(z))'' + \frac{3z}{2n^2}(f(z)g(z))' \right| \\
&+|f(z)| \left| \mathcal{J}_n(g)(z) - g(z) + \frac{3z^2}{2n^2} g''(z) + \frac{3z}{2n^2} g'(z) \right| \\
&+|g(z)| \left| \mathcal{J}_n(f)(z) - f(z) + \frac{3z^2}{2n^2} f''(z) + \frac{3z}{2n^2} f'(z) \right| \\
&+|g(z) - \mathcal{J}_n(g)(z)| \cdot |\mathcal{J}_n(f)(z) - f(z)| \\
&\le \frac{C_1(r,f,g)}{n^3} + \|f\|_r \cdot \frac{C_2(r,g)}{n^3} + \|g\|_r \cdot \frac{C_3(r,f)}{n^3} + \frac{C_4(r,g)}{n^2} \cdot \frac{C_5(r,f)}{n^2} \\
&\le \frac{C(r,f,g)}{n^3}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $|z| \le r$, with $C(r, f, g) > 0$ independent of n and depending on r, f, g. □

In what follows, the above theorem is used to obtain a lower estimate in the Grüss-type inequality.

Corollary 4.8. *Suppose that $1 \le r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. Then there exists an $n_0 \in \mathbb{N}$, depending only on r, f and g, such that*

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \ge \frac{1}{n^2} \cdot \frac{\|3e_2 f' \cdot g'\|_r}{2}, \quad n \in \mathbb{N}, n \ge n_0.$$

Proof. We can write

$$\begin{aligned}&\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z)\\ =&\frac{1}{n^2}\left\{-3z^2 f'(z)g'(z) + \frac{1}{n^2}\left[n^4\left(\mathcal{J}_n(fg)(z) - \mathcal{J}_n(f)(z)\mathcal{J}_n(g)(z) + \frac{3z^2}{n^2}f'(z)g'(z)\right)\right]\right\}.\end{aligned}$$

Applying to the above identity, the obvious inequality

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r,$$

and denoting $e_2(z) = z^2$, we obtain

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \geq \frac{1}{n^2}\left\{\|3e_2 f'g'\|_r - \frac{1}{n^2}\left[n^4\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r\right]\right\}.$$

Since f and g are not constant functions, we easily get $\|3e_2 f'g'\|_r > 0$.

Taking into account that by Theorem 4.4, we get

$$n^3\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r \leq C(r,f,g)$$

and that $\frac{1}{n} \to 0$, there exists an index n_0 (depending only on r, f, g), such that for all $n \geq n_0$, we have

$$\|3e_2 f'g'\|_r - \frac{1}{n}\left[n^3\left\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g) + \frac{3e_2}{n^2}f'g'\right\|_r\right] \geq \frac{\|3e_2 f'g'\|_r}{2} > 0,$$

which for all $n \geq n_0$ implies

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \geq \frac{1}{n^2}\cdot\frac{\|3e_2 f'g'\|_r}{2}.$$

The corollary is proved. □

As an immediate consequence of Corollary 4.7 and Corollary 4.8, we obtain the following exact estimate.

Corollary 4.9. *Suppose that $1 \leq r < R$ and $f, g : \mathbb{D}_R \to \mathbb{C}$ are analytic in $\mathbb{D}_R$. If f and g are not constant functions, then there exists $n_0 \in \mathbb{N}$ depending only on r, f and g, such that we have*

$$\|\mathcal{J}_n(fg) - \mathcal{J}_n(f)\mathcal{J}_n(g)\|_r \sim \frac{1}{n^2}, \quad n \in \mathbb{N}, n \geq n_0,$$

where the constants in the equivalence are independent of n but depend on r, f, g.

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