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The Sampson Laplacian on Negatively Pinched Riemannian Manifolds

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

We prove vanishing theorems for the kernel of the Sampson Laplacian, acting on symmetric tensors on a Riemannian manifold and estimate its first eigenvalue on negatively pinched Riemannian manifolds. Some applications of these results to conformal Killing tensors are presented.

Keywords: *Riemannian manifold, Sampson Laplacian, spectral and vanishing theorems, conformal Killing tensor.*
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1. Introduction

Let (M,g) be a Riemannian manifold. We regard it as a connected C^∞-manifold M of dimension $n \geq 2$ endowed with a metric tensor g and the Levi-Civita connection ∇. Let $TM := T^{(0,1)}M$ (resp., $T^*M := T^{(1,0)}M$) be its tangent (resp. cotangent) bundle where $T^{(p,q)}M = (\otimes^p T^*M) \otimes (\otimes^q TM)$. Let S^pM (resp. $\Lambda^r M$) be a subbundle of $T^{(p,0)}M$, consisting of covariant symmetric p-tensors (resp. differential r-forms) on M. Denote the vector spaces of their C^∞-sections by $C^\infty(T^{(p,q)}M)$, $C^\infty(S^pM)$ and $C^\infty(\Lambda^r M)$, respectively.

The *Lichnerowicz-type Laplacian* has the form (see [17, 24])

$$\Delta_L T = \bar{\Delta}\, T + t\, \Re_p(T) \tag{1.1}$$

for any $t \in \mathbb{R}$ and $T \in C^\infty(\otimes^p T^*M)$. In (1.1), $\bar{\Delta}$ is the *Bochner* (*rough*) *Laplacian* and $\Re_p$ is the *Weitzenböck curvature operator*, which in a known way depends linearly on the Riemann curvature tensor and the Ricci tensor of (M,g). In addition, the Weitzenböck curvature operator of Δ_L satisfies the following identities (see [20, p. 315])

$$g(\Re_p(T),\, T') = g(T,\, \Re_p(T'))$$

and

$$\mathrm{trace}_g\, \Re_p(T) = \Re_{p-2}(\mathrm{trace}_g T) \tag{1.2}$$

for any $T, T' \in C^\infty(\otimes^p T^*M)$. In particular, for $t=1$, (1.1) yields the formula $\Delta_L = \bar{\Delta} + \Re_p$ of the ordinary *Lichnerowicz Laplacian* (see [20]; [1, pp. 53–54]). We recall that the formula

$$\Delta_H\, \omega = \bar{\Delta}\, \omega + \Re_p(\omega)$$

for an arbitrary p-form $\omega \in C^\infty(\Lambda^p M)$ determines the well known *Hodge Laplacian* (see [1, p. 35]; [25, pp. 335; 347]). At the same time, the *Sampson Laplacian* Δ_S acting on C^∞-sections of the vector bundle S^pM has the following *Weitzenböck decomposition* (see [24, 28, 33]):

$$\Delta_S\, \varphi = \bar{\Delta}\, \varphi - \Re_p(\varphi) \tag{1.3}$$

for any $\varphi \in C^\infty(S^pM)$. Therefore, the differential operator Δ_S is also an example of the Lichnerowicz-type Laplacian for the special case when $t=-1$ and $T \in C^\infty(S^pM)$.

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Formulas of the type (1.1) are particularly important in the study of interactions between the geometry and topology of Riemannian manifolds. In fact, there exists a method, due to Bochner, of proving vanishing theorems for the null space of a Laplace operator admitting the Weitzenböck decomposition and furthermore of estimating its lowest eigenvalue (see [1, pp. 52–53]; [25, pp. 333–364]). This method mainly applies to compact manifolds. As an application of the Bochner technique, we recall the following theorem in [41]: If (M, g) is a closed (i.e., compact and without boundary) Riemannian manifold with positive (resp. negative) curvature operator of the second kind, $\overset{\circ}{R} : S_0^2 M \to S_0^2 M$, then it does not admit the Hodge Laplacian Δ_H with the non-degenerate null space, hence the Betti numbers $b_1(M) = \ldots = b_{n-1}(M) = 0$ (resp. the Tachibana numbers $t_1(M) = \ldots = t_{n-1}(M) = 0$, that we conclude from [35]).

Remark 1.1. The Riemann curvature tensor Rm induces an algebraic *curvature operator* $\overset{\circ}{R} : S_0^2 M \to S_0^2 M$ (see, for example, [19]). The symmetries of Rm imply that $\overset{\circ}{R}$ is a selfadjoint operator, with respect to the point-wise inner product on $S_0^2 M$. That is why, the eigenvalues of $\overset{\circ}{R}$ are all real numbers at each point $x \in M$. Thus, we say $\overset{\circ}{R}$ is *positive semidefinite* (resp. *positive-definite*), or simply $\overset{\circ}{R} \geq 0$ (resp. $\overset{\circ}{R} > 0$), if all the eigenvalues of $\overset{\circ}{R}$ are nonnegative (resp. positive). The properties and applications of $\overset{\circ}{R}$ were studied in [1, pp. 51–52]; [5, 6, 19, 23, 24, 41], etc.

In the present paper, we prove the vanishing theorem for the kernel $\ker \Delta_S$ of the Laplacian Δ_S on a closed Riemannian manifold (M, g) with negatively pinched sectional curvature. We find an estimate of its lower eigenvalue depending on the sign of the sectional curvature of (M, g). In addition, we give some applications of the above results.

In conclusion, recall that the Sampson Laplacian Δ_S is of fundamental importance in mathematical physics (e.g., [11, 26, 45]). Note also that the operator Δ_S acting on symmetric covariant 2-tensor fields appears in many problems in Riemannian geometry including Ricci flow (e.g., [3, 9, 16, 21]; [1, pp. 64; 133] and [7, pp. 109–110]). This article continues our study of the Sampson Laplacian, which we carried out in [24].

2. The Sampson Laplacian and its Weitzenböck curvature operator

Here, we define the differential operator $\delta^* : C^\infty(S^p M) \to C^\infty(S^{p+1} M)$ of degree one by the formula $\delta^* \varphi = (p+1)\mathrm{Sym}(\nabla \varphi)$ for an arbitrary $\varphi \in C^\infty(S^p M)$ and the standard point-wise symmetry operator $\mathrm{Sym} : T^* M \otimes S^p M \to S^{p+1} M$. Let $\delta : C^\infty(S^{p+1} M) \to C^\infty(S^p M)$ be the adjoint operator for δ^* (see [1, pp. 35, 434]). Then, in accordance with [28], we define the Laplacian

$$\Delta_S = \delta\, \delta^* - \delta^* \delta.$$

By [28], Δ_S admits the Weitzenböck decomposition (1.3). In addition, from (1.2) we conclude that $\Re_p : S^p M \to S^p M$ is a symmetric endomorphism. More properties of the operator Δ_S can be found in the following papers: [24, 30, 31, 32, 33, 42].

Let $S_0^p M$ be a vector bundle of traceless symmetric p-tensor on M, which is defined by the condition $\mathrm{trace}_g\, \varphi = 0$, where $\mathrm{trace}_g\, \varphi = \sum_i \varphi(e_i, e_i, X_3, \ldots, X_p)$ for any $\varphi \in C^\infty(S^p M)$ and an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M$ at an arbitrary point $x \in M$. Then from (1.2) and (1.3) we conclude that the following proposition is true.

Theorem 2.1. *The Sampson Laplacian Δ_S maps the vector space $C^\infty(S_0^p M)$ into itself.*

Using (1.3), we define the *Weitzenböck quadratic form* $Q_p : S^p M \times S^p M \to \mathbb{R}$ by the equality

$$Q_p(\varphi) = g\left(\Re_p(\varphi), \varphi\right) = R_{ij}\, \varphi^{i\, i_2 \ldots i_p} \varphi^{j}{}_{i_2 \ldots i_p} + (p-1)\, R_{ijkl}\, \varphi^{i\, l\, i_3 \ldots i_p} \varphi^{jk}{}_{i_3 \ldots i_p} \tag{2.1}$$

(see also [34]) for local components $\varphi_{i_1, \ldots, i_p}$, R_{ij} and R_{ijkl} of an arbitrary $\varphi_x \in S_0^p(T^* M)$, the Ricci tensor Ric and the Riemann curvature tensor Rm, respectively, and for any $\varphi \in S^p M$.

Next we prove two propositions on the quadratic form (2.1).

In the paper, we consider positive numbers $\delta > \varepsilon > 0$.

Theorem 2.2. *Let $\Delta_S : C^\infty(S_0^2 M) \to C^\infty(S_0^2 M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle $S_0^2 M$ of traceless symmetric 2-tensors on an n-dimensional $(n \geq 2)$ closed Riemannian manifold with negative sectional* *curvature. If $-\delta$ and $-\varepsilon$ are the minimum and maximum of the sectional curvature, then its Weitzenböck quadratic form $Q_2(\varphi)$ satisfies the inequalities*

$$-n\,\delta\,\|\varphi\|^2 \le Q_2(\varphi) \le -n\,\varepsilon\,\|\varphi\|^2$$

for any $\varphi \in C^\infty(S^2_0 M)$.

Proof. First, we consider (2.1) for $p=2$. Thus, for any point $x \in M$ and any $\varphi \in C^\infty(S^2_0 M)$ there exists an orthonormal eigen-frame $e_1,\ldots,e_n$ of T_xM such that $\varphi_x(e_i,e_j)=\mu_i\,\delta_{ij}$, where δ_{ij} is the Kronecker delta, and the following holds (see [1, p. 436]; [3, p. 388]):

$$Q_2(\varphi) = g\,(\Re_2(\varphi_x),\,\varphi_x) = 2\sum\nolimits_{i<j}\sec(e_i\wedge e_j)(\mu_i-\mu_j)^{\,2}.$$

Here, $\sec(e_i\wedge e_j) = R\,(e_i,\,e_j,\,e_i,\,e_j)$ is the *sectional curvature* $\sec\sigma_x$ of (M,g) in the direction of the tangent two-plane section $\sigma_x = \mathrm{span}\,\{e_i,\,e_j\}$ of T_xM at $x\in M$. If, in addition, there is a point $x \in M$, where the sectional curvature of (M,g) satisfies the inequalities

$$-\delta \le \sec(x) \le -\varepsilon$$

for all 2-planes $\pi(x)\subset T_xM$ and some constants $\delta>\varepsilon>0$, then from

$$R_{ij}\,\varphi^{i\,k}\varphi^{j}{}_{k} + R_{ijkl}\,\varphi^{i\,l}\varphi^{jk} = 2\sum\nolimits_{i<j}\sec(e_i\wedge e_j)\,(\,\mu_i-\mu_j)^{\,2}$$

we obtain the double inequalities (see [27])

$$-\,n\,\delta\,\|\varphi_x\|^2 \le R_{ij}\,\varphi^{i\,k}\varphi^{j}{}_{k} + R_{ijkl}\,\varphi^{i\,l}\varphi^{jk} \le -n\,\varepsilon\,\|\varphi_x\|^2, \tag{2.2}$$

where $\|\varphi_x\|^2 = \varphi^{ij}\,\varphi_{ij}$ for the local components φ_{ij}. In this case, from (2.2) we conclude that the quadratic form $Q_2(\varphi_x)$ is negative definite for all nonzero $\varphi_x \in S^2_0(T^*_xM)$. In particular, the equality $Q_2(\varphi_x)=0$ holds if and only if $\varphi_x = 0$.

Suppose that (M,g) is compact with negative sectional curvature and denote by $-\delta$ and $-\varepsilon$ the minimum and maximum of its sectional curvature of (M,g). Then from (2.1) we conclude that the Weitzenböck quadratic form satisfies the inequalities

$$-n\,\delta\|\varphi\|^2 \le Q_2(\varphi) \le -n\,\varepsilon\|\varphi\|^2$$

for any $\varphi\in C^\infty\left(S^2_0M\right)$. □

Next we will consider the case when $p\ge 3$. At the same time, let $x\in M$ be a point where the sectional curvature of (M,g) satisfies the inequalities

$$-\delta \le \sec\,\pi(x) \le -\varepsilon < 0$$

for all 2-plans $\pi(x)\subset T_xM$. We rewrite the double inequalities (2.2) in the form

$$-n\,\delta\|\varphi_x\|^2 - R_{ij}\,\varphi^{i\,k}\varphi^{j}{}_{k} \le R_{ijkl}\,\varphi^{i\,l}\varphi^{jk} \le -n\,\varepsilon\|\varphi_x\|^2 - R_{ij}\,\varphi^{i\,k}\varphi^{j}{}_{k},$$

where by [4, p. 81–82] the following inequalities hold:

$$-\,(n-1)\,\delta\,\|\varphi_x\|^2 \le R_{ij}\,\varphi^{ik}\varphi^{j}{}_{k} \le -\,(n-1)\,\varepsilon\,\|\varphi_x\|^2. \tag{2.3}$$

Then from (2.2) and (2.3) we obtain the following double inequalities:

$$(-\,n\,\delta + (n-1)\,\varepsilon)\ \|\varphi_x\|^2 \le R_{ijkl}\,\varphi^{i\,l}\varphi^{jk} \le (-\,n\,\varepsilon\, + (n-1)\,\delta)\,\|\varphi_x\|^2.$$

From the above we conclude that the following inequalities are satisfied:

$$(p-1)(-n\,\delta + (n-1)\,\varepsilon)\|\varphi_x\|^2 \le (p-1)\,R_{ijkl}\,\varphi^{i\,l i_3\ldots i_p}\varphi^{jk}_{i_3\ldots i_p} \le (p-1)(-n\,\varepsilon + (n-1)\,\delta)\|\varphi_x\|^2 \tag{2.4}$$

(see [4, p. 82]; [13, p. 91]) for local components $\varphi_{i_1\ldots i_p}$ of $\varphi_x \in S^p_0\,(T^*M)$ and $\|\varphi_x\|^2 = \varphi^{i_1\,\ldots i_p}\varphi_{\,i_1\,\ldots i_p}$. In turn, from (2.3) we deduce (see [4, p. 82]; [13, p. 90])

$$-\,(n-1)\,\delta\|\varphi_x\|^2 \le R_{ij}\,\varphi^{ii_2\ldots i_p}\varphi^{j}{}_{i_2\ldots i_p} \le -(n-1)\,\varepsilon\|\varphi_x\|^2. \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$\begin{aligned}((n-1)(p-1)\varepsilon - ((p-1)\,n + (n-1))\delta)\|\varphi_x\|^2 \\ \leq R_{ij}\varphi^{i\,i_2\ldots i_p}\varphi^{j}_{i_2\ldots i_p} + (p-1)R_{ijkl}\,\varphi^{i\,l\,i_3\ldots i_p}\varphi^{jk}_{i_3\ldots i_p} \\ \leq ((n-1)(p-1)\delta - ((p-1)\,n + (n-1))\varepsilon)\,\|\varphi_x\|^2.\end{aligned}$$

Suppose now that (M, g) is a closed Riemannian manifold with negative sectional curvature. Denote by $-\delta$ and $-\varepsilon$ the minimum and maximum of the sectional curvatures of (M, g). Based on the above result, we obtain the following.

Theorem 2.3. *Let $\Delta_S : C^\infty(S^p_0 M) \to C^\infty(S^p_0 M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle $S^p_0 M$ of traceless symmetric p-tensors $(p \geq 3)$ on an n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) with negative sectional curvature. If $-\delta$ and $-\varepsilon$ are the minimum and maximum of the sectional curvature of (M, g), then its Weitzenböck quadratic form satisfies the inequalities*

$$((n-1)\,(p-1)\,\varepsilon - (np-1)\,\delta)\;\|\varphi_x\|^2 \leq Q_p(\varphi) \leq ((n-1)\,(p-1)\,\delta - (np-1)\,\varepsilon\,)\,\|\varphi_x\|^2$$

for any $\varphi \in C^\infty(S^p_0 M)$.

Corollary 2.1. *Let (M, g) be an n-dimensional $(n \geq 2)$ closed Riemannian manifold with negative sectional curvature and $-\delta$ and $-\varepsilon$ are the minimum and maximum of the sectional curvature, then its curvature operator $\mathring{R} : S^2_0(M) \to S^2_0(M)$ satisfies the inequalities*

$$(-\,n\,\delta + (n-1)\,\varepsilon)\;\|\varphi\|^2 \leq g\big(\mathring{R}(\varphi),\,\varphi\big) \leq (-n\,\varepsilon + (n-1)\,\delta)\;\|\varphi\|^2$$

for any $\varphi \in C^\infty(S^2_0 M)$.

The inequality

$$(n-1)(p-1)\,\delta < (np-1)\,\varepsilon \tag{2.6}$$

implies the condition $Q_p(\varphi_x) < 0$ for an any nonzero $\varphi_x \in S^p_0(T^*_x M)$ at an arbitrary $x \in M$. Side by side, the inequalities $\varepsilon < \delta < 0$ and (2.6) can be rewritten in the following form:

$$1 < \delta/\varepsilon < \frac{np-1}{(n-1)(p-1)} = 1 + \frac{1}{n-1} + \frac{1}{p-1}.$$

In this case, the sectional curvature of the manifold (M, g) satisfies the inequalities

$$-\big(1 + \frac{1}{n-1} + \frac{1}{p-1}\big) < -\frac{\delta}{\varepsilon} \leq \frac{\sec}{\delta} \leq -1.$$

We can normalize the metric g on the manifold M so that the above double inequalities become

$$-\,\big(1 + \frac{1}{n-1} + \frac{1}{p-1}\big) < \sec \leq -1. \tag{2.7}$$

Then the following corollary holds.

Corollary 2.2. *Let $\Delta_S : C^\infty(S^p_0 M) \to C^\infty(S^p_0 M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle $S^p_0 M$ of traceless symmetric p-tensors $(p \geq 3)$ on an n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) with negatively pinched sectional curvature such that (2.7) hold. Then its Weitzenböck quadratic form $Q_p(\varphi)$ is negative definite for any $p \geq 2$ and $\varphi \in S^p_0 M$.*

On the other hand, the inequalities $\varepsilon < \delta$ and (2.6) can be rewritten in the equivalent form

$$1 - \frac{n+p-2}{np-1} < \frac{\varepsilon}{\delta} < 1.$$

Then the sectional curvature of our manifold (M, g) satisfies the inequalities

$$-1 \leq \frac{\sec}{\delta} \leq -\frac{\varepsilon}{\delta} < -1 + \frac{n+p-2}{np-1}.$$

We can normalize the metric g on the manifold (M, g) such that the above inequalities become

$$-1 \leq \sec \leq -\varepsilon < -1 + \frac{n+p-2}{np-1}.$$

Recall that a Riemannian manifold (M, g), whose sectional curvature satisfies the inequalities

$$-1 \leq \sec \leq -\varepsilon, \tag{2.8}$$

is said to be *negatively ε-pinched*.

Remark 2.1. More properties of Riemannian manifolds with negatively pinched sectional curvatures can be found, e.g., in [6, 15, 43, 44]. We know from [6, p. 313] that if (M, g) is a locally symmetric manifold with non-constant negative sectional curvature, then its sectional curvature is $1/4$-pinched. In our case, we have the pinched sectional curvature with $[-1, -\varepsilon] \subset [-1, -1/4]$ such that

$$-1 \leq \sec \leq -\varepsilon < -1 + \frac{n+p-2}{np-1} < -\frac{1}{4}.$$

Thus, there are no negative definite Weitzenböck quadratic forms Q_p of Δ_S on a Riemannian manifold with negatively $1/4$-pinched sectional curvature (see [30, 31, 32]).

On the other hand, M. Gromov and W. Thurston have proved a theorem on negatively ε-pinched Riemannian manifold (see [14]). Namely, for any integer $n \geq 4$ and $\varepsilon \in (0, 1)$, there exists a compact Riemannian manifold (M, g) of dimension n such that the sectional curvatures of (M, g) lie in the interval $[-1, -\varepsilon]$, but (M, g) does not admit a metric of constant negative sectional curvature (see [14]). Using this proposition, we obtain the following

Corollary 2.3. *There exist closed n-dimensional $(n \geq 4)$ Riemannian manifolds (M, g) with negatively pinched sectional curvature, different from compact hyperbolic spaces and such that the Weitzenböck quadratic forms $Q_p(\varphi)$ of their Sampson Laplacians $\Delta_S : C^\infty(S_0^p M) \to C^\infty(S_0^p M)$ are negative definite for any $p \geq 2$.*

3. Vanishing and spectral theorems for Sampson Laplacian

Let $\Delta_S : C^\infty(S_0^p M) \to C^\infty(S_0^p M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle $S_0^p M$ of *traceless symmetric p-tensors* $(p \geq 3)$ on an n-dimensional $(n \geq 2)$ compact Riemannian manifold (M, g). Then in accordance with the general theory (e.g., [8]), a real number $\lambda^{(p)}$, for which there is a symmetric p-tensor $\varphi \in C^\infty(S_0^p M)$ (not identically zero) such that $\Delta_S \varphi = \lambda^{(p)} \varphi$, is called an *eigenvalue* of the Sampson Laplacian $\Delta_S : C^\infty(S_0^p M) \to C^\infty(S_0^p M)$, and the corresponding symmetric p-tensor $\varphi \in C^\infty(S_0^p M)$ is called an *eigentensor* of the Sampson Laplacian Δ_S corresponding to $\lambda^{(p)}$. All nonzero eigentensors corresponding to a fixed eigenvalue $\lambda^{(p)}$ form a vector subspace of $S_0^p M$ called the *eigenspace* of the Sampson Laplacian corresponding to its eigenvalue $\lambda^{(p)}$.

Using the general theory of elliptic operators on a closed Riemannian manifold (M, g), it can be proved that Δ_S has a discrete spectrum, denoted by $\mathrm{Spec}^{(p)} \Delta_S$, consisting of real eigenvalues of finite multiplicity, which accumulate only at infinity (see also [8]). Moreover, an arbitrary eigenspace of Δ_S is finite-dimensional and the eigentensors corresponding to distinct eigenvalues are orthogonal. In general, the Sampson Laplacian Δ_S is not positive definite and, at the same time, its principal symbol has the form

$$\sigma(\Delta_S)(\theta, x)\, \varphi_x = -g(\theta, \theta)\, \varphi_x$$

for $\theta \in T_x^* M - \{0\}$ and $\varphi_x \in S_0^p(T_x^* M)$ at any $x \in M$, but its spectrum satisfies the condition $\mathrm{Spec}^{(p)} \Delta_S \subseteq [-C, \infty)$ for some constant C (see [12, p. 54]). In this case, we have

$$\mathrm{Spec}^{(p)} \Delta_S = \big\{ -\lambda_1^{(p)} \leq \ldots \leq -\lambda_r^{(p)} \leq 0 < \lambda_{r+1}^{(p)} \leq \lambda_{r+2}^{(p)} \leq \ldots \to \infty \big\}.$$

Next, we find the conditions for which the spectrum of Δ_S consists of positive numbers.

By direct calculations, we obtain from (1.1) the *Bochner-Weitzenböck formula*

$$\frac{1}{2} \Delta_g \| \varphi \|^2 = -g\left(\bar{\Delta}\, \varphi, \varphi\right) + \|\nabla \varphi\|^2 = -g(\Delta_S\, \varphi, \varphi) + \|\nabla \varphi\|^2 - Q_p(\varphi) \tag{3.1}$$

for an arbitrary $\varphi \in \mathrm{C}^\infty(S^pM)$ and the Beltrami Laplacian $\Delta_g = \mathrm{div} \circ \mathrm{grad}$, which is defined on C^∞-functions. Let (M, g) be a closed manifold with negative sectional curvature such that (2.8) hold. From (3.1) we deduce the integral equation

$$\int_M \left(\|\nabla \varphi\|^2 - Q_p(\varphi) \right) \; \mathrm{d}_g = 0 \tag{3.2}$$

for $Q_p(\varphi) = g\left(\Re_p(\varphi),\, \varphi\right)$ and an arbitrary $\varphi \in \ker \Delta_L$. Firstly, we consider the case when $p = 2$. In this case, from Theorem 2.2 we know that

$$Q_p(\varphi) \leq -n\,\varepsilon\,\|\varphi\|^2 < 0$$

for an arbitrary nonzero $\varphi \in \mathrm{C}^\infty(S^2_0M)$. Then from this inequality and (3.2) we conclude that $\varphi \equiv 0$, thus, the kernel of Δ_L is trivial. As a result, we obtain the following vanishing theorem.

Theorem 3.1. *Let $\Delta_S : \mathrm{C}^\infty(S^2_0M) \to \mathrm{C}^\infty(S^2_0M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle S^2_0M of traceless symmetric 2-tensors on an n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) with strictly negative sectional curvature, then $\ker \Delta_S$ is trivial.*

On the other hand, from (3.1) we deduce the integral inequality

$$\int_M g\left(\Delta_S\,\varphi,\, \varphi\right) \mathrm{d}_g \geq -\int_M Q_2(\varphi)\, \mathrm{d}_g \tag{3.3}$$

for any $\varphi \in \mathrm{C}^\infty(S^2_0M)$. In addition, if we suppose that a symmetric 2-tensor $\varphi \in \mathrm{C}^\infty(S^2_0M)$ be an eigentensor of the Sampson Laplacian Δ_S corresponding to $\lambda^{(2)}$, then we can rewrite (3.3) in the following form:

$$\lambda^{(2)} \int_M \|\varphi\|^2 \; \mathrm{d}_g \geq n\,\varepsilon \int_M \|\varphi\|^2\, \mathrm{d}_g. \tag{3.4}$$

From (3.4) we conclude that $\lambda^{(2)} \geq n\,\varepsilon > 0$. In this case, we have the following

Theorem 3.2. *Let (M, g) be an n-dimensional $(n \geq 2)$ closed Riemannian manifold with negative sectional curvature. If $-\varepsilon$ is the maximum value of its sectional curvature for some positive number ε, then $\mathrm{Spec}^{(2)}\Delta_S \subset [n\,\varepsilon, \infty)$ for the Sampson Laplacian $\Delta_S : \mathrm{C}^\infty(S^2_0M) \to \mathrm{C}^\infty(S^2_0M)$.*

Secondary, consider the case when $p \geq 3$. Furthermore, suppose that the inequalities (2.8) are satisfied at any point $x \in M$. In this case, from Theorem 2.3 conclude that if $-\delta$ and $-\varepsilon$ satisfy the inequality

$$(n-1)(p-1)\,\delta - (np-1)\,\varepsilon < 0,$$

then $Q\,(\varphi) < 0$ for any $\varphi \in \mathrm{C}^\infty(S^p_0M)$. From the last inequality and the integral equation (3.2) we obtain $\varphi \equiv 0$. Then the following vanishing theorem is true.

Theorem 3.3. *Let $\Delta_S : \mathrm{C}^\infty(S^p_0M) \to \mathrm{C}^\infty(S^p_0M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle S^p_0M of traceless symmetric p-tensors $(p \geq 3)$ on an n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) with negative sectional curvature. If $-\delta$ and $-\varepsilon$ are the minimum and maximum of the sectional curvature such that (2.6) are satisfied, then the kernel of Δ_S is trivial.*

Taking into account the above and Corollary 2.2, we obtain the following

Corollary 3.1. *Let $\Delta_S : \mathrm{C}^\infty(S^p_0M) \to \mathrm{C}^\infty(S^p_0M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle S^p_0M of traceless symmetric p-tensors $(p \geq 3)$ on an n-dimensional $(n \geq 2)$ closed Riemannian manifold (M, g) with negatively pinched sectional curvature such that (2.7) are satisfied. Then the kernel of Δ_S is trivial.*

In addition, taking into account the above and Corollary 2.3, we can formulate the following statement of existence a trivial kernel of the Sampson Laplacian Δ_S.

Corollary 3.2. *There exist closed n-dimensional $(n \geq 4)$ Riemannian manifolds (M, g) with negatively pinched sectional curvature, different from compact hyperbolic spaces and such that the kernels of their Sampson Laplacians $\Delta_S : \mathrm{C}^\infty(S^p_0M) \to \mathrm{C}^\infty(S^p_0M)$ are trivial.*

On the other hand, if we suppose that a symmetric p-tensor $\varphi \in \mathrm{C}^\infty(S^p_0 M)$ is an eigentensor of Δ_S corresponding to $\lambda^{(p)}$, then we can rewrite (3.2) in the following form:

$$\lambda^{(p)} \int_M \|\varphi\|^2 \, \mathrm{d}_g \geq (\,(n-1)\,(p-1)\,\varepsilon - (np-1)\,\delta\,) \int_M \|\varphi\|^2 \, \mathrm{d}_g. \tag{3.5}$$

In turn, from (3.5) we conclude that

$$\lambda^{(p)} \geq (n-1)\,(p-1)\,\varepsilon - (np-1)\,\delta.$$

In addition, if

$$(n-1)(p-1)\,\varepsilon > (np-1)\,\delta \tag{3.6}$$

for any $p \geq 3$ and $n \geq 2$ then from (3.5) we conclude that $\lambda^{(p)} > 0$. As a result, we obtain the following

Theorem 3.4. *Let $\Delta_S : \mathrm{C}^\infty(S^p_0 M) \to \mathrm{C}^\infty(S^p_0 M)$ be the Sampson Laplacian acting on C^∞-sections of the bundle $S^p_0 M$ of traceless symmetric p-tensors ($p \geq 3$) on an n-dimensional ($n \geq 2$) closed Riemannian manifold (M, g) with negative sectional curvature. If $-\delta$ and $-\varepsilon$ are the minimum and maximum of the sectional curvature of (M, g), then*

$$\mathrm{Spec}^{(p)} \Delta_S \subset [(n-1)(p-1)\,\varepsilon - (np-1)\,\delta,\ \infty).$$

In addition, if (3.6) are satisfied, then the spectrum of Δ_S consists of positive numbers.

4. Applications to the theory of conformal Killing tensors

Here, we give some applications of the above results. First, we will consider conformal Killing p-forms. Namely, *conformal Killing p-forms* (or, *conformal Killing-Yano p-tensors*) have been defined on n-dimensional Riemannian manifolds ($1 \leq p \leq n-1$) by S. Tachibana and T. Kashiwada (see [18, 40]) as a natural generalization of conformal Killing vector fields. Since then, these forms were extensively studied by many geometers. These studies were motivated by existence of various applications of conformal Killing p-forms (e.g., [2, 37]).

The vector space of conformal Killing p-forms on an n-dimensional closed Riemannian manifold (M, g) has a finite dimension $t_p(M)$ named the *Tachibana number* (e.g., [22, 29, 35]). The numbers $t_1(M), \ldots, t_{n-1}(M)$ are conformal scalar invariants of (M, g) and satisfy the duality theorem: $t_p(M) = t_{n-p}(M)$. The theorem is an analog of the well-known *Poincaré duality theorem* for the Betti numbers of a closed (M, g). Moreover, we proved in [35] that a) there exist closed Riemannian manifolds with nonzero Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$, b) Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are zero for a closed n-dimensional ($n \geq 2$) Riemannian manifold (M, g) with negative *curvature operator* $\mathring{R} : S^2_0 M \to S^2_0 M$ defined on the vector bundle $S^2_0 M$. Based on Corollary 2.1, we conclude that if

$$-1 \leq \sec < -1 + 1/n$$

then the curvature operator $\mathring{R}$ is negative definite. Therefore, the following theorem holds.

Theorem 4.1. *If (M, g) is an n-dimensional ($n \geq 2$) closed Riemannian manifold with negatively pinched sectional curvature such that*

$$-1 \leq \sec \leq -1 + 1/n,$$

then its Tachibana numbers $t_1(M), \ldots, t_{n-1}(M)$ are equal to zero.

Remark 4.1. The above theorem is a generalization of the following theorem from [43]: Let (M, g) be a closed Riemannian manifold with negatively pinched sectional curvature such that

$$-1 \leq \sec \leq -\varepsilon.$$

If the dimension of M is $n = 2m$ (resp., $n = 2m+1$) and $\varepsilon > 1/4$ (resp., $\varepsilon > 2(m-1)/(8m-5)$), then there are no conformal Killing 2-forms on the manifold. In this case, $t_2(M) = t_{n-2}(M) = 0$. In addition, the above theorem complements our theorem in [27] on the Tachibana numbers of compact Einstein manifolds. For results on conformally Killing forms on complete non-compact Riemannian manifolds, see [38].

Based on the main theorem from [14], we obtain the following.

Corollary 4.1. *There exist closed n-dimensional $(n \geq 4)$ Riemannian manifolds (M, g) with negatively pinched sectional curvature, different from compact hyperbolic spaces and such that their Tachibana numbers $t_1(M), \dots, t_{n-1}(M)$ are equal to zero.*

Next, we consider a *conformal Killing symmetric p-tensor* $(p \geq 2)$ that is a symmetric trace-free p-tensor $\varphi \in C^\infty(S_0^p M)$ satisfying the following condition: the trace-free part of $\delta^* \varphi$ equals to zero, which is equivalent to the following equation (see [10]; [39, p. 559])

$$\frac{1}{p+1}\,\delta^* \varphi = -\,\frac{p}{n+2\,(p-1)}\,g\,\circ\,\delta\,\varphi. \tag{4.1}$$

In local coordinates, (4.2) can be rewritten in the following form (see also [10]):

$$\nabla_{(i_0}\,\varphi_{\,i_1 i_2 \dots i_p)} = -\,\frac{p}{n+2\,(p-1)}\,g_{(i_0 i_1}\,\delta\,\varphi_{\,i_2 \dots i_p)},$$

where we write $\phi_{(i_0 i_1 \dots i_p)}$ for symmetric part of a tensor $\phi_{i_0 i_1 \dots i_p}$. Using the definition of the Sampson Laplacian and based on the formula (4.1), we obtain

$$\begin{aligned}\int_M g\,(\Delta_S\,\varphi,\ \varphi)\ \mathrm{d}_g &= \frac{1}{(p+1)}\int_M \|\delta^*\,\varphi\|^2\,\mathrm{d}_g - \frac{1}{(p-1)}\int_M \|\delta\,\varphi\|^2\,\mathrm{d}_g \\ &= -\frac{n-2(p-2)}{n+2\,(p-1)}\int_M \|\delta\,\varphi\|^2\,\mathrm{d}_g.\end{aligned}$$

In this case, for any conformal Killing tensor $\varphi \in C^\infty(S_0^p M)$ we defive the integral formula

$$\frac{n-2(p-2)}{n+2\,(p-1)}\int_M \|\delta\,\varphi\|^2\,\mathrm{d}_g + \int_M \big(\|\nabla\,\varphi\|^2 - Q_p(\varphi)\,\big)\,\mathrm{d}_g = 0. \tag{4.2}$$

Using (4.2) and based on Corollary 2.1, we obtain the following proposition.

Corollary 4.2. *There exist closed n-dimensional $(n \geq 4)$ Riemannian manifolds (M, g) with negatively pinched sectional curvature and different from compact hyperbolic spaces, which have no nonzero symmetric conformal Killing p-tensors for any $p \geq 2$.*

Remark 4.2. This corollary completes the vanishing theorem in [36] on conformally Killing symmetric tensors of order 2 on a compact Riemannian manifold and its generalization in the case of conformally Killing symmetric tensors of order $p \geq 2$ from [9, 16].

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