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OPTIMAL STEP STRESS ACCELERATED LIFE TESTING FOR THE LENGTH-BIASED EXPONENTIAL CUMULATIVE EXPOSURE MODEL

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Abstract: This paper considers a simple step stress accelerated life test for units modeled by a length-biased exponential distribution. The cumulative exposure model of time to failure holds in this accelerated life test model. The optimal test plan is constructed by determining the optimal stress change time. Parameters of the model are estimated by using the maximum likelihood estimation method. The corresponding approximate confidence intervals are obtained by using the asymptotic normality features of the maximum likelihood estimators. Theoretical outcomes are illustrated with simulation studies and a real data example.

Key words: Accelerated life test, cumulative exposure model,length-biased exponential distribution, maximum likelihood estimation, optimal test plan, step-stress model

1. Introduction

Studies based on the accelerated life tests (ALT) have been popular since they let the experimenters control the higher stress levels to be used for components or units in the life tests. It is clear that long lifetimes of highly reliable products make observing the experiment difficult. In such cases, ALT approaches provide higher than usual stress conditions for units/components. These tests are used for estimation of the lifetime of highly reliable components within an acceptable period (for more details Nelson [14] is recommended to the readers). Various types of ALT plans take part in reliability theory such as constant stress, step stress or progressive stress ALTs. These plans differ from each other depending on how to apply stresses to components. In a constant stress plan, the stress applied to a component does not vary with time. In contrast, there is a time point in step stress and more time point to increase stress levels in progressively stress. Studies on these different cases of ALTs take part in various studies by various authors.

Nelson [13] introduced the step-stress ALTs that allows test conditions to change during testing. Among step stress experiments, the cumulative exposure model (CEM) is one of the most useful and used models. A simple step stress model starts with initial low stress and if it does not fail in a predetermined time point, τ, the stress level is increased. Simple step stress models contain only one stress change point. The CEM defined by Nelson [13] for simple step-stress testing with stresses and is given as

$$F_0(t) = \begin{cases} F_1(t) & ,t \leq \tau \\ F_2(t-\tau+\tau') & ,t \geq \tau \end{cases} \tag{1.1}$$

where τ' (the equivalent start time) is the solution of $F_1(\tau) = F_2(\tau')$.

The ALT plans are considered for many different probability distributions by various authors. For instance; Miller and Nelson [12] considered optimum ALT plan under exponentially distributed lifetimes. Chung and Bai [4] studied ALT for log-normal lifetime distributions, Ebrahem and Al-Masri

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[6] considered ALT for log-logistic distribution, Ma and Meeker [11] explored ALT for log-location-scale distributions, Saxena et al. [16] considered ALT for Rayleigh distribution, Haghighi [10] considered ALT for an extension of the exponential distribution, Abbas and Firdos [1] studied ALT for Fréchet distribution. For more probability distribution, the literature list can be extended. On the other hand, ALT plans mostly are also considered for censored cases. Based on many different censoring schemes, ALT plans are considered for different probability models.
It is known that several probability models have been extensively used over the past decade in describing lifetime data. The exponential distribution is one of the major distributions for modeling lifetime datasets. For this purpose, many generalizations and modifications of exponential distribution take part in the literature. For example, generalized exponential distribution by Gupta and Kundu [9] and exponential-geometric distribution by Adamidis and Loukas [2] and etc. Length-biased distributions have great importance in reliability, biomedicine, and ecology among other distributions due to their greater flexibility in modeling data in different areas such as lifetime analysis, engineering, economics, finance, demography, actuarial and medical sciences (Akhter et al. [3]). Recently, Dara and Ahmed [5] proposed a new extension of exponential distribution denoted by "moment exponential distribution". Then, it is called as length-biased exponential (LBE) distribution by some authors.
The probability density (pdf) and distribution (cdf) function of the LBE distribution are given as

$$f(t) = \frac{t}{\theta^2} exp\left\{-\frac{t}{\theta}\right\}, \, x > 0, \, \theta > 0 \tag{1.2}$$

$$F(t) = 1 - \left[1 + \frac{t}{\theta}\right] exp\left\{-\frac{t}{\theta}\right\} \tag{1.3}$$

where θ is the scale parameter.

The LBE distribution has never been studied for any ALT plans before. In this study, we aimed to obtain parameter estimation of LBE distribution under simple step- stress cumulative exposure model. Maximum likelihood estimation (MLE) method is used to obtain point estimates and their credible intervals. For this purpose, in Section 2, we presented the model description. Following, MLE and approximate confidence intervals are obtained in Section 3. An optimization criterion to obtain optimum stress change time and its applications are given in Section 4. The simulation studies and a real data example are given to illustrate the theoretical outcomes in Sections 5 and 6, respectively.

2. Model Description

The lifetime of a test unit or component follows a LBE distribution under any constant stress. We assume that the scale parameter of the distribution is a log-linear function of stress. The following assumptions are provided for a LBE distributed lifetime units.

- Test procedure is done at stresses S_1 and $S_2 (S_1 < S_2)$ levels.
- Under any level of stress, the lifetime of a test unit follows a LBE distribution with the given cdf as

$$F_i(t) = 1 - \left[1 + \frac{t}{\theta_i}\right] exp\left\{-\frac{t}{\theta_i}\right\}$$

- The scale parameter θ_i is the log-linear function of stresses as $\log \theta_i = \beta_0 + \beta_1 S_i$ where $i = 1, 2$, β_0 and $\beta_1 (< 0)$ are unknown parameters depending on the nature of the product and the method of the test.
- All test units are independently and identically distributed variables from the LBE distribution.

• In this test, the cumulative exposure model which is defined by Nelson [14] for the simple step-stress testing with stresses S_1 and S_2 is used.
• n_i failure times $t_{ij}, j=1,2,\cdots,n_i$ of test units are observed under test operation at stress level $S_i, i=1,2$.
Based on the given assumptions above, length-biased exponential cumulative exposure (LBECE) model is given as follows. Firstly, the equivalent start time $\tau^{'}$ for the LBECE model which is the solution of $F_1(\tau)=F_2(\tau^{'})$ is equal to

$$\tau^{'}=\left(\frac{\theta_2}{\theta_1}\right)\tau$$

Then, by replacing $\tau^{'}$ in (1) the cdf of a test unit is obtained as

$$F_0(t)=\begin{cases} 1-\left[1+\frac{t}{\theta_1}\right]exp\left\{-\frac{t}{\theta_1}\right\} & ,t\leq\tau \\ 1-\left[1+\frac{t-\tau}{\theta_2}+\frac{\tau}{\theta_1}\right]exp\left\{-\frac{t-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} & ,t\geq\tau \end{cases} \tag{2.1}$$

and the pdf is given as

$$f_0(t)=\begin{cases} \frac{t}{\theta_1^2}exp\left\{-\frac{t}{\theta_1}\right\} & ,t\leq\tau \\ \frac{\theta_1 t+(\theta_2-\theta_1)\tau}{\theta_1\theta_2^2}exp\left\{-\frac{t-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} & ,t\geq\tau \end{cases} \tag{2.2}$$

3. Maximum Likelihood Estimation

This section considers obtaining MLEs of model parameters. Let t_{ij} denotes the observed failure time of a test component j under $i-$th stress level. Also, n_1 be the number of components failed at stress S_1 and n_2 at stress S_2. Corresponding likelihood function on the observed sample is given as

$$L(\theta)=\prod_{j=1}^{n_1}f_1(t_{1j},\theta)\prod_{j=1}^{n_2}f_2(t_{2j},\theta) \tag{3.1}$$

where $j=1,2,\cdots,n_i$ and $i=1,2$, $f_1(.)$ and $f_2(.)$ denotes the cases of the pdf due to the stress levels. By replacing Equation (2.2) in Equation (3.1) we obtain the likelihood function as

$$\begin{aligned} L(\theta_1,\theta_2)=\frac{1}{\theta_1^{2n_1+n_2}}\frac{1}{\theta_2^{2n_2}}\prod_{j=1}^{n_1}t_{1j}exp\left\{-\frac{t_{1j}}{\theta_1}\right\}\prod_{j=1}^{n_2}&\left(\theta_1 t_{2j}+(\theta_2-\theta_1)\tau\right) \\ &\times exp\left\{-\frac{t_{2j}-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right\} \end{aligned} \tag{3.2}$$

and the log-likelihood function is obtained as

$$\begin{aligned} \ell(\theta_1,\theta_2)=&-(2n_1+n_2)\log\theta_1-2n_2\log\theta_2+\sum_{j=1}^{n_1}\log(t_{1j})-\sum_{j=1}^{n_1}\frac{t_{1j}}{\theta_1} \\ &+\sum_{j=1}^{n_2}\log\left(\theta_1 t_{2j}+(\theta_2-\theta_1)\tau\right)-\sum_{j=1}^{n_2}\frac{t_{2j}-\tau}{\theta_2}-\frac{\tau n_2}{\theta_1} \end{aligned} \tag{3.3}$$

By replacing the relation $\log\theta_i=\beta_0+\beta_1 S_i$ in log-likelihood function, we obtain

$$\begin{aligned} \ell(\beta_0,\beta_1)=&-2n\beta_0-2\beta_1(n_1S_1+n_2S_2)-n_2(\beta_0+\beta_1S_1)+\sum_{j=1}^{n_1}\log(t_{1j}) \\ &-e^{-(\beta_0+\beta_1S_1)}\sum_{j=1}^{n_1}t_{1j}+\sum_{j=1}^{n_2}\log\left[e^{(\beta_0+\beta_1S_1)}(t_{2i}-\tau)+e^{(\beta_0+\beta_1S_2)}\tau\right] \\ &-e^{-(\beta_0+\beta_1S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)-n_2\tau e^{-(\beta_0+\beta_1S_1)} \end{aligned} \tag{3.4}$$

where $n=n_1+n_2$. To obtain the MLEs of the parameters, denoted by $\hat{\beta}_0$ and $\hat{\beta}_1$ we should equate the partial derivates of $\ell(\beta_0,\beta_1)$ to zero with respect to β_0 and β_1 respectively as given in the following

$$\frac{\partial\ell}{\partial\beta_0}=-2n-n_2+e^{-(\beta_0+\beta_1 S_1)}\sum_{j=1}^{n_1}t_{1j}+e^{-(\beta_0+\beta_1 S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)+n_2\tau e^{-(\beta_0+\beta_1 S_1)} \tag{3.5}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\beta_1}=&-2(n_1S_1+n_2S_2)-n_2S_1+S_1e^{-(\beta_0+\beta_1 S_1)}\sum_{j=1}^{n_1}t_{1j}+S_2e^{-(\beta_0+\beta_1 S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)\\&+S_1n_2\tau e^{-(\beta_0+\beta_1 S_1)}+\sum_{j=1}^{n_2}\frac{S_1e^{(\beta_0+\beta_1 S_1)}(t_{2j}-\tau)+S_2e^{(\beta_0+\beta_1 S_2)}\tau}{e^{(\beta_0+\beta_1 S_1)}(t_{2j}-\tau)+e^{(\beta_0+\beta_1 S_2)}\tau}\end{aligned} \tag{3.6}$$

These non-linear equations can not be solved analytically and some iterative methods are needed. Thus, approximate solutions of the system of these non-linear equations are the MLEs of the β_0 and β_1.

Approximate confidence intervals for MLEs of the parameters can be obtained by using the inverse of the asymptotic Fisher information matrix. The inverse Fisher information matrix is given as follows

$$F^{-1}=\begin{bmatrix}-E\left[\frac{\partial^2\ell}{\partial\beta_0^2}\right] & -E\left[\frac{\partial^2\ell}{\partial\beta_0\partial\beta_1}\right]\\ -E\left[\frac{\partial^2\ell}{\partial\beta_1\partial\beta_0}\right] & -E\left[\frac{\partial^2\ell}{\partial\beta_1^2}\right]\end{bmatrix}^{-1}_{(\beta_0,\beta_1)=(\hat{\beta}_0,\hat{\beta}_1)}=\begin{bmatrix}Var(\hat{\beta}_0) & Cov(\hat{\beta}_0\hat{\beta}_1)\\ Cov(\hat{\beta}_1\hat{\beta}_0) & Var(\hat{\beta}_1)\end{bmatrix}$$

where

$$\frac{\partial^2\ell}{\partial\beta_0^2}=-e^{-(\beta_0+\beta_1 S_1)}\sum_{j=1}^{n_1}t_{1j}-e^{-(\beta_0+\beta_1 S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)-n_2\tau e^{-(\beta_0+\beta_1 S_1)}$$

$$\frac{\partial^2\ell}{\partial\beta_0\partial\beta_1}=-S_1e^{-(\beta_0+\beta_1 S_1)}\sum_{j=1}^{n_1}t_{1j}-S_2e^{-(\beta_0+\beta_1 S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)-n_2S_1\tau e^{-(\beta_0+\beta_1 S_1)}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta_1^2}=&-S_1^2e^{-(\beta_0+\beta_1 S_1)}\sum_{j=1}^{n_1}t_{1j}-S_2^2e^{-(\beta_0+\beta_1 S_2)}\sum_{j=1}^{n_2}(t_{2j}-\tau)-n_2S_1^2\tau e^{-(\beta_0+\beta_1 S_1)}\\&+(S_1-S_2)^2e^{(\beta_0+\beta_1 S_1)}e^{(\beta_0+\beta_1 S_2)}\tau\sum_{j=1}^{n_2}\frac{t_{2j}-\tau}{\left[e^{(\beta_0+\beta_1 S_1)}(t_{2j}-\tau)+e^{(\beta_0+\beta_1 S_2)}\tau\right]^2}\end{aligned}$$

Thus,

$$-E\left[\frac{\partial^2\ell}{\partial\beta_0^2}\right]=n_1\gamma\big(3,\tau/\theta_1\big)+n_2\big[\Gamma(3,\tau/\theta_1)-(\tau/\theta_1)\Gamma(2,\tau/\theta_1)\big]+n_2\tau/\theta_1$$

$$-E\left[\frac{\partial^2\ell}{\partial\beta_0\partial\beta_1}\right]=S_1n_1\gamma\big(3,\tau/\theta_1\big)+S_2n_2\big[\Gamma(3,\tau/\theta_1)-(\tau/\theta_1)\Gamma(2,\tau/\theta_1)\big]+n_2S_1\tau/\theta_1$$

$$\begin{aligned}-E\left[\frac{\partial^2\ell}{\partial\beta_1^2}\right]=&S_1^2n_1\gamma\big(3,\tau/\theta_1\big)+S_2^2n_2\big[\Gamma(3,\tau/\theta_1)-(\tau/\theta_1)\Gamma(2,\tau/\theta_1)\big]+n_2S_1^2\tau/\theta_1\\&-\frac{(S_1-S_2)^2\tau n_2}{\theta_1}\left\{e^{-\tau/\theta_1}\left(1-\sqrt{\frac{\tau}{\theta_1}}W_{-\frac{1}{2},0}\big(\theta_1/\tau\big)\right)\right\}\end{aligned}$$

where $\theta_1 = e^{\beta_0+\beta_1 S_1}$ and $\theta_2 = e^{\beta_0+\beta_1 S_2}$. Also, $\gamma(a,b)$ denotes the lower incomplete gamma function and $\Gamma(a,b)$ denotes the upper incomplete gamma function as given in the following

$$\gamma(a,b) = \int_0^b t^{a-1}e^{-t}dt \quad \text{and} \quad \Gamma(a,b) = \int_b^\infty t^{a-1}e^{-t}dt$$

Further, the expression $W_{\lambda,\mu}(z)$ in $-E\left[\frac{\partial^2 \ell}{\partial \beta_1^2}\right]$ denotes the Whittaker functions and used for the solution of the integral $\int_{\tau/\theta_1}^\infty u^{-1}e^{-u}du$ (Gradshteyn and Ryzhik [8], Eq. 3.381.6, pg. 346).
It is knows that the MLEs under some regularity conditions are consistent and normally distributed (Godambe [7]). Thus, the a $100(1-\delta)\%$ asymptotic confidence intervals of β_0 and β_1 can be constructed by

$$\hat{\beta}_0 \mp Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\beta}_0)} \text{ and } \hat{\beta}_1 \mp Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\beta}_1)}$$

where Z_δ is 100 δth percentile of standard normal distribution $N(0,1)$.

4. Optimal Test Plan

The optimal test plan emphasizes an optimum stress change timepoint τ which determines the lifetime of lower stress level. In simple step stress accelerated life test plan, optimal stress change time is determined by minimizing the asymptotic variance of MLEs of a given log $100p$-th percentile at the design stress level S_0 (Ebrahem and Masri [6]). The $100p$-th percentile of the length-biased exponential distribution, denoted by $Q_p(S_0)$ at the design stress level S_0 is obtained as

$$Q_p(S_0) = -e^{\beta_0+\beta_1 S_0}\left[1 + W_{-1}\left(\frac{p-1}{e}\right)\right]$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function (i.e., the solution of the equation $W(z)e^{W(z)} = z$). The asymptotic variance (AV) of the MLEs of the log $100p$-th percentile at the design stress level can be obtained by using

$$\begin{aligned} AV\left\{\log\left[\hat{Q}_p(S_0)\right]\right\} &= AV\left\{-\hat{\beta}_0 - \hat{\beta}_1 S_0 - \log\left[1 + W_{-1}\left(\frac{p-1}{e}\right)\right]\right\} \\ &= H\Sigma H^T \end{aligned}$$

where

$$H = \left[\frac{\partial \log\left[\hat{Q}_p(S_0)\right]}{\partial \hat{\beta}_0} \quad \frac{\partial \log\left[\hat{Q}_p(S_0)\right]}{\partial \hat{\beta}_1}\right] = \begin{bmatrix} -1 & -S_0 \end{bmatrix}$$

and Σ is the variance-covariance matrix which is obtained by using the inverse of the Fisher information matrix. Thus, the asymptotic variance of the MLEs of the log $100p$-th percentile at the design stress level can be obtained as follows

$$AV\left\{\log\left[\hat{Q}_p(S_0)\right\} = \Sigma_{11} + 2S_0\Sigma_{12} + S_0^2\Sigma_{22}$$

The Σ_{ij} values are already given in Section 3. Consequently, the optimal stress change time, denoted by τ^* is the τ value that minimizing the $AV\left\{\log\left[\hat{Q}_p(S_0)\right\}$. The NMinimize option of Mathematica 11 is a very useful tool to obtain the optimal τ value that minimizing the asymptotic variance.

We performed a small numerical study to observe the existence and evaluate the optimal stress change time with minimizing $AV\left\{\log\left[\hat{Q}_p(S_0)\right\}$. For given values of design stress level S_0 and parameters β_0 and β_1, different combinations of two levels of stress S_1 and S_2 as ($S_1 < S_2$), we obtained the optimal stress-change times which provide variance optimality and reported in Table 1. We used the NMinimize option of Mathematica 11 for calculations.

TABLE 1. Optimal time, τ^*, changing stress for $s_0 = 0.25, \beta_0 = 2.5$ and $\beta_1 = -1.5$

	s_2				
s_1	1.25	1.50	1.75	2.00	2.50
0.30	5.55	6.46	7.32	8.15	9.74
0.40	4.38	5.17	5.92	6.63	7.99
0.50	3.44	4.12	4.77	5.38	6.54
0.75	1.78	2.28	2.74	3.16	3.96
1.00	0.80	1.19	1.52	1.82	2.37

It is observed that the optimal stress change time increases as parallel to increasing S_2 stress level when S_1 is fixed. On the other hand, decreases are observed on optimal stress change time in parallel to increasing on S_1 stress level when S_2 is fixed. These results are reasonable and acceptable.

5. Simulation Study

In this section, we provide a simulation study to illustrate the theoretical outcomes. We performed simulations and obtained the MLEs of the parameters and their corresponding confidence intervals.

We take the parameter values as $\beta_0 = 2.5$ and $\beta_1 = -1.5$, different stress levels as $(S_1, S_2) = (0.50, 1.50)$, $(S_1, S_2) = (0.50, 2.00)$, $(S_1, S_2) = (0.75, 1.50)$ and $(S_1, S_2) = (0.75, 2.00)$ and the stress change times τ as $4.12, 5.38, 2.28$ and 3.16, respectively. We consider different sample sizes as $n = 25, 50, 100, 250, 500$.

We first generate random samples of the LBECE model from the cdf in Eq. (2.1) with size n. Then, we generate 10 000 samples with each size n. We use R software (Team R.C. [15]) to perform this simulation. We obtained the maximum likelihood estimates of β_0 and β_1 with their mean squared errors (MSE), relative errors (RE), the %95 approximate confidence intervals (CI) and coverage probabilities (CP). We presented simulation results in Tables 2,3,4 and 5. The MSEs and REs for an arbitrary parameter can be obtained as follows

$$MSE_\xi = E_\xi[(\hat{\xi} - \xi)^2] \quad \text{and} \quad RE_\xi = \frac{|\xi - \hat{\xi}|}{\xi} 100\%$$

We observed that both estimates are obtained quite close to their actual values. In parallel to increase sample sizes, estimations are almost same with the actual values. As expected, MSEs and REs are getting smaller at the same time. Lengths of the approximate confidence intervals also decrease with increasing sample sizes. Coverage probabilities of the CIs have quite close to their actual value 0.95. In all cases of stress levels, consistent results are obtained. It is known that using the optimal stress change time makes estimations better than using arbitrary stress change times. However, it is clearly seen that differences between estimates for our examples are not very large. Of course, various combinations can be worth trying according to the needs of many engineering problems.

TABLE 2. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = -1.5, \tau = 4.12$ and $S_1 = 0.50, S_2 = 1.50$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.55967	0.00356	2.38671	1.58066	3.53867	96.33
	β_1	-1.55907	0.00349	3.93796	-2.36209	-0.75605	96.65
50	β_0	2.55195	0.00270	2.07806	1.87081	3.23310	96.48
	β_1	-1.54341	0.00188	2.89384	-2.10417	-0.98265	96.43
100	β_0	2.52732	0.00075	1.09283	2.05321	3.00143	96.29
	β_1	-1.52281	0.00052	1.52069	-1.91407	-1.13155	96.10
250	β_0	2.50917	0.00008	0.36662	2.21222	2.80611	95.54
	β_1	-1.50768	0.00006	0.51176	-1.75317	-1.26218	95.93
500	β_0	2.50381	0.00001	0.15248	2.2945	2.71312	95.88
	β_1	-1.50337	0.00001	0.22498	-1.67651	-1.33024	96.04

TABLE 3. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = -1.5, \tau = 5.38$ and $S_1 = 0.50, S_2 = 2$).

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.55861	0.00343	2.34421	1.81832	3.29889	97.01
	β_1	-1.54283	0.00183	2.85562	-2.04983	-1.03584	97.17
50	β_0	2.53178	0.00101	1.27110	2.01902	3.04453	96.73
	β_1	-1.52261	0.00051	1.50708	-1.87412	-1.17109	96.63
100	β_0	2.51489	0.00022	0.59551	2.15599	2.87379	96.25
	β_1	-1.51008	0.00010	0.67184	-1.75649	-1.26366	96.65
250	β_0	2.50750	0.00006	0.30019	2.28171	2.73330	96.11
	β_1	-1.50571	0.00003	0.38041	-1.66080	-1.35061	96.54
500	β_0	2.50312	0.00001	0.12484	2.3438	2.66244	95.87
	β_1	-1.50195	0.0000*	0.12979	-1.61142	-1.39247	96.33

(* denotes smaller values than $\times 10^{-5}$)

TABLE 4. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = 1.5, \tau = 2.28$ and $S_1 = 0.75, S_2 = 1.5$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.61059	0.01223	4.42372	1.06294	4.15825	97.85
	β_1	-1.58820	0.00778	5.88030	-2.76213	-0.41428	98.51
50	β_0	2.61029	0.01216	4.41170	1.52793	3.69266	96.98
	β_1	-1.58200	0.00672	5.46663	-2.40209	-0.76191	96.96
100	β_0	2.54484	0.00201	1.79351	1.80052	3.28915	95.86
	β_1	-1.53358	0.00113	2.23876	-2.09971	-0.96746	95.96
250	β_0	2.51936	0.00037	0.77450	2.05444	2.98428	95.88
	β_1	-1.51444	0.00021	0.96248	-1.86853	-1.16034	96.11
500	β_0	2.50866	0.00007	0.34635	2.18138	2.83594	95.71
	β_1	-1.50673	0.00005	0.44882	-1.75615	-1.25732	95.85

TABLE 5. The MLEs, MSEs, REs and approximate CIs of β_0 and β_1 based on 3000 replications.($\beta_0 = 2.5, \beta_1 = 1.5, \tau = 3.16$ and $S_1 = 0.75, S_2 = 2$)

n	Parameter	MLE	MSE	RE	Lower CI	Upper CI	CP
25	β_0	2.59675	0.00936	3.87000	1.59086	3.60264	98.14
	β_1	-1.56215	0.00386	4.14333	-2.19076	-0.93354	97.94
50	β_0	2.55287	0.00280	2.11473	1.85966	3.24607	96.58
	β_1	-1.53325	0.00111	2.21657	-1.96787	-1.09863	96.40
100	β_0	2.52397	0.00057	0.95869	2.04077	3.00716	96.14
	β_1	-1.51480	0.00022	0.98642	-1.81849	-1.21110	9627
250	β_0	2.51157	0.00013	0.46271	2.20816	2.81497	95.96
	β_1	-1.50817	0.00007	0.54448	-1.69906	-1.31727	96.27
500	β_0	2.50405	0.00002	0.16189	2.29010	2.71800	95.56
	β_1	-1.50243	0.00001	0.16186	-1.63711	-1.36775	96.02

6. Real Data Example

In this section, a real data set is presented to illustrate the theoretical outcomes. We used the data set of the amount of annual rainfall (in inches, from 1984 to 2008) recorded at the Los Angeles Civic Center that is available on the website of Los Angeles Almanac: www.laalmanac.com. Recently, Tarvirdizade and Ahmadpour [17] were used this data set in the reliability context. The data set is given as in the following;

$$12.82, 17.86, 7.66, 2.48, 8.08, 7.35, 11.99, 21.00, 7.36, 8.11, 24.35, 12.44, 12.40$$
$$31.01, 9.09, 11.57, 17.94, 4.42, 16.42, 9.25, 37.96, 13.19, 3.21, 13.53, 9.08$$

We fit this data set to LBE distribution and we obtain MLE of the scale parameter as $\hat{\theta} = 6.6114$. The corresponding Kolmogorov-Smirnov test statistics and associated p-values are obtained 0.16 and 0.915. Therefore, we can reject the null hypothesis that this dataset comes from the LBE distribution. Also, the estimated density and the empirical cdf plots support these observations (Figure 1). Then, we considered different stress levels S_1, S_2 and stress change times τ to exemplify our findings.

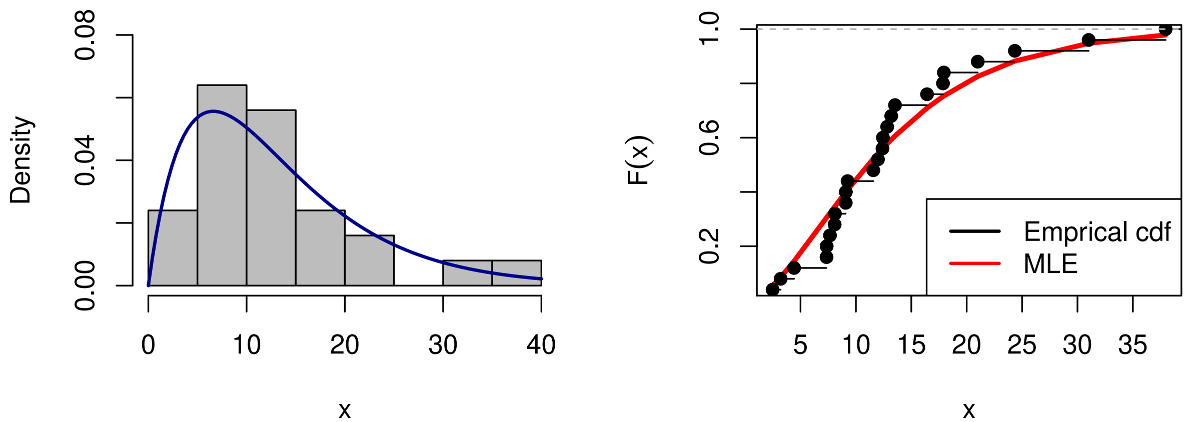

FIGURE 1. Estimated density and empirical cdf for real-data example fitted by the LBE distribution.

TABLE 6. Parameter estimates and their approximate confidence intervals for the real dataset.

S_1	S_2	τ	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\theta}_2$
0.5	1.5	7.5	2.4429 (1.7592;3.1266)	-0.5342 (-1.1007;0.0324)	8.8096	5.1639
		12.5	1.9753 (1.4205;2.5302)	-0.1141 (-0.7407;0.5125)	6.8092	6.0749
		15	1.8768 (1.3144;2.4392)	-0.0172 (-0.6987;0.7331)	6.5890	6.7034
0.5	2	7.5	2.3543 (1.7556;2.9529)	-0.3564 (-0.7341; 0.0213)	8.8118	5.1629
		12.5	1.9564 (1.4891;2.4236)	-0.0764 (-0.4941;0.3412)	6.8085	6.0711
		15	1.8796 (1.4172;2.3419)	-0.0118 (-0.4656;0.4892)	6.5895	6.7076
0.75	1.5	7.5	2.7112 (1.7614;3.6610)	-0.7131 (-1.4686;0.0423)	8.8141	5.1629
		12.5	2.0313 (1.1905;2.8722)	-0.1513 (-0.9868;0.6842)	6.8063	6.0761
		15	1.8681 (0.9765;2.7597)	-0.0233 (-0.9314;0.9780)	6.5900	6.7061
0.75	1.5	7.5	2.4974 (1.7614;3.2334)	-0.4279 (-0.8812;0.0254)	8.8147	5.1631
		12.5	1.9866 (1.3768;2.5964)	-0.0916 (-0.5927;0.4096)	6.8071	6.0710
		15	1.8758 (1.2502;2.5014)	-0.0129 (-0.5596;0.5854)	6.5897	6.6969

It is observed that estimates are getting closer to its MLE value under the normal conditions when stress change time equal to 15 for all stress levels. Similarly, estimations are worsening with decreasing stress change time. The mean of this real data set is 13.22. We may conclude that close values of stress change time to the mean of the sample help to obtain better estimates for these determined stress levels.

7. Conclusions

The importance of length-biased distributions in especially reliability studies and their greater flexibility in modeling data in different areas such as lifetime data analysis, engineering, actuarial etc. inspired to consider accelerated life test plans under this distribution. Therefore, the length-biased exponential distribution is used which is one of the most used length-biased distributions in the literature. As a first attempt based on LBE distribution, we considered a cumulative exposure model under simple step stress accelerated life test.

We see that maximum likelihood estimations are obtained using some iterative methods as in many inference problems. Therefore, approximate confidence intervals are used in place of exact ones. Nevertheless, performances of the estimator and its confidence intervals are quite well performed. In addition, different combinations of the stress levels are compared with simulations and real data examples. We see that for the high stress with fixed level, results for lower stress level gives better result than higher ones. Similarly, results for higher stress level gives better result in the case of fixed lower stress level. Even so, there are not very important differences between estimates for our combinations. We also obtained optimal stress change times to construct the best ALT plans for this cumulative exposure model. All simulations and real data studies were applied according to this optimal plan.

As open problems, this ALT plan can be extended for cumulative exposure models under multiple stress levels. Also, censored cases can be considered for these plans.

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