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Research Article

Elementary students' functional thinking: From recursive to correspondence

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Abstract

This study aims to identify elementary students' functional thinking processes in solving pattern problems. Previous studies showed that elementary students' functional thinking still often experience errors in solving pattern problems. The study of the functional thinking process in solving pattern problems is a fundamental key as a solution to find out the strengths and weaknesses of elementary school students, so that they are better prepared in generalizing relationships, representing and analyzing function behavior in advanced algebra classes. This study used a descriptive qualitative approach with a case study method. Participants of study was sixty-five elementary students who had not yet received generalization patterns material. The instruments were tasks and interview guidelines. Based on the task results, students who had correct answers were chosen using purposive sampling to be given an in-depth interview. The finding indicated that elementary students are able to think functionally in different ways. Students' functional thinking begins with recursive thinking in the pre-finding formula in the entry stage. Students find the formula by corresponding thinking in the attack stage. Finally, students use the formula to get inverse in the review stage.

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Introduction

Algebra is important in mathematics education (Kieran, 2007). Teaching algebra in mathematics education aims for students to have basic abilities in algebraic reasoning. Algebraic reasoning is a process of generalizing mathematical ideas from a set of specific examples, establish these generalizations through a discourse of argumentation, and express them in an increasingly formal and age-appropriate manner (Blanton & Kaput, 2005). Classify of algebraic thinking in three categories: arithmetic generalizations, functional thinking, and generalization and justification (Blanton & Kaput, 2005).

Algebra is one of the most important subjects, but many students experience difficulties (Kieran, 2007). A study conducted by Wijaya revealed that 14% of 367 students experienced errors in the mathematics process for the algebraic domain (Wijaya et al. 2014). This result is reinforced by another study, which revealed that algebra and calculation problems in PISA were more difficult for Indonesian students to solve than questions about numbers, geometry, and data (Stacey, 2011). The results of another study revealed that students experienced a pseudo error in solving algebra problems (Subanji & Nusantara, 2013). Furthermore, the algebraic abilities of Indonesian students are significantly lower than in them who are in other Southeast Asian countries such as Thailand, Singapore, and Malaysia (Jupri & Drijvers, 2014). The results of the study emphasize the importance of increasing students' abilities in algebra.

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One of algebraic thinking is functional thinking (Blanton & Kaput, 2005; Smith, 2008). Functional thinking is fundamental for algebraic thinking (Eisenmann, 2009; Kaiser & Willander, 2005; Lichti, 2018; Roth, 2019; Stephens et al. 2017; Tanışlı, 2011). Functional thinking is a central topic and important in mathematics. It is key for algebraic thinking because it involves a generalization of how quantities are related (Tanışlı, 2011). Functional thinking is fundamental to algebra and calculus (Wilkie, 2015). In line with this, functional thinking can serve as an important point of entry into algebra because it involves generalizing relationships between quantities; representing those relationships, or functions, in multiple ways using natural language, formal algebraic notation, tables, and graphs; and reasoning fluently with these representations in order to interpret and predict function behaviour (Stephens et al. 2017).

Many students have difficulty in functional thinking (Bush & Karp, 2013; Huntley et al. 2007; Jupri & Drijvers, 2014; Knuth, 2000; Wijaya et al. 2014). Some of these difficulties include 14% of 367 students experiencing errors in mathematical process for the algebraic domain (Wijaya et al. 2014), lack of understanding of functional representation (Bush & Karp, 2013), it can not interpret symbol manipulation (Huntley et al. 2007), difficulties in understanding algebra expression and variable (Jupri & Drijvers, 2014), difficulties in generalization and justification (Lannin, 2005).

Functional thinking is generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior (Blanton et al. 2011). Functional thinking as representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances (Smith, 2008). Functional thinking define as the process of building, describing, and reasoning with and about functions (Stephens et al. 2017). Functional thinking is closely related to the function concept (Blanton, 2008). From some of these definitions, functional thinking can be interpreted as thinking in generalizing two or more quantities with a function representation both verbally and symbolically.

Functional thinking consists of 3 types, recursive patterning, covariational thinking, and correspondence relationship. Recursive patterning involves finding variation within a sequence of values. Covariational thinking is based on analyzing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function's description (e.g., "as x increases by one, y increases by three"). Correspondence relationship is based on identifying a correlation between variables (e.g., "y is three times x plus 2") (Confrey & Smith, 1991; Smith, 2008).

Some mathematical education studies examine functional thinking (Blanton & Kaput, 2004, 2005; Stephens et al. 2017; Tanışlı, 2011; Warren et al. 2006; Wilkie & Clarke, 2016). Elementary students are capable not only of developing functional thinking but also of communicating their thinking both verbally and symbolically (Warren et al. 2006). Students were able to think functionally at the kindergarten level co-variationally and were able to think functionally as a correspondent in the 1st grade (Blanton & Kaput, 2004, 2005). Another research characterized functional thinking as (a) symbolizing quantities and operating with symbolized expression, (b) representing data graphically, (c) finding a functional relationship, (d) predicting unknown state using known data, and (e) identifying and describing numerical and geometrical pattern (Blanton & Kaput, 2005).

Five-grader students thought about covariation while working with the linear function tables (Tanışlı, 2011). Another study found that (a) students initially used a recursive approach and looked for a recursive pattern while they were investigating the function tables, (b) students found correspondence relationship with co-variational thinking and generalize with different ways, (c) the relationships between two quantities were defined as both multiplicative and additive relations by the students, (d) when the students were working on the function tables with general forms by $y = 2x - a$ dan $y = 3x - a$, they had more difficulties in the solution process than on the tables with $y = 2x + a$ dan $y = 3x + a$.

Three types and ten levels of students sophistication in functional thinking (Stephens et al. 2017), (a) type Variational Thinking with Recursive Pattern-Particular (L1), Recursive Pattern-General (L2); (b) type Covariational Thinking with Covariation Relationship (L3) and (c) Type Correspondence Thinking with Single Instantiation (L4), Functional-Particular (L5), Functional-Basic (L6), Functional-Emergent in a variable (L7), Functional-emergent in words (L8), Functional-condensed in variables (L9), and Functional-condensed in words (L10).

Four types of visual structure in functional thinking (figure 1) (a) type 1 as the same structure as the previous day's plant with three additional leaves, one on each stem (Wilkie & Clarke, 2016). This type based on a recursive generalization (b) type 2 as three stems, each with the same number of leaves as the day number and one central leaf. This type-based on correspondence generalization, (c) type 3 as One horizontal stem with twice the number of leaves as the day number and one extra, and a vertical stem with the same number of leaves as the day number. This type has correspondence generalization as a basic. (d) Type 4, which has two horizontal stems with the same number of leaves as the day number and a vertical stem with one extra leaf, is based on correspondence generalization.

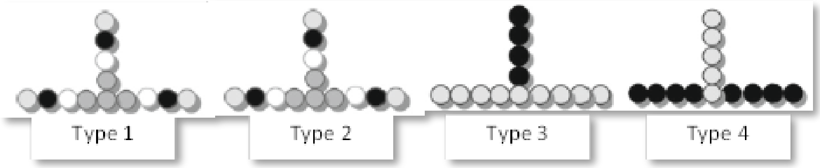


Figure 1.
Visual Structure Students in Functional Thinking (Wilkie & Clarke, 2016)

Students can think functionally at elementary school (Blanton & Kaput, 2004, 2005; Blanton et al. 2011; Warren et al. 2006; Warren & Cooper, 2005). At the kindergarten, students were able to think variationally, think correspondence in first grade, and symbolized in third-grade (Blanton & Kaput, 2004). In fourth-grade, students were able to develop functional thinking verbally and symbolically (Warren et al. 2006).

Competence in understanding patterns, relationships, and functions as a basis for the development of functional thinking is taught from grade 3 to grade 5 in elementary schools (NCTM, 2000). Different in the Indonesian curriculum (K-13) (Amir et al. 2019; Kemendikbud, 2016), these competencies began to be taught in junior high schools (secondary school). Mathematical curriculum for elementary schools in Indonesia emphasizes the strengthening of numbers, geometry, and data competencies, but it is suspected that elementary students in Indonesia are able to think functional when given a problem.

Problem of Study

Researchers guess that elementary students can think functionally using different processes in solving mathematics problems. It is how students generalize the relationship between covarying quantities and represent the generalization. The process that occurs is different depending on the type of functional thinking, namely from recursive to correspondence. However, based on the identification of literature conducted by researchers, studies have not been found that identify the thinking process in terms of the functional thinking of differences in type. Functional thinking processes generally consist of the stages of entry, attack, review. Thus, the research problem of this study is to identify elementary students' functional processes of thinking in solving pattern problems. The sub-research problems are formulated as follows.

- How the students functional thinking types in solving pattern problems?
- How the students' functional thinking process at the entry stage in solving pattern problems?
- How the students' functional thinking process at the attack stage in solving pattern problems?
- How the students' functional thinking process at the review stage in solving problem patterns

Method

Research Design

This study identified elementary students' functional thinking processes in solving pattern problems. The data was acquired from the students' answers in solving functional thinking problems and in-depth interviews. This study applied a descriptive qualitative approach with a case study method; this method allows searching in a selected subject in detail (Cohen et al. 2000). In a qualitative approach, problems explored to find in-depth understanding (Creswell, 2012). This approach is carried out to identify the processes used by students in functional thinking. Students Functional thinking in this study observed from students' relationship generalizing process between two quantities, represent generalizations and use generalizations in inverse.

Research Sample

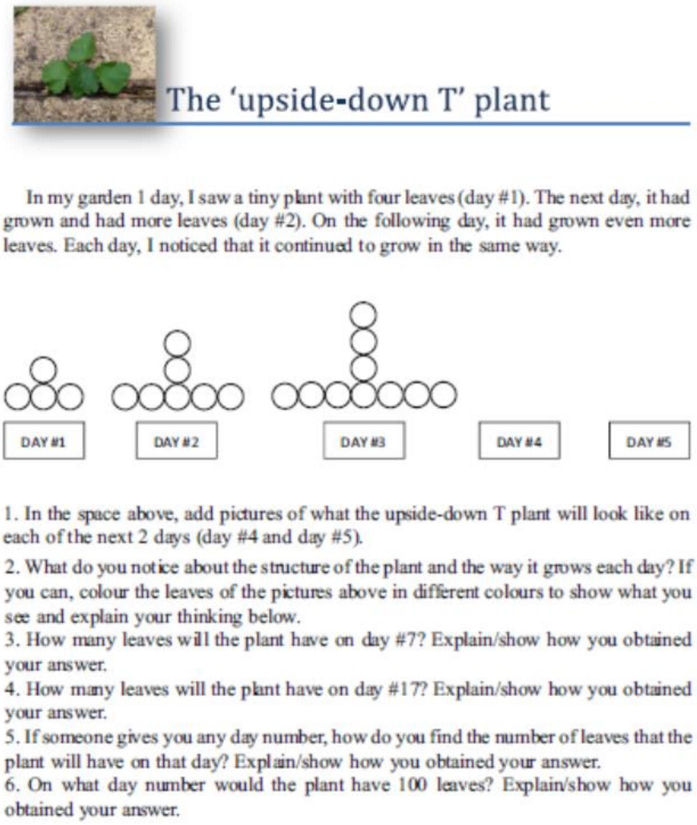
The participants in this study were 65 elementary students in Mataram, Lombok, Western Nusa Tenggara, Indonesia. A functional thinking task was given to sixty-five students and then chosen students with purpose sampling for interviewed one student who had the right answer and used recursive and correspondence thinking (Fraenkel et al. 2012). The interview was conducted to obtain in-depth data about the students' processes in functional thinking.

Instrument and Procedure

An instrument used in this study was a task and interview protocol. A task used in this study was a contextual task about a pattern to measure students' functional thinking. It is adapted from Wilkie and Clarke (Figure 2) (Wilkie & Clarke, 2016). The task consists of the growth context of leaves number from the first day to the third day, and then students are asked to determine the number of leaves on the 4th day, 5th day, 7th day, and 17th day. In addition, students were also asked to determine the relationship between days and leave numbers. Interview protocol used in this study contains questions to students about steps taken by students in solving these problems.

Before used, an instrument was analyzed to check validity and reliability. Checking the validity of the instrument was done by expert validity through interview content analysis while checking the reliability of the instrument was done by small-scale trials to students. Content analysis interviews were conducted by two mathematical education specialists to assess the suitability of content, construction, and language. The results of the content analysis interview were 93%, then, through a series of revisions to expert suggestions. So the instrument is declared valid and could be used for research. Checking the reliability of the instrument was tested for ten elementary students and analyzed for reliability by Alpha Cronbach criteria. From the Cronbach's Alpha test results, the obtained value of α is 0.70 that meets the high criteria. The high criteria of Cronbach's Alpha mean that an instrument was reliable.

This study was conducted by providing functional thinking tasks to 65 elementary students. Results of the 65 student answers are then examined and analyzed to obtain true and false student data. Students who have the right answers are grouped according to the different processes they used. Then one student was selected from each group with different processes for in-depth interviews. The task results and interview results are analyzed and then used to identify students' processes in functional thinking.


The 'upside-down T' plant

In my garden 1 day, I saw a tiny plant with four leaves (day #1). The next day, it had grown and had more leaves (day #2). On the following day, it had grown even more leaves. Each day, I noticed that it continued to grow in the same way.

DAY #1 DAY #2 DAY #3 DAY #4 DAY #5

1. In the space above, add pictures of what the upside-down T plant will look like on each of the next 2 days (day #4 and day #5).
2. What do you notice about the structure of the plant and the way it grows each day? If you can, colour the leaves of the pictures above in different colours to show what you see and explain your thinking below.
3. How many leaves will the plant have on day #7? Explain/show how you obtained your answer.
4. How many leaves will the plant have on day #17? Explain/show how you obtained your answer.
5. If someone gives you any day number, how do you find the number of leaves that the plant will have on that day? Explain/show how you obtained your answer.
6. On what day number would the plant have 100 leaves? Explain/show how you obtained your answer.

Figure 2.
Functional Thinking Task Developed by Wilkie and Clarke (2020)

Data Analysis

The data obtained from task was examined and analyzed with: (a) thinking stage consists of Entry, Attack, and Review (Mason et al. 2010), (b) generalizing process consist of relating, searching and extending (Ellis, 2007), and (c) functional thinking types consist of recursive pattern, covariation relationship and correspondence relationship (Confrey & Smith, 1991).

Table 1.

Mason, Ellis, Smith Data Analysis

<table>
<tr><th>Type/model</th><th>Mason thinking stage</th><th>Ellis generalize process</th><th>Activity</th><th>Example</th></tr>
<tr><td rowspan="3">Recursive Pattern: identify variation in a single sequence of values, indicating how to obtain a number in a sequence given the previous number or numbers</td><td>Entry</td><td>Relating</td><td rowspan="3">- Searching for information from a problem like a picture or an object
- Try to find out what is known and what asked in the problems
- Determine the value of a quantity by looking at changes from the previous value
- Generalize relationship two varying quantities by looking at difference one number with the previous number in one quantity.
- Represent relationship two varying quantities by a verbal, symbolic, table, or graphic.
- Determine inverse by summing different recursively.</td><td rowspan="3"><table><tr><td>x</td><td>y</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>6</td></tr><tr><td>4</td><td>p</td></tr></table>- Found p by eight because knowing the next number pattern increase by 2</td></tr>
<tr><td>Attack</td><td>Searching</td></tr>
<tr><td>Review</td><td>Extending</td></tr>
<tr><td rowspan="3">Co-variation Relationship: identify how two quantities vary concerning each other as an expression of the change in one quantity given a (unit) change in the related quantity</td><td>Entry</td><td>Relating</td><td rowspan="3">- Searching for information from a problem like a picture or an object
- Try to find out what is known and what asked in the problems
- Determine the value of a quantity by looking at changes in the first quantity, followed by changes in the second quantity.
- Generalize relationship two varying quantities by looking at changes in the first quantity, followed by changes in the second quantity.
- Represent relationship two varying quantities by a verbal, symbolic, table, or graphic
- Determine inverse by summing different recursively</td><td rowspan="3"><table><tr><td>x</td><td>y</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>6</td></tr><tr><td>4</td><td>p</td></tr></table>- Found p by eight because of knowing pattern "x increase by 1, y increase by 2."</td></tr>
<tr><td>Attack</td><td>Searching</td></tr>
<tr><td>Review</td><td>Extending</td></tr>
<tr><td rowspan="3">Correspondence relationship: correlation between corresponding pairs of independent and dependent variables typically expressed as a function rule</td><td>Entry</td><td>Relating</td><td rowspan="3">- Searching for information from a problem like a picture or an object
- Try to find out what is known and what asked in the problems
- Determine the value of a quantity by looking at correspondence relationship two varying quantities.
- Generalize relationship two varying quantities by looking at correspondence relationship.
- Represent relationship two varying quantities by a verbal, symbolic, table, or graphic
- Determine inverse using formula or correspondent rule found earlier.</td><td rowspan="3"><table><tr><td>x</td><td>y</td></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>4</td></tr><tr><td>3</td><td>6</td></tr><tr><td>4</td><td>p</td></tr></table>- Found 8 by knowing correspondence relationship "y equal 2 times x (y = 2x)"</td></tr>
<tr><td>Attack</td><td>Searching</td></tr>
<tr><td>Review</td><td>Extending</td></tr>
</table>

Data analysis employed Milles and Huberman's descriptive qualitative method, which consists of data reduction, data presentation, verification, and conclusion (Miles & Huberman, 1994). Data reduction produced a transcript from students' tasks answer, which was classified according to the processes students used, and examine in-depth with an interview — data presentation by describing students' processes in functional thinking. Conclusions are based on the results of identifying students' processes in functional thinking.

Results and Discussion

This section presents the elementary students' functional thinking in solving pattern problems. Functional thinking is based on students' process in solving problems of generalizing relationships two quantities, representation, and reasoning in analyze function behavior. More ever, it is based on the thinking stage consist of entry, attack, and review (Mason et al. 2010); and generalization action consists of relating, searching, and extending (Ellis, 2007).

Functional Thinking Types

Based on the task answer given to sixty-five students, it is found that ten students have the right answer, while fifty-five other students have an incomplete and false answer. Ten students have right answer to analyze its found two models consists of a recursive pattern and correspondence relationship (Confrey & Smith, 1991).

Table 2.
Students Answer Criteria

Right/wrong	Wrong answer	Incomplete answer	Right answer
Numbers	45	10	10

Table 3.
Functional Thinking Types

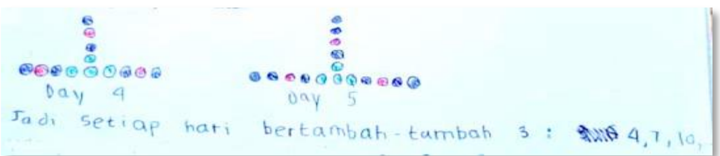
Student right answer	Functional thinking types	Number
10	Recursive pattern	8
	Correspondence relationship	2

Based on the students' answer, we found two student used recursive thinking firstly and correspondence thinking finally. To found more information about how functional thinking this student, the research gives an interview in dept. Recursive to correspondence symbolic rule in functional thinking include relationship generalizing process of days (day#) and leave number initially recursively, but after going through the process, generalizing process is done by correspondence, representing generalization result symbolically, and reasoning in determine inverse by using result symbolic generalization (formula). This rule is analyzed in three-stages, entry, attack, and review stage.

Entry Stage

First, in the entry stage, students find information by reading problems, searching for information from leaf drawing contained in problems then understand the questions in the problems. Information obtained by students in reading problems is about leaf pattern whose increase every day and has a "T" upside-down form. Students suppose the pattern is that leaves number increase by three leaves, i.e., one leaf in each branch (there are three branches). To understand the problems, students read the problems they understand in detail what the question asked about.

Students do the relating process by relating information conducted in problems. Students did the relating process is relating process by object consist of form and properties. In relating object by properties, students relates object (leaf) properties at first, second and third day with leaves number every day. In relating object by form, students observe form or leave drawing that has the same shape ("T" upside-down) but has different numbers.



Translation
So, the leave increase by 3 every day: 4, 7, 10,..

Figure 3.
Recursive Thinking to Find Leaf Number in Next Day

Attack Stage

Second, at the attack stage, students plan a strategy used to solve problems. From the students' observations at the entry stage, students find that the leaves increase by three every day and the increment of one leaf at each branch. Students draw leaves on 4th day, first, draw leaves on 1st day and give a green color to each leaf, then add one leaf to each branch (there are three branches) and give a blue color to get a leaf picture on 2th day. Second, add one more leaf to each branch by giving a red color to get leaves on 3rd day. Finally, add one more leaf to each branch by giving a blue color to get leaves on 4th day. The same thing is done by students to draw leaves on the 5th day by adding three leaves and giving color to each leaf growth.

Students carry out the generalizing process of searching some patterns by finding the different number of leaves on an adjacent day and observing whether the difference is the same or not. From searching for some patterns process, students found that each day, the leaves increased by three. Furthermore, students carry out extending the generalizing process by continuing by knowing leaves number increased by three every day. Students continued by adding three leaves as many as the specified number of days. In extending by a continuing process, students conduct a generalization process in correspondence in which are able to find the relationship between days and leaf number. Students understand leaf patterns from the first, second, and third days by doing correspondence.

$Day\#1 = 4\ leaves = 1 + 3 \times 1$

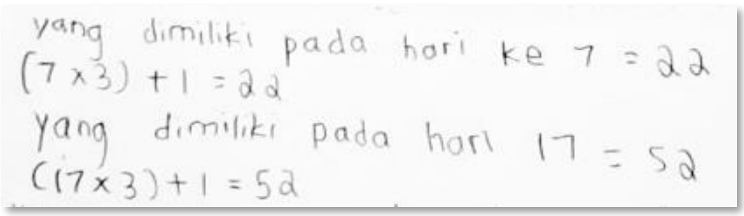
$Day\#2 = 7\ leaves = 1 + 3 \times 2$

$Day\#3 = 10\ leaves = 1 + 3 \times 3$

.

.

$Day\#7 = 22\ leaves = 1 + 3 \times 7$



Translation
Leave number at 7th day = 22
22 come from (7 x3) + 1 = 22
Leave number at 17th day = 52
52 come from (17 x 3) + 1 = 52

Figure 4.
Correspondence Thinking to Find Formula for Leaf Number

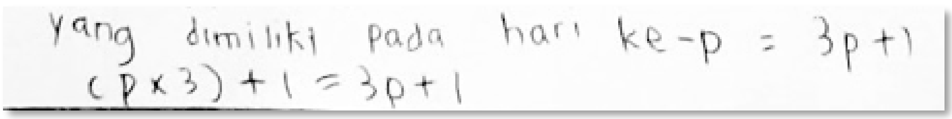
Students represent relationship generalization of days and leaf numbers symbolically. It is by the number of the leaf = 3p + 1. The representation showed that students used "correspondence thinking" formally. Students also determined inverse using rule or formula that they found at the generalizing process. First, they began with supposing for a specified day with variable h. Second, they wrote a rule or formula that they had found before for 100 leaves. Third, they subtracted both segments by one to get 3h = 99. Finally, students divided both segments by three to get h = 33.

$suppose, day = h$

$3h + 1 = 100$

$3h = 99 \quad substract\ by\ -1$

$h = 33 \quad devide\ by\ 3$

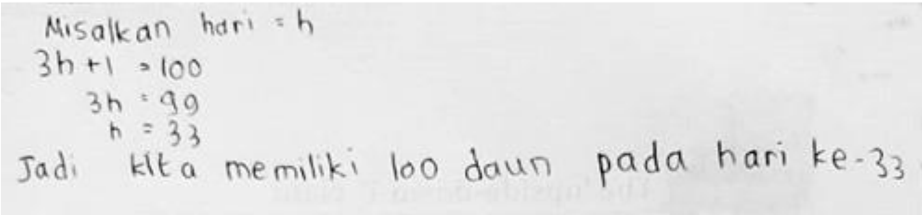


Translation
Leave number at pth day = 3p + 1 (p x 3) + 1 = 3p + 1

Figure 5.
Formula Finding for Leaf Number

Review Stage

Third, at the review stage, students checked the answers obtained by observing the answers and calculation process. Students made a written conclusion on the answer to the inverse problem by writing, "so we have 100 leaves on day 33".



Translation
For example, day = h
3h + 1 = 100
3h = 99
H = 33
So that, we have 100 leaves on 33^{th} day

Figure 6.
Using Formula to Find the Day (Inverse)

Discussion and Conclusion

Functional thinking is a part of algebraic thinking (Blanton & Kaput, 2005), and it is fundamental (Eisenmann, 2009; Kaiser & Willander, 2005; Lichti, 2018; Lichti & Roth, 2019; Stephens et al. 2017; Tanışlı, 2011). Functional thinking is generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with the various representations to analyze function behavior (Blanton et al. 2011).

Students can think functionally at elementary school while at the kindergarten, students can think variationally, think correspondence at first grade, and symbolized at the 3^{rd} grade (Blanton & Kaput, 2004, 2005; Blanton et al. 2011; Warren et al. 2006; Warren & Cooper, 2005). In 4^{th} grade, students can develop functional thinking verbally and symbolically (Warren et al. 2006).

From data analyzed, this study found that there is one student who is able to think functionally in recursive firstly and correspondence. Finally, we called it by recursive to symbolic correspondence rule. At recursive to correspondence symbolic rule, students can generalize two varying quantities initially recursively, and then after the calculation process, students can generalize relationships correspondently, represent generalization result symbolically, and determine inverse using formula or rule generalizing result.

The result showed that elementary students could think functionally both recursively and correspondently. This result matches the result showed by a previous study that students can think correspondently at the first-grade and represent generalization verbally and symbolically at the third and fourth-grade (Blanton & Kaput, 2004; Warren et al. 2006). Students carried out the relating process by an object, which was conducted by observing information from problems like leaf drawing from the first to the third day. Students' observation of leaf leads to different visualization structure. Four visualizing structure types in functional thinking consist of type 1, type 2, type 3, and type 4 (Wilkie & Clarke, 2016). In this study, elementary students have visualization structures in functional thinking type 1 (Figure 3). Type 1, as same structure as the previous day's plant with three additional leaves, one on each stem. This type is based on the recursive generalization process.

Students applied recursively in this study by paying attention to patterns in quantity before and after. Students generalize relationships by explicit strategy by counting and recursive; and non-explicit strategy by contextual and guest-and-check (Lannin, 2005). Students represent the result of generalization verbally and symbolically. It is according to a previous study was found that students in first grade can learn to think in sophisticated ways about variable quantities and variable notation (Blanton et al. 2017). Students show correspondence verbal rule in this study by generalizing pattern correspondence. Students used explicit contextual strategy for generalizing patterns (Lannin, 2005) by constructing the rule based on information in the situation. Information obtained by students in the context of the situation is a form of leaf growth that has an inverted T shape (T upside-down) and has a center leaf.

Students show recursive to correspondence symbolic rule in this study by initially observe pattern recursively by leaf increased by three each day. After knowing the pattern recursively before, students can determine the relationship between variables correspondently. Students used recursive non-explicit and guest-and-check explicit processes for generalizing pattern. This strategy also according to a study was found that students initially used a recursive approach and looked for a recursive pattern and then found a correspondence relationship in different ways (Tanıslı, 2011).

Recursive verbal rule and correspondence verbal rule in this study used by the student to find "near generalization" and "far generalization" (Stacey, 1989). Recursive to correspondence symbolic rule used by students to find "formal generalization" (Amit & Neria, 2008). The previous study showed the representation of generalization varying quantities in a different way; there are by a verbal, table, graph, or symbolic (Blanton et al. 2017; Blanton & Kaput, 2004, 2005; Warren et al. 2006). This study showed that elementary students represent generalization in two ways that are verbally and symbolically.

Based on the findings and discussion concluded that students could generalize function by combining recursive and correspondence thinking. Students' functional thinking process at the entry stage students begin to solve patterns of problems with the type of recursive thinking, at this stage students are said to be pre-finding a formula. In the attack stage, the type of thinking students leads to think correspondence. Students have used formulas in solving pattern problems but still use symbolic formulas. At the review stage, students have used formulas by connecting to each other in solving pattern problems.

Recommendations

For Further Studies

In this study, the researcher recommends for further studies to do the following: (a) Subsequent functional thinking research examines the specifically correspondence of functional thinking type. The finding shows that students think functionally by correspondence type initially begin with recursive type. It is rare for students who think functionally as correspondents from the beginning to work on the problems. Students found the general rule for relationships of days and leave numbers by remembering arithmetic sequence concepts. So for further studies can be carried out for students' correspondence functional thinking from the firs timing working on the problem. (b) it can also be considered to examine students' functional thinking in solving non-linear problems. There may be different functional thinking processes and strategies.

For Applicants

Recursive to correspondence symbolic rule is a process undertaken by students in this study. With these processes, elementary students can develop their functional thinking and direct them to develop algebraic thinking in the future. Therefore, for **applicants**, the educators in elementary schools realize that students are able to think functionally in several types. Furthermore, it need to present problems in the form of pictorial and non-pictorial problems, problems for non-linear patterns, and problems that require students to use different processes, including strategies in solving pattern problems, giving these problems will make functional thinking more varied. These processes can be considered in preparing lesson plans such as to make a task.

Limitations of Study

The researcher realizes that this study has limitation. Its include: (a) this study used the qualitative approach with case study method, so the conclusion obtained cannot be generalized for other case, (b) instrument in this study consist of one problems, (c) participant of this study taken from one location and subject were only taken for one participant, (d) problem in this study only for linier pattern.

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