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HARARY ENERGY OF COMPLEMENT OF LINE GRAPHS OF REGULAR GRAPHS

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Abstract. The Harary matrix of a graph G is defined as $H(G) = [h_{ij}]$, where $h_{ij} = \frac{1}{d(v_i,v_j)}$, if $i \neq j$ and $h_{ij} = 0$, otherwise, where $d(v_i, v_j)$ is the distance between the vertices v_i and v_j in G. The H-energy of G is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be H-equienergetic if they have same H-energy. In this paper we obtain the H-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus constructs pairs of H-equienergetic graphs of same order and having different H-eigenvalues.

1. Introduction

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertices of G be labeled as $v_1, v_2, \dots, v_n$. The *adjacency matrix* of a graph G is the square matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ are the *adjacency eigenvalues* of G, and they are labeled as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. These form the *adjacency spectrum* of G [4]. Two graphs are said to be cospectral if they have same spectra.

The *distance* between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of the shortest path joining v_i and v_j. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. A graph G is said to be r*-regular graph* if all of its vertices have same degree equal to r.

The *Harary matrix* [9] of a graph G is a square matrix $H(G) = [h_{ij}]$ of order n, where

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$$h_{ij} = \begin{cases} \frac{1}{d(v_i, v_j)}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

The Harary matrix was used in the study of molecules in the quantitative structure property relationship (QSPR) models [9].

The *Harary index* defined as the sum of the reciprocal of the distances between all pairs of vertices and it can be derived from the Harary matrix. It has interesting properties in structure-property correlations [11, 16].

The eigenvalues of $H(G)$ labeled as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ are said to be the *Harary eigenvalues* or *H-eigenvalues* of G and their collection is called *Harary spectrum* or *H-spectrum* of G. Two non-isomorphic graphs are said to be *H-cospectral* if they have same H-spectra.

The *Harary energy* or *H-energy* of a graph G, denoted by $HE(G)$, is defined as [5]

$$HE(G) = \sum_{i=1}^{n} |\xi_i| \ . \tag{1}$$

The Harary energy is defined in full analogy with the *ordinary graph energy* $E(G)$, defined as [6]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{2}$$

The ordinary graph energy has a relation with the total π-electron energy of a molecule in quantum chemistry [10]. Bounds for the Harary energy of a graph are reported in [3, 5].

Two connected graphs G_1 and G_2 are said to be *Harary equienergetic* or *H-equienergetic* if $HE(G_1) = HE(G_2)$. The H-equienergetic graphs are reported in [12, 13]. The distance energy of complements of iterated line graphs of regular graphs has been obtained in [8]. In this paper we use similar technique of [8] to obtain the H-energy of the complement of line graphs of certain regular graphs and thus construct H-equienergetic graphs having different H-spectra.

The *complement* of a graph G is a graph $\overline{G}$, with vertex set same as of G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* of G, denoted by $L(G)$ is the graph whose vertices corresponds to the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. For $k = 1, 2, \ldots$ the k-th iterated line graph of G is defined as $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) = G$ and $L^1(G) = L(G)$ [7].

If G is a regular graph of order n_0 and of degree r_0 then the line graph $L(G)$ is a regular graph of order $n_1 = (n_0 r_0)/2$ and of degree $r_1 = 2r_0 - 2$. Consequently the order and degree of $L^k(G)$ are [1, 2]

$$n_k = \frac{r_{k-1} n_{k-1}}{2} \tag{3}$$

and

$$r_k = 2r_{k-1} - 2, \tag{4}$$

where n_i and r_i stands for order and degree of $L^i(G)$, $i = 0, 1, \ldots$.

Therefore

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{5}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2) \tag{6}$$

We need following results.

Theorem 1. *[4] If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.*

Theorem 2. *[15] If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of $L(G)$ are*

$$\lambda_i + r - 2, \qquad i = 1, 2, \ldots, n, \qquad \textit{and}$$

$$-2, \qquad n(r-2)/2 \ \textit{times}.$$

Theorem 3. *[14] Let G be an r-regular graph of order n. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then the adjacency eigenvalues of $\overline{G}$, the complement of G, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.*

Theorem 4. *[3] Let G be an r-regular graph of order n and let $diam(G) \leq 2$. If $r, \lambda_2, \ldots, \lambda_n$ are the adjacency eigenvalues of G, then its H-eigenvalues are $\frac{1}{2}(n + r - 1)$ and $\frac{1}{2}(\lambda_i - 1)$, $i = 2, 3, \ldots, n$.*

Lemma 5. *[8] Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then $diam\left(\overline{L^k(G)}\right) = 2$, $k \geq 1$.*

2. Results

Theorem 6. *Let G be an r-regular graph of order n. If $r \leq \frac{n-1}{2}$, then*

$$HE\left(\overline{L(G)}\right) = r(n-2).$$

Proof. Let the adjacency eigenvalues of G be $r, \lambda_2, \ldots, \lambda_n$. From Theorem 2, the adjacency eigenvalues of $L(G)$ are

$$\left.\begin{array}{lll} 2r-2, & \text{and} & \\ \lambda_i + r - 2, & i = 2, 3, \ldots, n, & \text{and} \\ -2, & n(r-2)/2 \ \text{times}. & \end{array}\right\} \tag{7}$$

From Theorem 3 and Eq. (7), the adjacency eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} (nr/2)-2r+1, & \text{and} & \\ -\lambda_i - r + 1, & i=2,3,\ldots,n, & \text{and} \\ 1, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{8}$$

The graph $\overline{L(G)}$ is a regular graph of order $nr/2$ and of degree $(nr/2)-2r+1$. Since $r \leq \frac{n-1}{2}$, by Lemma 5, $diam\left(\overline{L(G)}\right) = 2$. Therefore by Theorem 4 and Eq. (8), the H-eigenvalues of $\overline{L(G)}$ are

$$\left.\begin{array}{rll} (nr-2r)/2, & \text{and} & \\ -(\lambda_i + r)/2, & i=2,3,\ldots,n, & \text{and} \\ 0, & n(r-2)/2 \ \text{ times.} & \end{array}\right\} \tag{9}$$

All adjacency eigenvalues of a regular graph of degree r satisfy the condition $-r \leq \lambda_i \leq r$ [4]. Therefore $\lambda_i + r \geq 0$, $i = 1,2,\ldots,n$. Therefore by (9),

$$\begin{aligned} HE\left(\overline{L(G)}\right) &= \frac{nr-2r}{2} + \sum_{i=2}^{n} \frac{(\lambda_i + r)}{2} + |0| \times \frac{n(r-2)}{2} \\ &= r(n-2) \qquad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r. \end{aligned}$$

□

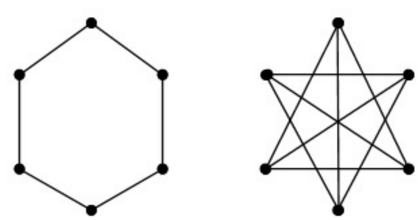

FIGURE 1. Cycle C_6 and $\overline{L(C_6)}$.

Example 7. *Consider the cycle C_6. It satisfies the conditions of Theorem 6. Complement of $L(C_6)$ is shown in the Figure 1. The H-eigenvalues of $\overline{L(C_6)}$ are 4, 0, −0.5, −0.5, −1.5, −1.5. Hence $HE(\overline{L(C_6)}) = 8$ and by Theorem 6 also, $HE(\overline{L(C_6)}) = 8$.*

Corollary 8. *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$HE(\overline{L^k(G)}) = r_{k-1}(n_{k-1} - 2).$$

Proof. If $r_0 \leq \frac{n_0-1}{2}$, then by Eqs. (3) and (4), we have

$$r_1 = 2r_0 - 2 \leq n_0 - 3 \leq \frac{1}{2}\left(\frac{n_0 r_0}{2} - 1\right) = \frac{n_1 - 1}{2}.$$

Hence

$$r_{k-1} \leq \frac{n_{k-1} - 1}{2}.$$

Therefore by Theorem 6,

$$HE\left(\overline{L^k(G)}\right) = HE\left(\overline{L(L^{k-1}(G))}\right) = r_{k-1}(n_{k-1} - 2).$$

□

Corollary 9. *Let G be a regular graph of order n_0 and of degree r_0. Let n_k and r_k be the order and degree respectively of the k-th iterated line graph $L^k(G)$, $k \geq 1$. If $r_0 \leq \frac{n_0-1}{2}$, then*

$$HE\left(\overline{L^k(G)}\right) = \left[\frac{n_0}{2^{k-1}} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2)\right] - 2(2^{k-1} r_0 - 2^k + 2).$$

3. H-EQUIENERGETIC GRAPHS

If G_1 and G_2 are the regular graphs of same order and of same degree. Then $L(G_1)$ and $L(G_2)$ are of the same order and of same degree. Further their complements are also of same order and of same degree.

Lemma 10. *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then $\overline{L(G_1)}$ and $\overline{L(G_2)}$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Proof. Follows from Eqs. (7), (8) and (9). □

Lemma 11. *Let G_1 and G_2 be regular graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are H-cospectral if and only if G_1 and G_2 are cospectral.*

Theorem 12. *Let G_1 and G_2 be regular, non H-cospectral graphs of the same order n and of the same degree r. If $r \leq \frac{n-1}{2}$, then for $k \geq 1$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non H-cospectral, H-equienergetic graphs of equal order and of equal number of edges.*

Proof. Follows from Lemma 11 and Corollary 9. □

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