Eastern Anatolian Journal of Science
Volume VI, Issue II, 2020, 22-34

Eastern Anatolian Journal of Science

The Multiplicity of Eigenvalues of A Vectorial Singular Diffusion Equation with Discontinuous Conditions

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Abstract

In this paper, we are studied *m*-dimensional vectorial diffusion equation with jump conditions inside a finite interval. We obtain some conclusions about multiplicity of the eigenvalues based on the estimation of solutions. Asymptotic formules of eigenfunctions in each interval are obtained. Also, properties related to the characteristic function of the problem are given and proven. We prove that, under certain conditions on potential matrix, the problem can only have a finite number of eigenvalues with multiplicity *m*.

Keywords: Vectorial Diffusion problem, Multiplicity eigenvalues, Jump conditions.

1. *Introduction*

Consider the m-dimensional vectorial singular diffusion equations

$$-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y,\quad x\in(0,\pi)\setminus\{a_1,a_2\} \tag{1}$$

$$y'(0)=\theta \tag{2}$$

$$y'(\pi)=\theta \tag{3}$$

$$y(a_1+0)=\alpha_1 y(a_1-0) \tag{4}$$

$$y'(a_1+0)=\beta_1 y'(a_1-0)+i\lambda\gamma_1 y(a_1-0) \tag{5}$$

$$y(a_2+0)=\alpha_2 y(a_2-0) \tag{6}$$

$$y'(a_2+0)=\beta_2 y'(a_2-0)+i\lambda\gamma_2 y(a_2-0). \tag{7}$$

Received:20.08.2020
Accepted:09.10.2020
Published:16.12.2020
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Cite this article as: A. Ergün, The Multiplicity of Eigenvalues of A Vectorial Singular Diffusion Equation with Discontinuous Conditions, *Eastern Anatolian Journal of Science, Vol. 6, Issue 2, 22-34, 2020.*

Where λ is the spectral parameter and $y=(y_1,y_2,...y_m)^T$ is an m-dimensional vector function, $q(x)\in L_2[0,\pi]$, $p(x)\in W_2^1[0,\pi]$, $a_1,a_2\in(0,\pi)$, $a_1<a_2$, $|\alpha_1-1|^2+\gamma_1^2\neq 0$, $|\alpha_2-1|^2+\gamma_2\neq 0$, $\left(\beta_i=\dfrac{1}{\alpha_i}(i=1,2)\right)$. The potential matrix $(2\lambda p(x)+q(x))$ is an $m\times m$ symmetric matrix function. θ denotes the m-dimensional zero vector.

Differential operators are defined as singular and regular. In 1946, Titchmarsh studied spectral theory of second order singular differential operators (Titchmars 1932). In 1984, the studies on the spectral theory of singular differential operators were conducted by Levitan (Levitan 1984). Singular differential operators with conditions of discontinuity are often used in mathematical physics, geophysics and natural sciences. Some studies were performed for the diffusion equation (Gasymov 1981-Koyunbakan 2007). In general, these problems are associated with discontinuous material properties. For example; it is used to in determining the parameters of the electricity line in electronics (Vakanas 1994). It is used to determine geophysical models for the release of the earth (Muller 1985). The discontinuity here is the reflection of the shear waves at the base of the earth's crust. In 1999, Kong, Wu and Zettl studied equivalence between the algebraic and geometric multiplicities of any eigenvalue of regular Sturm-Liouville problems. C. L. Shen and C.T. Shies (Shen and Shies 1999) studied the multiplicity of eigenvalues of the *m*-dimensional vectorial Sturm-Liouville problem

$$-y''+Q(x)y=\lambda y\ ,\ y(0)=y(1)=\theta$$

where Q is continuous $m\times m$ Jacobi matrix-valued function defined on $0\leq x\leq 1$.
Q. Kong (Kong 2002) generalized to the case when Q is real symmetric. In 2007, C.F Yang, Z.Y. Huang and X.P. Yang (Yang, Huang and Yang 2007)

extended the result of paper by Shen and Shies, 1999 to the Sturm-Liouville equations with a weight function, a leading coefficient and general seperation conditions. However, there are no such result for the discontinuous problem $(1)-(7)$. In this study, we first define the characteristic function of the eigenvalues of vectorial problem $(1)-(7)$. Following this, we prove the conclusion that the eigenvalues of the problem coincide with the zeros of characteristic function. Then, we show the asymptotic forms of the solutions. We obtain some results about multiplicity of the eigenvalues.

2. *Characteristic Function and Asymptotics of Solution*

Denote $H=L_2\left(I,\mathbb{C}^m\right)$ the Hilbert space of vector-values functions with the scalar product

$$(f,g)=\int_0^{a_1} g_1^* f_1 dx+\int_{a_1}^{a_2} g_2^* f_2 dx+\int_{a_2}^{\pi} g_r^* f_r dx=\int_0^{\pi} g^* f dx,$$

where $f=(f_1,f_2,...,f_m)^T$, $g=(g_1,g_2,...g_m)^T$ and $f_i,g_i\in L^2(I)$, $f_1(x)=f(x)\big|_{(0,a_1)}$, $f_2(x)=f(x)\big|_{(a_1,a_2)}$ and $f_r(x)=f(x)\big|_{(a_2,\pi)}$. We can define an operator L associated with the problem $(1)-(7)$ on H

$$L:-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y\ ,\ y\in D(L)$$

$$D(L)=\left\{y\in H;\ y,y'\in AC\left[I,\mathbb{C}^m\right]\right\},\ Ly\in L^2\left[I,\mathbb{C}^m\right]$$

$$y'(0)=y'(\pi)=\theta$$

$$y(a_1+0)=\alpha_1 y(a_1-0) \qquad ,$$

$$y'(a_1+0)=\beta_1 y'(a_1-0)+i\lambda\gamma_1 y(a_1-0)$$

$$y(a_2+0)=\alpha_2 y(a_2-0),$$

$$y'(a_2+0)=\beta_2 y'(a_2-0)+i\lambda\gamma_2 y(a_2-0).$$

Lemma 2.1. The operator L is self-adjoint.

Proof. The proof is similar to the scaler case in (Wang, Sun and Zettl 2007). We consider the problem on the three intervals $(0,a_1)$, (a_1,a_2) and (a_2,π) respectively. Where θ_m denotes $m\times m$ zero matrix and E_m denotes $m\times m$ identify matrix. On $(0,a_1)$, the matrix initial value problem

$$\begin{cases}-Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y\ ,\ x\in(0,a_1)\\ \phi_1(0,\lambda)=E_m\ ,\ \phi_1'(0,\lambda)=\theta_m\end{cases} \tag{8}$$

has a unique solution $\phi_1(x,\lambda)$. What's more, for any fixed $x\in(0,a_1)$, $\phi_1(x,\lambda)$ is an entire matrix function in λ (Agranovich 1963, pp. 17). By variation of constants, we have

$$\phi_1(x,\lambda)=\cos\lambda x E_m$$

$$+\frac{1}{\lambda}\int_0^x \sin\lambda(x-t)\left(2\lambda p(t)+q(t)\right)\phi_1(t,\lambda)dt. \tag{9}$$

On (a_1,a_2) the matrix initial value problem

$$\begin{cases}-Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y\ ,\ x\in(a_1,a_2)\\ \phi_2(a_1+0)=\alpha_1\phi_2(a_1-0)\\ \phi_2'(a_1+0)=\beta_1\phi_2'(a_1-0)+i\lambda\gamma_1\phi_2(a_1-0)\end{cases} \tag{10}$$

has a unique solution $\phi_2(x,\lambda)$. What's more, for any fixed $x\in(a_1,a_2)$, $\phi_2(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have (Ergün and Amirov 2019)

$$\begin{aligned}\phi_2(x,\lambda)&=\left(\alpha^+ +\frac{\gamma_1}{2}\right)e^{i\lambda x}+\left(\alpha^- -\frac{\gamma_1}{2}\right)e^{i\lambda(2a_1-x)}\\ &+\alpha^+\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}J(t)\phi_2(t,\lambda)dt\\ &-\frac{1}{2}\alpha^-\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}J(t)\phi_2(t,\lambda)dt\\ &-i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}J(t)\phi_2(t,\lambda)dt\\ &+i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x+t-2a_1)}{\lambda}J(t)\phi_2(t,\lambda)dt\\ &+\int_{a_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)\phi_2(t,\lambda)dt\end{aligned} \tag{11}$$

where $\alpha_1^{\pm}(x)=\frac{1}{2}(\alpha_1\pm\beta_1)$, $J(t)=2\lambda p(t)+q(t)$,

or

$$\begin{aligned}\phi_2(x,\lambda) &= \alpha_1 \cos\lambda(x-a_1)\phi_1(a_1-0)E_m \\ &+ i\gamma_1 \sin\lambda(x-a_1)\phi_1(a_1-0)E_m \\ &+ \frac{\beta_1}{\lambda}\sin\lambda(x-a_1)\phi_1'(a_1-0)E_m \\ &+ \int_{a_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}\left(2\lambda p(t)+q(t)\right)\phi_2(\mathrm{t},\lambda)\,dt.\end{aligned} \tag{12}$$

On (a_2,π) the matrix initial value problem

$$\begin{cases} -Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y \;,\; x\in(a,\pi) \\ \phi_3(a_2+0)=\alpha_2\phi_3(a_2-0) \\ \phi_3'(a_2+0)=\beta_2\phi_3'(a_2-0)+i\lambda\gamma_2\phi_3(a_2-0) \end{cases} \tag{13}$$

has a unique solution $\phi_3(x,\lambda)$. What's more, for any fixed $\forall x\in(a_2,\pi)$, $\phi_3(x,\lambda)$ is an entire matrix function in λ. By variation of constants, we have

$$\begin{aligned}\phi_3(x,\lambda) &= \alpha_1^+\alpha_2^+ e^{i\lambda x} + \alpha_1^-\alpha_2^- e^{i\lambda(2a_1-2a_2+x)} \\ &+ \alpha_1^+\alpha_2^- e^{i\lambda(2a_2-x)} + \alpha_1^-\alpha_2^+ e^{i\lambda(2a_1-x)} + \frac{\gamma_1\alpha_2^+}{2}e^{i\lambda x} \\ &- \frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_1-2a_2+x)} + \frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_2-x)} \\ &- \frac{\gamma_1\alpha_2^+}{2}e^{i\lambda(2a_1-x)} - \frac{\gamma_1\gamma_2}{4}e^{i\lambda x} - \frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)} \\ &- \frac{\gamma_2\alpha_1^+}{2}e^{i\lambda(2a_2-x)} - \frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-x)} + \frac{\gamma_2\alpha_1^+}{2}e^{i\lambda x} \\ &+ \frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-2a_2+x)} - \frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)} + \frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)} \\ &+ \left(\alpha_1^+\alpha_2^+ + \frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \left(-\alpha_1^+\alpha_2^- + \frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_2)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \left(-\alpha_1^-\alpha_2^+ - \frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &- \alpha_2^-\int_{a_1}^{a_2}\frac{\sin\lambda(x+t-2a_2)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \left(\alpha_1^-\alpha_2^- - \frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(2a_1-2a_2+x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \alpha_2^+\int_{a_1}^{a_2}\frac{\sin\lambda(x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &- \frac{i}{2}\left(\gamma_1\alpha_2^+ - \gamma_2\alpha_1^+\right)\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \frac{i}{2}\left(-\gamma_1\alpha_2^- - \gamma_2\alpha_1^-\right)\int_0^{a_1}\frac{\cos\lambda(x+t-2a_2)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \frac{i}{2}\left(\gamma_1\alpha_2^+ + \gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda(x+t-2a_1)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda(x+t-2a_2)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \frac{i}{2}\left(\gamma_1\alpha_2^- + \gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda(2a_1-2a_2+x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &- \frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda(x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt \\ &+ \int_{a_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}J(t)\phi_3(t,\lambda)\,dt\end{aligned}$$

(14)

where $\alpha_2^{\pm}(x)=\frac{1}{2}(\alpha_2\pm\beta_2)$ or

$$\begin{aligned}\phi_3(x,\lambda) &= \alpha_2\cos\lambda(x-a_2)\phi_2(a_2-0,\lambda)E_m \\ &+ i\gamma_2\sin\lambda(x-a_2)\phi_2(a_2-0,\lambda)E_m \\ &+ \frac{\beta_2}{\lambda}\sin\lambda\beta(x-a_2)\phi_2'(a_2-0,\lambda)E_m \\ &+ \int_{a_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}\left(2\lambda p(t)+q(t)\right)\phi_3(\mathrm{t},\lambda)\,dt.\end{aligned} \tag{15}$$

Let

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda),\; x\in(0,a_1) \\ \phi_2(x,\lambda), x\in(a_1,a_2) \\ \phi_3(x,\lambda), x\in(a_2,\pi)\end{cases}.$$

Then, any solution of the equations (1) satisfying boundary condition (2) and jump conditions $(4)-(7)$ can be expressed as

$$y(x,\lambda)=\phi(x,\lambda)c_1=\begin{cases}\phi_1(x,\lambda)k_1, & x\in(0,a_1) \\ \phi_2(x,\lambda)k_1, & x\in(a_1,a_2) \\ \phi_3(x,\lambda)k_1, & x\in(a_2,\pi)\end{cases} \tag{16}$$

where k_1 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem $(1)-(7)$, then $k_1\neq\theta$ and $y(x,\lambda)$ satisfies the boundary condition at $x=\pi$, that is,

$$y'(\pi,\lambda)=\phi'(\pi,\lambda)k_1=\phi_3'(\pi,\lambda)k_1=\theta$$

Thus we get

$$\det\left(\phi_3'(\pi,\lambda)\right)=0 .$$

Similarly, on (a_2,π), consider the matrix initial value problem

$$\begin{cases}-Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y\ ,\ x\in(a_2,\pi)\\ \psi_3(\pi,\lambda)=E_m\ ,\psi_3'(\pi,\lambda)=\theta_m\end{cases}$$

(17)

The problem (17) has a unique solution $\psi_3(x,\lambda)$. What's more, for any fixed $x\in(a_2,\pi)$, $\psi_3(x,\lambda)$ is an entire matrix function in λ.

Consider the matrix initial value problem on (a_1,a_2),

$$\begin{cases}-Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y\ ,\ x\in(a_1,a_2)\\ \psi_2'(a_2-0,\lambda)=\alpha_2\psi_3'(a_2+0,\lambda)-i\lambda\alpha_2\gamma_2\psi_2(a_2-0,\lambda)\\ \psi_2(a_2-0,\lambda)=\beta_2\psi_3(a_2+0,\lambda)\end{cases} \quad (18)$$

The problem (18) has a unique solution $\psi_2(x,\lambda)$. What's more, for any fixed $x\in(a_1,a_2)$, $\psi_2(x,\lambda)$ is an entire matrix function in λ.

Consider the matrix initial value problem on $(0,a_1)$,

$$\begin{cases}-Y''+\left(2\lambda p(x)+q(x)\right)Y=\lambda^2 Y\ ,\ x\in(a_1,\pi)\\ \psi_1'(a_1-0,\lambda)=\alpha_1\psi_2'(a_1+0,\lambda)-i\lambda\alpha_1\gamma_1\psi_1(a_1-0,\lambda)\\ \psi_1(a_1-0,\lambda)=\beta_1\psi_2(a_1+0,\lambda)\end{cases} \quad (19)$$

The problem (19) has a unique solution $\psi_1(x,\lambda)$. What's more, for any fixed $x\in(0,a_1)$, $\psi_1(x,\lambda)$ is an entire matrix function in λ. Let

$$\psi(x,\lambda)=\begin{cases}\psi_1(x,\lambda),\ x\in(0,a_1)\\ \psi_2(x,\lambda),x\in(a_1,a_2)\\ \psi_3(x,\lambda),x\in(a_2,\pi)\end{cases}$$

Then, any solution of the equations (1) satisfying boundary condition (3) and jump conditions $(4)-(7)$ can be expressed as

$$y(x,\lambda)=\psi(x,\lambda)c_2=\begin{cases}\psi_1(x,\lambda)k_2,\ x\in(0,a_1)\\ \psi_2(x,\lambda)k_2,x\in(a_1,a_2)\\ \psi_3(x,\lambda)k_2,x\in(a_2,\pi)\end{cases} \quad (20)$$

where k_2 is an arbitrary m-dimensional constant vector. If λ is an eigenvalue of the problem $(1)-(7)$, then $k_2\neq\theta$ and $y(x,\lambda)$ satisfies the boundary condition at $x=0$, that is,

$$y'(0,\lambda)=\psi'(0,\lambda)k_2=\psi_1'(0,\lambda)k_2=\theta$$

Thus, we get

$$\det\left(\psi_1'(0,\lambda)\right)=0.$$

Let $\Delta_j(\lambda)=W\left(\phi_j(x,\lambda),\psi_j(x,\lambda)\right)$ be the Wronskian of solution matrices $\phi_j(x,\lambda)$ and $\psi_j(x,\lambda)$, $j=1,2,3$, that is,

$$\Delta_1(\lambda)=\begin{vmatrix}\phi_1(x,\lambda) & \psi_1(x,\lambda)\\ \phi_1'(x,\lambda) & \psi_1'(x,\lambda)\end{vmatrix},$$

$$\Delta_2(\lambda)=\begin{vmatrix}\phi_2(x,\lambda) & \psi_2(x,\lambda)\\ \phi_2'(x,\lambda) & \psi_2'(x,\lambda)\end{vmatrix}, \quad (21)$$

$$\Delta_3(\lambda)=\begin{vmatrix}\phi_3(x,\lambda) & \psi_3(x,\lambda)\\ \phi_3'(x,\lambda) & \psi_3'(x,\lambda)\end{vmatrix}.$$

Lemma 2.2. For any $\lambda\in\mathbb{C}$

$$\Delta_1(\lambda)=\Delta_2(\lambda)=\Delta_3(\lambda).$$

Proof. Because the Wronskian of the solution matrices $\phi_j(x,\lambda)$ and $\psi_j(x,\lambda)$ is independent of x,

$$\Delta_3(\lambda)=\Delta_3(\lambda)\big|_{x=a_2+0}=\begin{vmatrix}\phi_3(a_2+0,\lambda) & \psi_3(a_2+0,\lambda)\\ \phi_3'(a_2+0,\lambda) & \psi_3'(a_2+0,\lambda)\end{vmatrix}$$

$$\begin{aligned}&=\alpha_2\beta_2\psi_2'(a_2-0,\lambda)\phi_2(a_2-0,\lambda)\\&+i\lambda\alpha_2\gamma_2\psi_2(a_2-0,\lambda)\phi_2(a_2-0,\lambda)\\&-\alpha_2\beta_2\phi_2'(a_2-0,\lambda)\psi_2(a_2-0,\lambda)\\&-i\lambda\alpha_2\gamma_2\phi_2(a_2-0,\lambda)\psi_2(a_2-0,\lambda)\\&=\begin{vmatrix}\phi_2(a_2-0,\lambda) & \psi_2(a_2-0,\lambda)\\ \phi_2'(a_2-0,\lambda) & \psi_2'(a_2-0,\lambda)\end{vmatrix}\\&=\begin{vmatrix}\phi_2(x,\lambda) & \psi_2(x,\lambda)\\ \phi_2'(x,\lambda) & \psi_2'(x,\lambda)\end{vmatrix}_{x=a_2-0}=\Delta_2(\lambda)\end{aligned}$$

$$\Delta_2(\lambda)=\Delta_2(\lambda)\big|_{x=a_1+0}=\begin{vmatrix}\phi_2(a_1+0,\lambda) & \psi_2(a_1+0,\lambda)\\ \phi_2'(a_1+0,\lambda) & \psi_2'(a_1+0,\lambda)\end{vmatrix}$$

$$=\alpha_1\beta_1\psi_1'(a_1-0,\lambda)\phi_1(a_1-0,\lambda)$$
$$+i\lambda\alpha_1\gamma_1\psi_1(a_1-0,\lambda)\phi_1(a_1-0,\lambda)$$
$$-\alpha_1\beta_1\phi_1'(a_1-0,\lambda)\psi_1(a_1-0,\lambda)$$
$$-i\lambda\alpha_1\gamma_1\phi_1(a_1-0,\lambda)\psi_1(a_1-0,\lambda)$$
$$=\begin{vmatrix}\phi_1(a_1-0,\lambda) & \psi_1(a_1-0,\lambda)\\ \phi_1'(a_1-0,\lambda) & \psi_1'(a_1-0,\lambda)\end{vmatrix}$$
$$=\begin{vmatrix}\phi_1(x,\lambda) & \psi_1(x,\lambda)\\ \phi_1'(x,\lambda) & \psi_1'(x,\lambda)\end{vmatrix}_{x=a_1-0}=\Delta_1(\lambda)$$

the proof is completed.

Denote $\Delta(\lambda)=\Delta_1(\lambda)=\Delta_2(\lambda)=\Delta_3(\lambda)$, we have the following lemma.

Lemma 2.3. λ is an eigenvalue of $(1)-(7)$ if any only if $\Delta(\lambda)=0.$

Proof. *Necessity*. Assume that λ_0 is an eigenvalue of $(1)-(7)$. $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0, then by (16) we have

$$y(x,\lambda_0)=\phi(x,\lambda_0)k_3=\begin{cases}\phi_1(x,\lambda_0)k_3, & x\in(0,a_1)\\ \phi_2(x,\lambda_0)k_3, & x\in(a_1,a_2)\\ \phi_3(x,\lambda_0)k_3, & x\in(a_2,\pi)\end{cases} \quad (22)$$

$$y(x,\lambda_0)=\psi(x,\lambda_0)k_4=\begin{cases}\psi_1(x,\lambda_0)k_4, & x\in(0,a_1)\\ \psi_2(x,\lambda_0)k_4, & x\in(a_1,a_2)\\ \psi_3(x,\lambda_0)k_4, & x\in(a_2,\pi)\end{cases}$$

(23)

$k_3,\ k_4$ are m-dimensional nonzero constant vector.

So from (22) and (23) we have

$$\left.\begin{aligned}\phi_1(x,\lambda_0)k_3&=\psi_1(x,\lambda_0)k_4\\ \phi_1'(x,\lambda_0)k_3&=\psi_1'(x,\lambda_0)k_4\end{aligned}\right\}x\in(0,a_1).$$

By direct simplification, we get

$$\begin{pmatrix}\phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0)\end{pmatrix}\cdot\begin{pmatrix}k_3\\ k_4\end{pmatrix}=\begin{pmatrix}\theta\\ \theta\end{pmatrix}.$$

Because $k_3,\ k_4\neq 0$, the coefficient determinant of above linear siystem of equations

$$\begin{vmatrix}\phi_1(x,\lambda_0) & -\psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & -\psi_1'(x,\lambda_0)\end{vmatrix}$$
$$=(-1)^m\begin{vmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{vmatrix}$$
$$=(-1)^m\Delta_1(\lambda_0)=\Delta_2(\lambda_0)=\Delta_3(\lambda_0)=\Delta(\lambda_0)=0$$

Sufficiency. If for some $\lambda_0\in DL,\ \Delta(\lambda_0)=0$. Then the linear siystems of equations

$$\begin{pmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{pmatrix}\cdot\begin{pmatrix}k_3\\ k_4\end{pmatrix}=\begin{pmatrix}\theta\\ \theta\end{pmatrix}$$
$$\begin{pmatrix}\phi_2(x,\lambda_0) & \psi_2(x,\lambda_0)\\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0)\end{pmatrix}\cdot\begin{pmatrix}k_3\\ k_4\end{pmatrix}=\begin{pmatrix}\theta\\ \theta\end{pmatrix}$$
$$\begin{pmatrix}\phi_3(x,\lambda_0) & \psi_3(x,\lambda_0)\\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0)\end{pmatrix}\cdot\begin{pmatrix}k_3\\ k_4\end{pmatrix}=\begin{pmatrix}\theta\\ \theta\end{pmatrix}$$

have nonzero solutions. By a direct computation, we get

$$\left.\begin{aligned}\phi_1(x,\lambda_0)k_3&=-\psi_1(x,\lambda_0)k_4\\ \phi_1'(x,\lambda_0)k_3&=-\psi_1'(x,\lambda_0)k_4\end{aligned}\right\}x\in(0,a_1)$$
$$\left.\begin{aligned}\phi_2(x,\lambda_0)k_3&=-\psi_2(x,\lambda_0)k_4\\ \phi_2'(x,\lambda_0)k_3&=-\psi_2'(x,\lambda_0)k_4\end{aligned}\right\}x\in(a_1,a_2)$$
$$\left.\begin{aligned}\phi_3(x,\lambda_0)k_3&=-\psi_3(x,\lambda_0)k_4\\ \phi_3'(x,\lambda_0)k_3&=-\psi_3'(x,\lambda_0)k_4\end{aligned}\right\}x\in(a_2,\pi).$$

Denote

$$y(x,\lambda_0)=\begin{cases}\phi_1(x,\lambda_0)k_3=-\psi_1(x,\lambda_0)k_4, & x\in(0,a_1)\\ \phi_2(x,\lambda_0)k_3=-\psi_2(x,\lambda_0)k_4, & x\in(a_1,a_2)\\ \phi_3(x,\lambda_0)k_3=-\psi_3(x,\lambda_0)k_4, & x\in(a_2,\pi)\end{cases}.$$

We note that $y(x,\lambda_0)$ satisfies the boundary condition $(2),(3)$ and jump condition $(4)-(7)$. That is, $y(x,\lambda_0)$ is the eigenfunctions corresponding to λ_0. Thus λ_0 is an eigenvalue of the problem $(1)-(7)$.

Remark 2.4. As two especial case

$$\Delta(\lambda)=\begin{vmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{vmatrix}_{x=0}$$
$$=\begin{vmatrix}E_m & \psi_1(0,\lambda_0)\\ \theta_m & \psi_1'(0,\lambda_0)\end{vmatrix}=\det(\psi_1'(0,\lambda))$$
$$\Delta(\lambda)=\begin{vmatrix}\phi_3(x,\lambda_0) & \psi_3(x,\lambda_0)\\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0)\end{vmatrix}_{x=\pi}$$
$$=\begin{vmatrix}\phi_3(\pi,\lambda_0) & E_m\\ \phi_3'(\pi,\lambda_0) & \theta_m\end{vmatrix}=(-1)^m\det(\phi_3'(\pi,\lambda)).$$

Definition 2.5. $\Delta(\lambda)$ will be called the characteristic function of the eigenvalues of the problem $(1)-(7)$.

Definition 2.6. The algebraic multiplicity of an eigenvalue λ is defined to be the order of λ as a zero of $\Delta(\lambda)$. The geometric multiplicity of λ as an eigenvalue of the problem $(1)-(7)$ is defined to be the number of linearly independent solutions of the boundary value problem.

If we denote $2m\times 2m$ matrices

$$A(x,\lambda_0)=\begin{pmatrix}\phi_1(x,\lambda_0) & \psi_1(x,\lambda_0)\\ \phi_1'(x,\lambda_0) & \psi_1'(x,\lambda_0)\end{pmatrix},$$

$$B(x,\lambda_0)=\begin{pmatrix}\phi_2(x,\lambda_0) & \psi_2(x,\lambda_0)\\ \phi_2'(x,\lambda_0) & \psi_2'(x,\lambda_0)\end{pmatrix}$$

and

$$C(x,\lambda_0)=\begin{pmatrix}\phi_3(x,\lambda_0) & \psi_3(x,\lambda_0)\\ \phi_3'(x,\lambda_0) & \psi_3'(x,\lambda_0)\end{pmatrix}$$

the rank of matrix $A(x,\lambda_0)$ as $R(A(x,\lambda_0))$. Similarly, $B(x,\lambda_0)$ as $R(B(x,\lambda_0))$ and $C(x,\lambda_0)$ as $R(C(x,\lambda_0))$.

Corollary 2.7. The geometric multiplicity of λ_0 as an eigenvalue of the problem $(1)-(7)$ is equal to $2m-R(A(x,\lambda_0))$ or $2m-R(B(x,\lambda_0))$ or $2m-R(C(x,\lambda_0))$.

Corollary 2.8. $R(A(x,\lambda_0)), R(B(x,\lambda_0))$ or $R(C(x,\lambda_0))$ is at least equal to m, so the geometric multiplicity of λ_0 varies from 1 to m. When the geometric multiplicity of an eigenvalue is m, we say the eigenvalue has maximal (full) multiplicity. In this study, we refer multiplicity as the geometric multiplicity.

An entire function of non-integer order has an infinite set of zeros. The zeros of an analytic function which does not vanish identically are isolated (Boas 1954). $\psi_1'(0,\lambda)$ and $\phi_3'(\pi,\lambda)$ are entire function of order $\frac{1}{2}$ matrices. The sums and products of such functions are entire of order not exceeding $\frac{1}{2}$. Hence, the determinants of $\psi_1'(0,\lambda)$ and $\phi_3'(\pi,\lambda)$, that is, the characteristic functions are also non-integer.

Lemma 2.9. Eigenvalues for $(1)-(7)$ are real. The boundary value problem $(1)-(7)$ has a countable number of eigenvalues that grow unlimitedly, when that are ordered according to their absolute value.

The norm of a constant matrix as well as the norm of a matrix function A is denoted by $\|A\|$.

$A(x)=(a_{ij})_{i,j=1}^{m}: I\rightarrow M_{mxm}$, , for any $x\in I$, the norm of $A(x)$ may be taken as

$$\|A(x)\|=\max_{1\le i\le m}\sum_{j=1}^{m}|a_{ij}| \tag{24}$$

Let $\lambda = s^2, s=\sigma+i\tau, \sigma,\tau\in\mathbb{R}$. I have the following three lemmas.

Lemma 2.10. When $|\lambda|\rightarrow\infty$, the following asymptotic formulas hold on $0<x<a_1$,

$$\phi_1(x,\lambda)=\cos(\lambda x)E_m+O\left(|\lambda|^{-1}e^{|\sigma|x}\right) \tag{25}$$

$$\phi_1'(x,\lambda)=-\lambda\sin(\lambda x)E_m+O\left(e^{|\sigma|x}\right) \tag{26}$$

Proof. See the paper by Agranovich 1963.

Lemma 2.11. When $|\lambda|\rightarrow\infty$, $\phi_2(x,\lambda)$ and $\phi_2'(x,\lambda)$ have the following asymptotic formulas on $a_1<x<a_2$,

$$\phi_2(x,\lambda)=\frac{1}{2}\left(\alpha_1^{+}+\frac{\gamma_1}{2}\right)\exp\left(-i\left(\lambda x-\beta(x)\right)\right)E_m\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{27}$$

$$\phi_2'(x,\lambda)=\frac{1}{2}\left(\alpha_1^{+}+\frac{\gamma_1}{2}\right)\left(p(x)-\lambda\right)i\exp\left(-i\left(\lambda x-\beta(x)\right)\right)E_m+O(1) \tag{28}$$

where $\beta(x)=\int_{a_1}^{x}p(t)\,dt$.

Proof. Since $\phi_2(x,\lambda)$ is the solution of boundary value problem (10), we have

$$\phi_2(x,\lambda)=\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)\cos(\lambda x-\beta(x))E_m$$
$$+\left(\alpha_1^- -\frac{\gamma_1}{2}\right)\cos(\lambda(2a_1-x)-\beta(x))E_m+O\left(\frac{1}{\lambda}e^{\sigma x}\right)$$

$$\phi_2(x,\lambda)=\frac{1}{2}\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)e^{i(\lambda x-\beta(x))}E_m$$
$$+\frac{1}{2}\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)e^{-i(\lambda x-\beta(x))}E_m$$
$$+\frac{1}{2}\left(\alpha_1^- -\frac{\gamma_1}{2}\right)e^{i(\lambda(2a_1-x)-\beta(x))}E_m \tag{29}$$
$$+\frac{1}{2}\left(\alpha_1^- -\frac{\gamma_1}{2}\right)e^{-i(\lambda(2a_1-x)-\beta(x))}E_m+O\left(\frac{1}{\lambda}e^{\sigma x}\right)$$

Let $f(x,\lambda):=O\left(\frac{1}{\lambda}e^{\sigma x}\right)$ and note that

$$\phi_2(x,\lambda)=\frac{1}{2}\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)e^{-i(\lambda x-\beta(x))}E_m+(1+g(x,\lambda))$$

.

From at simple computation at equation (29), we get

$$g(x,\lambda)=e^{2i(\lambda x-\beta(x))}E_m+\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}e^{2i\lambda a_1}E_m$$
$$+\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}e^{2i[\lambda(x-a_1)+\beta(x)]}E_m+\frac{e^{i(\lambda x-\beta(x))}}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}f(x,\lambda)E_m$$

Let's examine $g(x,\lambda)=O\left(\frac{1}{\lambda}\right)$ accuracy.

$$|g(x,\lambda)|\le\left|e^{2i(\lambda x-\beta(x))}E_m\right|$$
$$+\left|\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}e^{2i\lambda a_1}E_m\right|$$
$$+\left|\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}e^{2i[\lambda(x-a_1)+\beta(x)]}E_m\right|+\left|\frac{e^{i(\lambda x-\beta(x))}}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}f(x,\lambda)E_m\right|$$

$$\le e^{-2\sigma x}E_m+\left|\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}\right|e^{-2\sigma a_1}E_m$$
$$+\left|\frac{\left(\alpha_1^- -\frac{\gamma_1}{2}\right)}{\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)}\right|e^{-2\sigma x}E_m+\frac{c}{\lambda}e^{-\sigma x}e^{\sigma x}E_m$$

Furthermore , $\sigma>\varepsilon|\lambda|$, $\varepsilon>0$ in D. Thus, $-\sigma<-\varepsilon|\lambda|$ and $e^{-2\sigma x}<e^{-\varepsilon|\lambda|x}$

Since $\frac{x}{e^x}\to 0$, $x<ce^x\ (c>0)$. Thus,

$e^{-2\sigma x}<\frac{c}{\varepsilon|\lambda|x}$. We get

$g(x,\lambda)=O\left(\frac{1}{\lambda}\right)$ $\lambda\to\infty$. Hence,

$$\phi_2(x,\lambda)=$$
$$\frac{1}{2}\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)\exp(-i(\lambda x-\beta(x)))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right)$$

$|\lambda|\to\infty$. Derivativing both sides of (27) and using the first formula (29), we could get the formula of (28) similarly.

Lemma 2.12. When $|\lambda|\to\infty$, $\phi_3(x,\lambda)$ and $\phi_3'(x,\lambda)$ have the following asymptotic formulas on $a_2<x<\pi$,

$$\phi_3(x,\lambda)=A\exp(-i\lambda T(x))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right) \tag{30}$$

$$\phi_3'(x,\lambda)=$$
$$A(p(x)-\lambda)i\exp(-i\lambda T(x))E_m+O(1) \tag{31}$$

where $A=\frac{1}{2}\left(\alpha_2^-\left(\alpha_1^- -\frac{\gamma_2}{2}\right)\right)$, $\alpha_2^{\mp}=\frac{1}{2}(\alpha_2\mp\beta_2)$

$T(x)=((2a_1-2a_2+x)-\beta(x))$.

Proof. Since $\phi_3(x,\lambda)$ is the solution of boundary value problem (13), we have

$$\phi_3(x,\lambda) = A_3\cos\left(\lambda\left((2a_1-2a_2+x)-\beta(x)\right)\right)E_m$$
$$-A_1\cos\left(\lambda\left(2a_1-2a_2-w^+(x)\right)\right)E_m$$
$$+A_1\cos\left(\lambda\left(2a_1+w^-(x)\right)\right)E_m \quad (32)$$
$$+A_2\cos\left(\lambda\left(2a_1+w^-(x)\right)\right)E_m + O\left(\frac{1}{\lambda}e^{\sigma(2a_1-2a_2+x)}\right)$$

We get

$$\phi_3(x,\lambda) =$$
$$\frac{A_3}{2}e^{i(\lambda((2a_1-2a_2+x)-\beta(x)))}E_m + \frac{A_3}{2}e^{-i(\lambda((2a_1-2a_2+x)-\beta(x)))}E_m$$
$$-\frac{A_1}{2}e^{i(\lambda((2a_1-2a_2+x)-w^+(x)))}E_m - \frac{A_1}{2}e^{-i(\lambda((2a_1-2a_2+x)-w^+(x)))}E_m$$
$$+\frac{A_1}{2}e^{i(\lambda((2a_1-x)+w^-(x)))}E_m + \frac{A_1}{2}e^{-i(\lambda((2a_1-x)+w^-(x)))}E_m$$
$$+\frac{A_2}{2}e^{i(\lambda((2a_1-x)+w^-(x)))}E_m + \frac{A_2}{2}e^{-i(\lambda((2a_1-x)+w^-(x)))}E_m$$
$$+O\left(\frac{1}{\lambda}e^{\sigma(2a_1-2a_2+x)}\right)$$

(33)

where $w^{\pm}(x) = \int_{a_2}^{x} p(t)dt \pm \int_{0}^{a_1} p(t)dt$,

$$A_1 = -\frac{\gamma_2}{2}\left(\alpha_1^+ + \frac{\gamma_1}{2}\right),\ A_2 = -\alpha_2^-\left(\alpha_1^+ + \frac{\gamma_1}{2}\right),$$
$$A_3 = -\alpha_2^-\left(\alpha_1^- - \frac{\gamma_1}{2}\right).$$

Let $f(x,\lambda) := O\left(\frac{1}{\lambda}e^{\sigma(2a_1-2a_2+x)}\right)$ and note that

$$\phi_3(x,\lambda) = \frac{A_3}{2}e^{-i(\lambda((2a_1-2a_2+x)-\beta(x)))}E_m + (1+g(x,\lambda))$$

From at simple computation at equation (32), we get

$$g(x,\lambda) =$$
$$e^{2i(\lambda((2a_1-2a_2+x)-\beta(x)))}E_m - \frac{A_1}{A_3}e^{i(2\lambda((2a_1-2a_2+x)-(w^+(x)+\beta(x))))}E_m$$
$$-\frac{A_1}{A_3}e^{i(w^+(x)-\beta(x))}E_m + \frac{A_1}{A_3}e^{i((4\lambda a_1-2a_2+(w^-(x)+\beta(x))))}E_m$$
$$+\frac{A_1}{A_3}e^{i((2\lambda a_2+2\lambda x+(\beta(x)-w^-(x))))}E_m + \frac{A_2}{A_3}e^{i((4\lambda a_1-2a_2+(w^-(x)+\beta(x))))}E_m$$
$$+\frac{A_2}{A_3}e^{i((2\lambda a_2+2\lambda x+(\beta(x)-w^-(x))))}E_m$$
$$+\frac{2e^{i(\lambda((2a_1-2a_2+x)-\beta(x)))}}{A_3}f(x,\lambda)E_m$$

Let's examine $g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ accuracy.

$$|g(x,\lambda)|$$
$$\leq \left|e^{2i(\lambda((2a_1-2a_2+x)-\beta(x)))}E_m\right| + \left|\frac{A_1}{A_3}e^{i(2\lambda((2a_1-2a_2+x)-(w^+(x)+\beta(x))))}E_m\right|$$
$$+\left|\frac{A_1}{A_3}e^{i(w^+(x)-\beta(x))}E_m\right| + \left|\frac{A_1}{A_3}e^{i((4\lambda a_1-2a_2+(w^-(x)+\beta(x))))}E_m\right|$$
$$+\left|\frac{A_1}{A_3}e^{i((2\lambda a_2+2\lambda x+(\beta(x)-w^-(x))))}E_m\right| + \left|\frac{A_2}{A_3}e^{i((4\lambda a_1-2a_2+(w^-(x)+\beta(x))))}E_m\right|$$
$$+\left|\frac{A_2}{A_3}e^{i((2\lambda a_2+2\lambda x+(\beta(x)-w^-(x))))}E_m\right|$$
$$+\left|\frac{2e^{i(\lambda((2a_1-2a_2+x)-\beta(x)))}}{A_3}f(x,\lambda)E_m\right|$$

$$\leq e^{-2\sigma(2a_1-2a_2+x)}E_m + \left|\frac{A_1}{A_3}\right|e^{-2\sigma(2a_1-2a_2+x)}E_m + \left|\frac{A_1}{A_3}\right|e^{-2\sigma x}E_m$$
$$+\left|\frac{A_1}{A_3}\right|e^{-2\sigma(2a_1-2a_2+x)}E_m + \left|\frac{A_1}{A_3}\right|e^{-2\sigma a_2}E_m + \left|\frac{A_2}{A_3}\right|e^{-2\sigma(2a_1-2a_2+x)}E_m$$
$$+\left|\frac{A_2}{A_3}\right|e^{-2\sigma a_2}E_m + \frac{c}{\lambda}e^{-2\sigma(2a_1-2a_2+x)}e^{2\sigma(2a_1-2a_2+x)}E_m$$

Furthermore, $\sigma > \varepsilon|\lambda|$, $\varepsilon > 0$ in D. Thus, $-\sigma < -\varepsilon|\lambda|$ and $e^{-2\sigma(2a_1-2a_2+x)} < e^{-\varepsilon|\lambda|(2a_1-2a_2+x)}$

Since $\frac{x}{e^{(2a_1-2a_2+x)}} \to 0$, $x < ce^{(2a_1-2a_2+x)}$ $(c>0)$. Thus,

$e^{-2\sigma(2a_1-2a_2+x)} < \frac{c}{\varepsilon|\lambda|(2a_1-2a_2+x)}$. We get

$g(x,\lambda) = O\left(\frac{1}{\lambda}\right)$ $\lambda \to \infty$. Hence,

$$\phi_3(x,\lambda) = A\exp(-i\lambda T(x))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right),$$

$|\lambda| \to \infty$. Derivativing both sides of (30) and using the first formula (32), we could get the formula of (31) similarly.

3. *Multiplicities of Eigenvalues of the Vectorial Problem*

This section of the paper includes the conditions of the potential matrix function $(2\lambda p(x)+q(x))$, under certain conditions, the problem that presented with $(1)-(7)$ equations have only a finite number of eigenvalues with multiplicity m. Where $p(x)\in W_2^1[0,\pi]$, $p(x)=\{p_{ij}(x)\}_{i,j=1}^{m}$, $q(x)\in L_2[0,\pi]$ and $q(x)=\{q_{ij}(x)\}_{i,j=1}^{m}$.

Theorem 3.1. Let $m\ge 2$. Assume that, for some $i,j\in\{1,2,...,m\}$ with $i\neq j$ either

(i)

$$\int_0^{a_1} p_{ij}(x)dx+M_1\int_{a_1}^{a_2} p_{ij}(x)dx+M_2\int_{a_2}^{\pi} p_{ij}(x)dx\neq 0$$
$$\int_0^{a_1} q_{ij}(x)dx+M_1\int_{a_1}^{a_2} q_{ij}(x)dx+M_2\int_{a_2}^{\pi} q_{ij}(x)dx\neq 0 \quad (34)$$

or

(ii)

$$\int_0^{a_1}[p_{ii}(x)-p_{jj}(x)]dx+M_1\int_{a_1}^{a_2}[p_{ii}(x)-p_{jj}(x)]dx$$
$$+M_2\int_{a_2}^{\pi}[p_{ii}(x)-p_{jj}(x)]dx\neq 0$$
$$\int_0^{a_1}[q_{ii}(x)-q_{jj}(x)]dx+M_1\int_{a_1}^{a_2}[q_{ii}(x)-q_{jj}(x)]dx$$
$$+M_2\int_{a_2}^{\pi}[q_{ii}(x)-q_{jj}(x)]dx\neq 0$$

(35)

Then, with finitely many exceptions. The multiplicities of the eigenvalues of the problem $(1)-(7)$ are at most $m-1$.

where $M_1=\frac{1}{4}\left(\alpha_1^{+}+\frac{\gamma_1}{2}\right)^2$,

$M_2=\frac{1}{4}\left(\alpha_2^{-}\left(\alpha_1^{-}-\frac{\gamma_2}{2}\right)\right)^2$.

Proof. (i) We assume that (34) holds. Suppose to the contrary, that there exists a sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ whole multiplicities are all m. Obviously, $\lambda_n\to\infty$ as $n\to\infty$. From the equations in (13). Denoting $\phi_3(x,\lambda)=\{y_{ij}^{+}(x)\}_{i,j=1}^{m}$, when $\lambda=\lambda_n$ for $n=1,2,...$, we get

$$y_{ii}^{+\prime\prime}(x)+(\lambda-(2\lambda p_{ii}(x)+q_{ii}(x)))y_{ii}^{+}(x)$$
$$-\sum_{k\neq i}(2\lambda p_{ii}(x)+q_{ii}(x))y_{ki}^{+}(x)=0 \quad (36)$$

and

$$y_{ij}^{+\prime\prime}(x)+(\lambda-(2\lambda p_{ii}(x)+q_{ii}(x)))y_{ij}^{+}(x)$$
$$-\sum_{k\neq j}(2\lambda p_{ii}(x)+q_{ii}(x))y_{kj}^{+}(x)=0 \quad (37)$$

Multiplying (36) and (37) by $y_{ij}^{+}(x)$ and $y_{ii}^{+}(x)$ respectively, then subtracting one from the other and using (30), nothing that the eigenvalues of the problem are all real, we have

$$\left(y_{ii}^{+\prime}(x)y_{ij}^{+}(x)-y_{ii}^{+}(x)y_{ij}^{+\prime}(x)\right)'$$
$$=\sum_{k\neq i}(2\lambda p_{ik}(x)+q_{ik}(x))\left(y_{ki}^{+}(x)y_{ij}^{+}(x)-y_{ii}^{+}(x)y_{kj}^{+}(x)\right)$$
$$=(2\lambda p_{ij}(x)+q_{ij}(x))\left[y_{ij}^{+}(x)y_{ji}^{+}(x)-y_{ii}^{+}(x)y_{ij}^{+}(x)\right]$$
$$+\sum_{k\neq i,j}(2\lambda p_{ij}(x)+q_{ij}(x))\left(y_{ki}^{+}(x)y_{ij}^{+}(x)-y_{ii}^{+}(x)y_{kj}^{+}(x)\right)$$
$$=-(2\lambda p_{ij}(x)+q_{ij}(x))M_2\exp(-i\lambda T(x))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right)$$

(38).

Similarly, from the equations in (10), denoting $\phi_2(x,\lambda)=\{y_{ij}^{-}(x)\}_{i,j=1}^{m}$, we get

$$\left(y_{ii}^{-\prime}(x)y_{ij}^{-}(x)-y_{ii}^{-}(x)y_{ij}^{-\prime}(x)\right)'=$$
$$-(2\lambda p_{ij}(x)+q_{ij}(x))M_1\exp(-i(\lambda x-\beta(x)))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right)$$

(39).

Similarly, from the equations in (8), denoting $\phi_1(x,\lambda)=\{y_{ij}^{0}(x)\}_{i,j=1}^{m}$, we get

$$\left(y_{ii}^{0\prime}(x)y_{ij}^{0}(x)-y_{ii}^{0}(x)y_{ij}^{0\prime}(x)\right)'$$

$$=-\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)\left[\cos^{2}(\lambda x)\right]+O\left(\frac{1}{\lambda}\right) \quad (40)$$

When λ is an eigenvalue with multiplicity m, we have $\phi_3'(\pi,\lambda)=0_m$. By integrating both sides of (38) from a_2 to π, for $\lambda_n\to\lambda$ and $n\to\infty$, we obtain

$$-\left(y_{ii}^{+\prime}(x)y_{ij}^{+}(x)-y_{ii}^{+}(x)y_{ij}^{+\prime}(x)\right)=$$

$$=\int_{a_2}^{\pi}\left[-s_{ij}(x)M_2\cos^{2}(\lambda T(x))E_m\left(1+O\left(\frac{1}{\lambda}\right)\right)\right]dx$$

(41)

By integrating both sides of (39) from a_1 to a_2 and applying the boundary condition

$$-\left(y_{ii}^{-\prime}(x)y_{ij}^{-}(x)-y_{ii}^{-}(x)y_{ij}^{-\prime}(x)\right)=$$

$$=\int_{a_1}^{a_2}\left[-s_{ij}(x)M_1\cos^{2}(\lambda x-\beta(x))\left(1+O\left(\frac{1}{\lambda}\right)\right)\right]dx$$

(42)

By integrating both sides of (40) from 0 to a_1 and applying the boundary condition $\phi_1'(0,\lambda)=0_m$, we obtain, for $\lambda_n\to\lambda$ and $n\to\infty$,

$$\left(y_{ii}^{0\prime}(x)y_{ij}^{0}(x)-y_{ii}^{0}(x)y_{ij}^{0\prime}(x)\right)=$$

$$-\int_{0}^{a_1}\left[s_{ij}(x)\left[\cos^{2}(\lambda x)\right]+O\left(\frac{1}{\lambda}\right)\right]dx \quad (43)$$

Sum the above (41), (42) and (43), then use the boundary conditions at point $x=a_1$ and $x=a_2$, we get

$$0=-\int_{0}^{a_1}\left[s_{ij}(x)\left[\cos^{2}(\lambda x)\right]+O\left(\frac{1}{\lambda}\right)\right]dx$$

$$+\int_{a_1}^{a_2}\left[-s_{ij}(x)\left[M_1\cos^{2}(\lambda x-\beta(x))\right]\right]dx$$

$$+\int_{a_2}^{\pi}\left[-s_{ij}(x)\left[M_2\cos^{2}(\lambda T(x))\right]\right]dx+O\left(\frac{1}{\lambda}\right)$$

By a simple computation, we see that

$$\int_{0}^{a_1}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx+M_1\int_{a_1}^{a_2}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx$$

$$+M_2\int_{a_2}^{\pi}\left(2\lambda p_{ij}(x)+q_{ij}(x)\right)dx$$

$$=-\int_{0}^{a_1}\left[s_{ij}(x)\cos 2\lambda x\right]dx$$

$$-\int_{a_1}^{a_2}\left[s_{ij}(x)\left[M_1\cos 2(\lambda x-\beta(x))\right]\right]dx$$

$$-\int_{a_2}^{\pi}\left[s_{ij}(x)\left[M_2\cos 2(\lambda T(x))\right]\right]dx+O\left(\frac{1}{\lambda}\right) \quad (44)$$

$$=-2\lambda\int_{0}^{a_1}p_{ij}(x)\cos(2\lambda x)\,dx-\int_{0}^{a_1}q_{ij}(x)\cos(2\lambda x)\,dx$$

$$-2\lambda M_1\int_{a_1}^{a_2}p_{ij}(x)\cos 2\lambda x\cos 2\beta(x)dx$$

$$-M_1\int_{a_1}^{a_2}q_{ij}(x)\cos 2\lambda x\cos 2\beta(x)dx$$

$$-2\lambda M_1\int_{a_1}^{a_2}p_{ij}(x)2\lambda x\sin 2\beta(x)dx$$

$$-M_1\int_{a_1}^{a_2}q_{ij}(x)\sin 2\lambda\sin 2\beta(x)dx$$

$$-2\lambda M_2\int_{a_2}^{\pi}p_{ij}(x)\cos 2\lambda(2a_1-2a_2+x)\cos 2\beta(x)dx$$

$$-M_2\int_{a_2}^{\pi}q_{ij}(x)\cos 2\lambda(2a_1-2a_2+x)\cos 2\beta(x)dx$$

$$-2\lambda M_2\int_{a_2}^{\pi}p_{ij}(x)\sin 2\lambda(2a_1-2a_2+x)\sin 2\beta(x)dx$$

$$-M_2\int_{a_2}^{\pi}q_{ij}(x)\sin 2\lambda(2a_1-2a_2+x)\sin 2\beta(x)dx$$

Then, we obtain, for $\lambda_n\to\infty$ and $n\to\infty$,

$$= -2\lambda \int_0^{a_1} p_{ij}(x)\cos(2\lambda x)\,dx$$

$$-2\lambda M_1 \int_{a_1}^{a_2} p_{ij}(x)\cos 2\lambda x \cos 2\beta(x) dx$$

$$-2\lambda M_1 \int_{a_1}^{a_2} p_{ij}(x) 2\lambda x \sin 2\beta(x) dx$$

$$-2\lambda M_2 \int_{a_2}^{\pi} p_{ij}(x)\cos 2\lambda(2a_1 - 2a_2 + x)\cos 2\beta(x) dx$$

$$-2\lambda M_2 \int_{a_2}^{\pi} p_{ij}(x)\sin 2\lambda(2a_1 - 2a_2 + x)\sin 2\beta(x) dx$$

By Riemann-Lebesgue Lemma, the right side of (44) approaches 0 as $\lambda_n = \lambda$ and $n \to \infty$. This implies that

$$\int_0^{a_1} p_{ij}(x) dx + M_1 \int_{a_1}^{a_2} p_{ij}(x)\,dx + M_2 \int_{a_2}^{\pi} p_{ij}(x)\,dx = 0$$

$$\int_0^{a_1} q_{ij}(x) dx + M_1 \int_{a_1}^{a_2} q_{ij}(x)\,dx + M_2 \int_{a_2}^{\pi} q_{ij}(x)\,dx = 0$$

We have reached a contradiction. The conclusion for this case is proved.

(ii) Next, we assume that

$$\int_0^{a_1} \left(2\lambda p_{ij}(x) + q_{ij}(x)\right) dx + M_1 \int_{a_1}^{a_2} \left(2\lambda p_{ij}(x) + q_{ij}(x)\right) dx$$

$$+M_2 \int_{a_2}^{\pi} \left(2\lambda p_{ij}(x) + q_{ij}(x)\right) dx = 0$$

or

$$\int_0^{a_1} s_{ij}(x) dx + M_1 \int_{a_1}^{a_2} s_{ij}(x)\,dx + M_2 \int_{a_2}^{\pi} s_{ij}(x)\,dx = 0, \forall i \neq j$$

,

where $s_{ij}(x) = \left(2\lambda p_{ij}(x) + q_{ij}(x)\right)$ and

$$\int_0^{a_1} \left[s_{ii}(x) - s_{jj}(x)\right] dx + M_1 \int_{a_1}^{a_2} \left[s_{ii}(x) - s_{jj}(x)\right] dx$$

$$+M_2 \int_{a_2}^{\pi} \left[s_{ii}(x) - s_{jj}(x)\right] dx \neq 0$$

Without loss of generality, we assume that for $i = 1,\ j = 2$

$$\int_0^{a_1} \left[s_{11}(x) - s_{22}(x)\right] dx + M_1 \int_{a_1}^{a_2} \left[s_{11}(x) - s_{22}(x)\right] dx$$

$$+M_2 \int_{a_2}^{\pi} \left[s_{11}(x) - s_{22}(x)\right] dx \neq 0$$

$$T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

and $y = Tz$. Then, the problem $(1)-(7)$ becomes

$$\left.\begin{aligned} & z'' + \left(\lambda^2 - R(x)\right) z = 0 \\ & z'(0) = z'(\pi) = 0 \\ & z(a_1 + 0) = \alpha_1 z(a_1 - 0) \\ & z'(a_1 + 0) = \beta_1 z'(a_1 - 0) + i\lambda\gamma_1 z(a_1 - 0) \\ & z(a_2 + 0) = \alpha_2 z(a_2 - 0) \\ & z'(a_2 + 0) = \beta_2 z'(a_2 - 0) + i\lambda\gamma_2 z(a_2 - 0) \end{aligned}\right\} \qquad (45)$$

where $R(x) = T^{-1} S(x) T$. By a simple computation, we get

$$R(x) = \begin{bmatrix} \frac{1}{2}(s_{11} + s_{22}) + s_{12} & \frac{1}{2}(s_{22} - s_{11}) & * & * & * \\ \frac{1}{2}(s_{22} - s_{11}) & \frac{1}{2}(s_{11} + s_{22}) + s_{12} & * & * & * \\ * & * & q_{33} & \cdots & \\ * & * & \vdots & \ddots & \\ * & * & & \cdots & q_{mm} \end{bmatrix}(x)$$

We note that the two problems $(1)-(7)$ and (45) have exactly the same spectral structure. Denote $R(x) = \left\{r_{ij}(x)\right\}_{i,j=1}^{m}$. Since

$$\int_0^{a_1} r_{12}(x)dx + M_1\int_{a_1}^{a_2} r_{12}(x)\,dx + M_2\int_{a_2}^{\pi} r_{12}(x)\,dx =$$

$$\int_0^{a_1}\left[s_{11}(x)-s_{22}(x)\right]dx + M_1\int_{a_1}^{a_2}\left[s_{11}(x)-s_{22}(x)\right]dx$$

$$+M_2\int_{a_2}^{\pi}\left[s_{11}(x)-s_{22}(x)\right]dx \neq 0$$

By part (i), the conclusion of the theorem holds for the problem (45), and hence holds for the problem $(1)-(7)$. The proof is completed.

4. Discussion

In this study vectorial singular diffusion equations with discontinuous conditions is considered. Firstly, important definitions and lemmas which are used frequently in characteristic function and asymptotics of solutions of vectorial singular diffusion equations with discontinuous conditions operators are given. Finally, the theorem about multiplicity of eigenvalues of a vectorial singular diffusion equations with discontinuous conditions is given and proved.

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