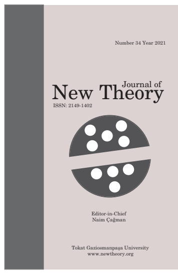
Journal of New Theory
ISSN: 2149-1402

34 (2021) 28-36
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



Best Proximity Point Results on Strong b-Metric Spaces

Mustafa Aslantaş[1]

Article History
Received: 20 Aug 2020
Accepted: 15 Mar 2021
Published: 30 Mar 2021
Research Article

Abstract — In the present paper, we prove the two best proximity point results on strong b-metric spaces with coefficient λ by introducing two new concepts which are named as BW b-contraction and proximal BW b-contraction. Thus, we generalise and improve many results available in the literature. To support our main results, some nontrivial and illustrative examples are given.

Keywords — *Best proximity point, BW-contraction, strong b-metric space*

Mathematics Subject Classification (2020) — 54E05, 54E35

1. Introduction and Preliminaries

In 1922, Banach [1] proved a fundamental theorem as named the Banach contraction principle, which is considered the beginning of fixed point theory on metric space. Due to its applicability, this principle has been extended and generalized by many authors in various ways [2–11]. In this sense, Boyd and Wong [12] obtained a fixed point theorem, a well-known generalization of the Banach contraction principle, as follows:

Theorem 1.1. Let $(\mho, \vartheta)$ be a complete metric space and $T : \mho \to \mho$ be a mapping such that

$$\vartheta(T\varsigma, T\varrho) \leq \psi\left(\vartheta(\varsigma, \varrho)\right)$$

for all $\varsigma, \varrho \in \mho$ where $\psi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous from the right function such that $\psi(\gamma) < \gamma$ for all $\gamma > 0$. Then, T has a unique fixed point $\xi \in \mho$.

The set of functions ψ is denoted Ψ.

On the other hand, Czerwik [13, 14] introduced the notion of a b-metric which is a generalization of a metric with a view of generalizing the Banach contraction principle.

Definition 1.2. Let $\mho$ be a non-empty set and $\vartheta : \mho \times \mho \to [0, +\infty)$ be a function such that for all $\varsigma, \varrho, \xi \in \mho$,

(1) $\vartheta(\varsigma, \varrho) = 0$ if and only if $\varsigma = \varrho$,
(2) $\vartheta(\varsigma, \varrho) = \vartheta(\varrho, \varsigma)$,
(3) $\vartheta(\varsigma, \xi) \leq \lambda[\vartheta(\varsigma, \varrho) + \vartheta(\varrho, \xi)]$.

Then ϑ is called a b-metric on ς and (ς, ϑ) is called a b-metric space.

[1]maslantas@karatekin.edu.tr (Corresponding Author)
[1]Department of Mathematics, Science Faculty, Çankırı Karatekin University, Çankırı, Turkey

It is clear that every metric space is a b-metric space, but not conversely. Indeed, let us consider the set $\mho = \mathbb{R}$ is endowed with the b-metric defined as $\vartheta(\varsigma, \varrho) = (\varsigma - \varrho)^2$ for all $\varsigma, \varrho \in \mho$. Then, $(\mho, \vartheta)$ is a b-metric space, but it is not a standard metric space. Note that b-metric may not be continuous. To remedy this deficiency, Kirk and Shahzad [15] introduced a strong b-metric space as follows:

Definition 1.3. [15] Let $\mho$ be a nonempty set. A map $\vartheta : \mho \times \mho \to [0, \infty)$ is a strong b-metric on $\mho$ if for all $\varsigma, \varrho, \xi \in \mho$ and $\lambda \geq 1$ the following conditions hold:
(i) $\varsigma = \varrho$ if and only if $\vartheta(\varsigma, \varrho) = 0$;
(ii) $\vartheta(\varsigma, \varrho) = \vartheta(\varrho, \varsigma)$;
(iii) $\vartheta(\varsigma, \xi) \leq \vartheta(\varsigma, \varrho) + \lambda\vartheta(\varrho, \xi)$.
Moreover, the triple $(\mho, \lambda, \vartheta)$ is called a strong b-metric space.

Lemma 1.4. [15] Every strong b-metric is continuous.

Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ. Each strong b-metric ϑ on $\mho$ generates T_0 topology τ_ϑ, which has, as a base, the family open p-balls

$$B(\varsigma, \varepsilon) = \{\varrho \in \mho : \vartheta(\varsigma, \varrho) < \varepsilon\}$$

for all $\varsigma \in \mho$ and $\varepsilon > 0$ A sequence $\{\varsigma_n\}$ in ς is said to be a Cauchy sequence if

$$\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0.$$

A sequence $\{\varsigma_n\}$ converges to a point ς in $\mho$ if and only if

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma) = 0.$$

$(\mho, \vartheta)$ is said to be complete if every Cauchy sequence $\{\varsigma_n\}$ in $\mho$ converges with respect to τ_ϑ to a point $\varsigma \in \mho$.

Recently, the fixed point theory has been extended and generalized in different ways for nonself mappings $T : \Gamma \to \Lambda$, where Γ and Λ are the subsets of a metric space $(\mho, \vartheta)$. Indeed, if $\Gamma \cap \Lambda = \emptyset$, it cannot have a solution of equation $T\varsigma = \varsigma$. Hence, it is sensible to investigate if there is a point ς such that $\vartheta(\varsigma, T\varsigma)$ is minimum. The concept of best proximity point has been emerged with this idea. A point ς is called a best proximity point if $\vartheta(\varsigma, T\varsigma) = \vartheta(\Gamma, \Lambda)$. Since every best proximity point is a natural generalization of fixed point, many authors have studied this topic [16–19].

Now, we recall some fundamental definitions and results on strong b-metric spaces which are useful for our main results.

Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$. We denote the following subsets of Γ and Λ, respectively,

$$\Gamma_0 = \{\varsigma \in \Gamma : \vartheta(\varsigma, \varrho) = \vartheta(\Gamma, \Lambda) \text{ for some } \varrho \in \Lambda\}$$

and

$$\Lambda_0 = \{\varrho \in \Lambda : \vartheta(\varsigma, \varrho) = \vartheta(\Gamma, \Lambda) \text{ for some } \varrho \in \Gamma\}$$

where $\vartheta(\Gamma, \Lambda) = \inf\{\vartheta(\varsigma, \varrho) : \varsigma \in \Gamma \text{ and } \varrho \in \Lambda\}$.

Definition 1.5. [20] Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$ with $\Gamma_0 \neq \emptyset$. Then, the pair (Γ, Λ) is said to exhibit the weak p-property if

$$\left.\begin{array}{l} \vartheta(\varsigma_1, \varrho_1) = \vartheta(\Gamma, \Lambda) \\ \vartheta(\varsigma_2, \varrho_2) = \vartheta(\Gamma, \Lambda) \end{array}\right\} \Longrightarrow \vartheta(\varsigma_1, \varsigma_2) \leq \vartheta(\varrho_1, \varrho_2)$$

for all $\varsigma_1, \varsigma_2 \in \Gamma_0$ and $\varrho_1, \varrho_2 \in \Lambda_0$.

Definition 1.6. [21] Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ, a mapping $T : \Gamma \to \Lambda$ is said to be a proximal contraction if there exists a nonnegative number $\alpha < 1$ such that, for all $u_1, u_2, \varsigma_1, \varsigma_2$ in Γ,

$$\left.\begin{array}{l}\vartheta(u_1, T\varsigma_1) = \vartheta(\Gamma, \Lambda)\\ \vartheta(u_2, T\varsigma_2) = \vartheta(\Gamma, \Lambda)\end{array}\right\} \Longrightarrow \vartheta(u_1, u_2) \leq \alpha\vartheta(\varsigma_1, \varsigma_2)$$

Definition 1.7. Let $(\mho, \vartheta)$ be a strong b-metric space with coefficient λ and Γ and Λ be nonempty subsets of $\mho$. If every sequence $\{\varrho_n\}$ in Λ satisfying the condition $\vartheta(\varsigma, \varrho_n) \to \vartheta(\varsigma, \Lambda)$ for some ς in Γ has a subsequence $\{\varrho_{n_k}\}$ such that $\varrho_{n_k} \to \varrho \in \Lambda$, then Λ is called an approximately compact with respect to Γ.

In the present paper, we prove two best proximity point results on strong b-metric spaces with coefficient λ by introducing two new concepts which are named as BW b-contraction and proximal BW b-contraction. Thus, we generalize and improve many results avaliable in the literature. Besides, to support our main results, some nontrivial and illustrative examples are given.

2. Best Proximity Point Results with Weak p-Property

We begin the following new concept of BW b-contraction mapping in this section.

Definition 2.1. Let $(\mho, \vartheta)$ be strong b-metric space with coefficient λ, Γ and Λ be nonempty subsets of $\mho$ and $T \to \Gamma \to \Lambda$ be a mapping. T is called BW b-contraction mapping if there exists $\psi \in \Psi$ such that

$$\vartheta(T\varsigma, T\varrho) \leq \psi\left(\vartheta(\varsigma, \varrho)\right)$$

for all $\varsigma, \varrho \in \Gamma$

Theorem 2.2. Let Γ and Λ be closed subsets of complete strong b-metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a BW b-contraction mapping. Assume that the pair (Γ, Λ) has the weak p-property and $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Proof. Let $\varsigma_0 \in \Gamma_0$ be an arbitrary point. Since $T\varsigma_0 \in T(\Gamma_0) \subseteq \Lambda_0$, there exists $\varsigma_1 \in \Gamma_0$ such that

$$\vartheta(\varsigma_1, T\varsigma_0) = \vartheta(\Gamma, \Lambda)$$

Similarly, there exists $\varsigma_2 \in \Gamma_0$ such that

$$\vartheta(\varsigma_2, T\varsigma_1) = \vartheta(\Gamma, \Lambda)$$

Since (Γ, Λ) has the weak p-Property, we have

$$\vartheta(\varsigma_1, \varsigma_2) \leq \vartheta(T_0, T\varsigma_1)$$

Continuing this process, we can construct a sequence $\{\varsigma_n\}$ such that

$$\vartheta(\varsigma_{n+1}, T\varsigma_n) = \vartheta(\Gamma, \Lambda) \tag{1}$$

and

$$\vartheta(\varsigma_n, \varsigma_{n+1}) \leq \vartheta(T\varsigma_{n-1}, T\varsigma_n) \tag{2}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\varsigma_{n_0} = \varsigma_{n_0+1}$, then the proof is done. Assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \mathbb{N}$. Using contractivity of T, we get

$$\begin{aligned}\vartheta(\varsigma_n, \varsigma_{n+1}) &\leq \vartheta(T\varsigma_{n-1}, T\varsigma_n) \\ &\leq \psi\left(\vartheta(\varsigma_{n-1}, \varsigma_n)\right) \\ &< \vartheta(\varsigma_{n-1}, \varsigma_n)\end{aligned} \tag{3}$$

for all $n \geq 1$. Thus, $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is a nonincreasing sequence in $\mathbb{R}$. Therefore, the sequence $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is convergent. Hence, there is $u \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = u.$$

We want to show that $u = 0$. Suppose that $u > 0$.

$$\begin{aligned} 0 &< u = \lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) \\ &\leq \lim_{n\to\infty} \vartheta\left(T\varsigma_{n-1}, T\varsigma_n\right) \\ &\leq \lim_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n)) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n)) \\ &\leq \psi(u) \\ &< u. \end{aligned}$$

This is a contradiction. Therefore, we have $\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = 0$. After that, we want to show that $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$. Assume the contrary. Hence, there is two subsequences $\{\varsigma_{n_k}\}$ and $\{\varsigma_{m_k}\}$ with $m_k > n_k \geq k$ and $\varepsilon > 0$ such that

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \geq \varepsilon \tag{4}$$

for all $k \geq 1$, where m_k is the smallest natural number satisfying (10) corresponding n_k. Therefore, we get

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) < \varepsilon.$$

Hence, we have

$$\begin{aligned} \varepsilon &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \\ &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \\ &< \varepsilon + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \end{aligned} \tag{5}$$

Letting $k \to \infty$ in (5), $\lim_{k\to\infty} \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) = \varepsilon$. Also, we have

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \leq \lambda\vartheta(\varsigma_{n_k}, \varsigma_{n_k+1}) + \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) + \lambda\vartheta(\varsigma_{m_k+1}, \varsigma_{m_k}) \tag{6}$$

and

$$\vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \leq \lambda\vartheta(\varsigma_{n_k+1}, \varsigma_{n_k}) + \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) + \lambda\vartheta(\varsigma_{m_k}, \varsigma_{m_k+1}). \tag{7}$$

Taking limit as $k \to \infty$ in (6) and (7), we get

$$\lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) = \varepsilon.$$

Then, we have

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_{n_k}, \varsigma_{m_k})) \\ &\leq \psi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

This is a contradiction. Hence, $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$ and so $\{\varsigma_n\}$ is a Cauchy sequence in Γ. Similarly, it can be seen that $\{T\varsigma_n\}$ is a Cauchy sequence in Λ. Since $(\mho, \vartheta)$ is a complete strong b-metric space with coefficient λ and Γ and Λ are closed subsets of $\mho$, there exist $\varsigma^* \in \Gamma$ and $\varrho^* \in \Lambda$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*) = 0$$

and

$$\lim_{n\to\infty} \vartheta(T\varsigma_n, \varrho^*) = 0$$

From Lemma 1.4, letting $n \to \infty$ in (1), we have

$$\vartheta(\varsigma^*, \varrho^*) = \vartheta(\Gamma, \Lambda). \tag{8}$$

Now, assume that $\varsigma^* \neq \varsigma_n$ for all $n \in \dot{\mathbb{N}}$. Then, we have

$$\begin{aligned} \vartheta(\varrho^*, T\varsigma^*) &= \lim_{n\to\infty} \vartheta(T\varsigma_n, T\varsigma^*) \\ &\leq \limsup_{n\to\infty} \psi(\vartheta(\varsigma_n, \varsigma^*)) \\ &< \vartheta(\varsigma_n, \varsigma^*) \end{aligned}$$

Assume that $\varsigma^* = \varsigma_n$ for some $n \in \mathbb{N}$. Then, we can find a subsequence $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $\varsigma^* \neq \varsigma_{n_k}$ for all $k \in \mathbb{N}$ and so we can consider this subsequence in the above steps. Hence, we have $\vartheta(\varrho^*, T\varsigma^*) = 0$ and so $\varrho^* = T\varsigma^*$. Thus, ς^* is a best proximity point of T. □

Example 2.3. Let $\mho = \mathbb{N}$ and $\vartheta : \mho \times \mho \to \mathbb{R}$ be a function defined by

$$\vartheta(\varsigma, \varrho) = \begin{cases} 0 & , \quad \varsigma = \varrho \\ 3 & , \quad \varsigma, \varrho \in \{2n-1 : n \geq 1\} \text{ and } \varsigma \neq \varrho \\ 1 & , \quad \text{otherwise} \end{cases}$$

It is clear that $(\mho, \vartheta)$ is a strong b-metric space with coefficient $\lambda \geq 2$. Now, we will show that $(\mho, \vartheta)$ is complete. Indeed, let $\{\varsigma_n\}$ be a Cauchy sequence. Then, for every $\varepsilon > 0$, we have

$$\vartheta(\varsigma_n, \varsigma_m) < \varepsilon$$

for all $m, n \geq n_0$. Hence, we get $\varsigma_n = \varsigma_m = \varsigma$ for all $m, n \geq n_0$. Define the sets $\Gamma = \{2n-1 : n \geq 1\}$ and $\Lambda = \{2n : n \geq 1\}$. Then, we have $\Gamma = \Gamma_0$, $\Lambda = \Lambda_0$, and $\vartheta(\Gamma, \Lambda) = 1$. Moreover, (Γ, Λ) has the weak p-Property. Let $T : \Gamma \to \Lambda$ and $\psi : [0, \infty) \to [0, \infty)$ be mappings defined as

$$T(2n-1) = 2n$$

for all $n \geq 1$ and

$$\psi(t) = \frac{t}{3}$$

for all $t \in [0, \infty)$. Then, it can be seen that $\psi \in \Psi$ and T is a BW b-contraction mapping. Furthermore, we have $T(\Gamma_0) \subseteq \Lambda_0$. Hence, all the hypotheses of Theorem 2.2 are satisfied. Therefore, T has a best proximity point in Γ.

Taking $\Gamma = \Lambda = \mho$ in Theorem 2.2, we give the following fixed point result.

Corollary 2.4. Let $(\mho, \vartheta)$ be a complete strong b-metric space with coefficient λ and $T : \mho \to \mho$ be a BW b-contraction mapping. Then, T has a fixed point in $\mho$.

If we take $\lambda = 1$ in Theorem 2.2 and Corollary 2.4 , we obtain the following results, respectively.

Corollary 2.5. Let Γ and Λ be closed subsets of complete metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a BW b-contraction mapping. Assume that the pair (Γ, Λ) has the weak p-Property and $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Corollary 2.6. Let $(\mho, \vartheta)$ be a complete metric space and $T : \mho \to \mho$ be a BW b-contraction mapping. Then, T has a fixed point in $\mho$.

3. Best Proximity Point Results with Proximal Contraction

Definition 3.1. Let Γ and Λ be subsets of strong b-metric space $(\mho, p)$ and $T : \Gamma \to \Lambda$ be a mapping. T is called a proximal BWb-contraction if the following condition satisfies

$$\left.\begin{array}{l}\vartheta(u_1, T\varsigma_1) = \vartheta(\Gamma, \Lambda)\\ \vartheta(u_2, T\varsigma_2) = \vartheta(\Gamma, \Lambda)\end{array}\right\} \Longrightarrow \vartheta(u_1, u_2) \leq \psi(\vartheta(\varsigma, \varrho))$$

for all $u_1, u_2, \varsigma_1, \varsigma_2 \in \Gamma$.

Theorem 3.2. Let $(\mho, \vartheta)$ be a complete strong b-metric space with coefficient λ, and Γ and Λ be closed subsets of $\mho$ with $\Gamma_0 \neq \emptyset$. Assume that $T : \Gamma \to \Lambda$ be a mapping satisfying $T(\Gamma_0) \subseteq \Lambda_0$ and Λ is an approximately compact w.r.t Γ. If T is a proximal BW b-contraction, then T has a best proximity poit in Γ.

Proof. Let $\varsigma_0 \in \Gamma_0$ be an arbitrary point. Since $T\varsigma_0 \in T(\Gamma_0) \subseteq \Lambda_0$, there exists $\varsigma_1 \in \Gamma_0$ such that

$$\vartheta(\varsigma_1, T\varsigma_0) = \vartheta(\Gamma, \Lambda)$$

Similarly, there exists $\varsigma_2 \in \Gamma_0$ such that

$$\vartheta(\varsigma_2, T\varsigma_1) = \vartheta(\Gamma, \Lambda)$$

Since T is a proximal BW b-contraction, we have

$$\vartheta(\varsigma_1, \varsigma_2) \leq \psi(\vartheta(\varsigma_0, \varsigma_1))$$

Continuing this process, we can construct a sequence $\{\varsigma_n\}$ in Γ such that

$$\vartheta(\varsigma_{n+1}, T\varsigma_n) = \vartheta(\Gamma, \Lambda) \tag{9}$$

and

$$\vartheta(\varsigma_n, \varsigma_{n+1}) \leq \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $\vartheta(\varsigma_{n_0}, \varsigma_{n_0+1}) = 0$, then the proof is done. Assume that $\vartheta(\varsigma_n, \varsigma_{n+1}) > 0$ for all $n \in \mathbb{N}$. Hence, we have

$$\begin{array}{rcl}\vartheta(\varsigma_n, \varsigma_{n+1}) & \leq & \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & < & \vartheta(\varsigma_{n-1}, \varsigma_n)\end{array}$$

for all $n \geq 1$. Thus, $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is a nonincreasing sequences in $\mathbb{R}$. Therefore, the sequence $\{\vartheta(\varsigma_n, \varsigma_{n+1})\}$ is convergent. Hence, there is $u \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = u.$$

We want to show that $u = 0$. Suppose that $u > 0$.

$$\begin{array}{rcl}0 & < & u = \lim\limits_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1})\\ & \leq & \lim\limits_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & \leq & \limsup\limits_{n\to\infty} \psi(\vartheta(\varsigma_{n-1}, \varsigma_n))\\ & \leq & \psi(u)\\ & < & u.\end{array}$$

This is a contradiction. Therefore, we have $\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma_{n+1}) = 0$. After that, we want to show that $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$. Assume the contrary. Hence, there are two subsequences $\{\varsigma_{n_k}\}$ and $\{\varsigma_{m_k}\}$ with $m_k > n_k \geq k$ and $\varepsilon > 0$ such that

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \geq \varepsilon \tag{10}$$

for all $k \geq 1$, where m_k is the smallest natural number satisfying (10) corresponding n_k. Therefore, we get

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) < \varepsilon.$$

Thus, we have

$$\begin{aligned} \varepsilon &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \\ &\leq \vartheta(\varsigma_{n_k}, \varsigma_{m_k-1}) + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \\ &< \varepsilon + \lambda\vartheta(\varsigma_{m_k-1}, \varsigma_{m_k}) \end{aligned} \tag{11}$$

Letting $k \to \infty$ in (11), $\lim_{k\to\infty} p(\varsigma_{n_k}, \varsigma_{m_k}) = \varepsilon$. Also, we have

$$\vartheta(\varsigma_{n_k}, \varsigma_{m_k}) \leq \lambda\vartheta(\varsigma_{n_k}, \varsigma_{n_k+1}) + \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) + \lambda\vartheta(\varsigma_{m_k+1}, \varsigma_{m_k}) \tag{12}$$

and

$$\vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \leq \lambda\vartheta(\varsigma_{n_k+1}, \varsigma_{n_k}) + \vartheta(\varsigma_{n_k}, \varsigma_{m_k}) + \lambda\vartheta(\varsigma_{m_k}, \varsigma_{m_k+1}). \tag{13}$$

Taking limit as $k \to \infty$ in (12) and (13), we get

$$\lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) = \varepsilon.$$

Then, we have

$$\begin{aligned} \varepsilon &= \lim_{k\to\infty} \vartheta(\varsigma_{n_k+1}, \varsigma_{m_k+1}) \\ &\leq \lim\sup_{n\to\infty} \psi(\vartheta(\varsigma_{n_k}, \varsigma_{m_k})) \\ &\leq \psi(\varepsilon) \\ &< \varepsilon. \end{aligned}$$

This is a contradiction. Hence, $\lim_{n,m\to\infty} \vartheta(\varsigma_n, \varsigma_m) = 0$ and so $\{\varsigma_n\}$ is a Cauchy sequence in Γ. Since (ς, ϑ) is a complete strong b-metric space, and Γ is a closed subset of $\mho$, there exists $\varsigma^* \in \Gamma$ such that

$$\lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*) = 0 \tag{14}$$

Also, we have

$$\begin{aligned} \vartheta(\varsigma^*, \Lambda) &\leq \vartheta(\varsigma^*, T\varsigma_n) \\ &\leq \lambda p(\varsigma^*, \varsigma_{n+1}) + \vartheta(\varsigma_{n+1}, T\varsigma_n) \\ &= \lambda\vartheta(\varsigma^*, \varsigma_{n+1}) + \vartheta(\Gamma, \Lambda) \\ &\leq \lambda\vartheta(\varsigma^*, \varsigma_{n+1}) + \vartheta(\varsigma^*, \Lambda) \end{aligned}$$

From (14), we get $\vartheta(\varsigma^*, T\varsigma_n) \to \vartheta(\varsigma^*, \Lambda)$ as $n \to \infty$. Since Λ is an approximately compact concerning Γ, there exists a subsequence $\{T\varsigma_{n_k}\}$ of $\{T\varsigma_n\}$ such that

$$T\varsigma_{n_k} \to \varrho^*$$

for some $\varrho^* \in \Lambda$. Letting $n \to \infty$ in (9), we have

$$\vartheta(\varsigma^*, \varrho^*) = \vartheta(\Gamma, \Lambda).$$

Besides, since $T\varsigma^* \in \Lambda_0$, there exists $\xi \in \Gamma_0$ such that

$$\vartheta(\xi, T\varsigma^*) = \vartheta(\Gamma, \Lambda)$$

Now, assume that $\varsigma^* \neq \varsigma_n$ for all $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\vartheta(\varsigma^*, \xi) &= \lim_{n\to\infty} \vartheta(\varsigma_{n+1}, \xi)\\ &\leq \lim_{n\to\infty} \psi(\vartheta(\varsigma_n, \varsigma^*))\\ &< \lim_{n\to\infty} \vartheta(\varsigma_n, \varsigma^*)\\ &= 0\end{aligned}$$

Assume that $\varsigma^* = \varsigma_n$ for some $n \in \mathbb{N}$. Then, we can find a subsequence $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $\varsigma^* \neq \varsigma_{n_k}$ for all $k \in \mathbb{N}$ and so we can consider this subsequence in the above steps. Therefore, $\varsigma^* = \xi$ and so T has a best proximity point in Γ. □

Example 3.3. Let $\mho = [0,1] \cup [2,\infty)$ and $\vartheta : \mho \times \mho \to \mathbb{R}$ be a function defined as

$$\vartheta(\varsigma, \varrho) = \begin{cases} 0 & , \quad \varsigma = \varrho \\ 3 & , \quad \varsigma, \varrho \in [0,1] \text{ and } \varsigma \neq \varrho \\ |\varsigma - \varrho| & , \quad \text{otherwise} \end{cases}$$

Then, $(\mho, \vartheta)$ is a complete strong b-metric space with coefficient $\lambda \geq 2$. Define the sets $\Gamma = [0,1]$ and $\Lambda = [2,\infty)$, then we have $\vartheta(\Gamma, \Lambda) = 1$, $\Gamma_0 = \{1\}$ and $\Lambda_0 = \{2\}$. Let $T : \Gamma \to \Lambda$ be a mapping defined as

$$T\varsigma = \begin{cases} 2 & , \quad \varsigma = 1 \\ \varsigma + 2 & , \quad \varsigma \in [0,1) \end{cases}$$

for all $\varsigma \in \Gamma$. Then, we have $T(\Gamma_0) \subseteq \Lambda_0$. Further, we define a function $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. Then, it can be seen that $\psi \in \Psi$ and T is a proximal *BW* b-contraction mapping. Moreover, we have $T(\Gamma_0) \subseteq \Lambda_0$. Hence, all the hypotheses of Theorem 3.2 are satisfied. Therefore, T has a best proximity point in Γ.

If we take $\lambda = 1$ in Theorem 3.2, we obtain the following results, respectively.

Corollary 3.4. Let Γ and Λ be closed subsets of complete metric space $(\mho, \vartheta)$ with $\Gamma_0 \neq \emptyset$ and $T : \Gamma \to \Lambda$ be a proximal *BW* b-contraction mapping satisfying $T(\Gamma_0) \subseteq \Lambda_0$. Then, T has a best proximity point in Γ.

Note that if we take $\Gamma = \Lambda = \mho$ in Definition 3.1, then the proximal *BW* b-contraction mapping becomes *BW* b-contraction mapping. Therefore, we can obtain Corollary 2.5 and Corollary 2.6 from Theorem 3.2.

4. Conclusion

The applications of the fixed point theorems comprise diverse disciplines of mathematics, statistics, and engineering dealing with various problems such as the theory of differential equations, approximation theory, potential theory, functional analysis, and topology. In this paper, we obtain some best proximity point results on strong b-metric spaces and present some generalizations of the fixed point results.

References

[1] S. Banach, *Sur Les Opérations Dans Les Ensembles Abstraits et Leur Applications Aux Équations Intégrales*, Fundamenta Mathematicae 3(1) (1922) 133-181.

[2] M. Abbas, T. Nazir, *Fixed Point of Generalized Weakly Contractive Mappings in Ordered Partial Metric Spaces*, Fixed Point Theory and Application 2012(1) (2012) 1-19.

[3] M. Aslantaş, H. Şahin, D. Türkoğlu, *Some Caristi Type Fixed Point Theorems*, The Journal of Analysis (2020) 1-15. https://doi.org/10.1007/s41478-020-00248-8

[4] L. Ćirić, B. Samet, H. Aydi, C. Vetro, *Common Fixed Points of Generalized Contractions on Partial Metric Spaces and An Application*, Applied Mathematics and Computation 218(6) (2011) 2398-2406.

[5] M. Riaz, S. T. Tehrim, *A Robust Extension of VIKOR Method for Bipolar Fuzzy Sets Using Connection Numbers of SPA Theory Based Metric Spaces*, Artificial Intelligence Review 54 (2020) 561-591.

[6] M. Riaz, S. T. Tehrim, *Bipolar Fuzzy Soft Mappings with Application to Bipolar Disorders*, International Journal of Biomathematics 12(07) (2019) 1-31.

[7] M. Riaz, F. Smarandache, A. Firdous, A. Fakhar, *On Soft Rough Topology with Multi-attribute Group Decision Making*, Mathematics 7(1) (2019) 1-18.

[8] S. T. Tehrim, M. Riaz, *A Novel Extension of TOPSIS to MCGDM with Bipolar Neutrosophic Soft Topology*, Journal of Intelligent and Fuzzy Systems 37(4) (2019) 5531-5549.

[9] M. Riaz, M. R. Hashmi, *Fixed Points of Fuzzy Neutrosophic Soft Mapping with Decision-making*, Fixed Point Theory and Applications 2018(7) (2018) 1-10.

[10] S. Romaguera, *Fixed Point Theorems for Generalized Contractions on Partial Metric Spaces*, Topology and Its Applications 159(1) (2012) 194-199.

[11] S. Romaguera, *A Kirk Type Characterization of Completeness for Partial Metric Spaces*, Fixed Point Theory and Applications (2010) Article ID 493298, 6 pages.

[12] D. W. Boyd, J. S. Wong, *On Nonlinear Contraction*, Proceedings of the American Mathematical Society 20(2) (1969) 458-464.

[13] S. Czerwik, *Contraction Mappings in b-metric Spaces*, Acta Mathematica et Informatica Universitatis Ostraviensis 1(1) (1993) 5-11.

[14] S. Czerwik, *Nonlinear Set-valued Contraction Mappings in b-metric Spaces*, Atti del Seminario Matematico e Fisico dell'Universitá di Modena 46 (1998) 263-276.

[15] W. Kirk, N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer, Cham, 2014.

[16] İ. Altun, M. Aslantaş, H. Sahin, *Best Proximity Point Results for p-proximal Contractions*, Acta Mathematica Hungarica 162 (2020) 393-402.

[17] M. Aslantaş, H. Şahin, İ. Altun, *Best Proximity Point Theorems for Cyclic p-contractions with Some Consequences and Applications*, Nonlinear Analysis: Modelling and Control 26(1) (2021) 113-129.

[18] H. Şahin, *Best Proximity Point Theory on Vector Metric Spaces*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 70(1) 2021 130-142.

[19] H. Şahin, M. Aslantaş, I. Altun, *Feng–Liu Type Approach to Best Proximity Point Results for Multivalued Mappings*, Journal of Fixed Point Theory and Applications 22(11) (2020) 1-13.

[20] M. Gabeleh, *Global Optimal Solutions of Non-self Mappings*, Scientific Bulletin, Series A: Applied Mathematics and Physics 75(3) (2013) 67-74.

[21] S. S. Basha, *Best Proximity Points: Optimal Solutions*, Journal of Optimization Theory and Applications 151 (2011) 210-216.