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Some Divisibility Properties of Lucas Numbers

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Abstract. The Lucas number sequence is a popular number sequence that has been described as similar to the Fibonacci number sequence. A lot of research has been done on this number sequence. Some of these studies are on the divisibility properties of this number sequence. Carlitz (1964) examined the requirement that a given Lucas number can be divided by another Lucas number. After that, many studies have been done on this subject. In the present article, we obtain some divisibility properties of the Lucas Numbers. First, we examine the case $L_{(2n-1)m}/L_m$ and then we obtain $L_{(2n-1)m}$ using different forms of Lucas numbers.

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1. Introduction

L_n n*th* term of Lucas sequence with $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$, $n > 0$. The first few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, ...$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the quadratic equation $x^2 - x - 1 = 0$, then $L_n = \alpha^n + \beta^n$. It is obvious that $\alpha + \beta = 1$, $\alpha.\beta = -1$, $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$.

Many studies have been conducted on the divisibility properties of Lucas numbers. Carlitz [1] proved that "$L_m \mid L_n \Rightarrow n = (2k-1)m, m > 1$". Carlitz and Hunter [2] have reached the following equations;

$$\begin{aligned} L_{n-1}^4 + L_n^4 + L_{n+1}^4 &= 2\left[2.L_n^2 - 5(-1)^n\right]^2, \\ L_{n+1}^5 - L_n^5 - L_{n-1}^5 &= 5L_{n+1}L_nL_{n-1}\left[2.L_n^2 - 5(-1)^n\right], \\ L_{n+1}^7 - L_n^7 - L_{n-1}^7 &= 7L_{n+1}L_nL_{n-1}\left[2.L_n^2 - 5(-1)^n\right]^2. \end{aligned}$$

Also, Carlitz [3] obtained $\lfloor \alpha^k L_n + \frac{1}{2} \rfloor = L_{n+k}$, $n \geq k+2$, $k \geq 2$.

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Hoggatt [5] achieved the following equations in his work titled Lucas triangle;

$$\begin{aligned} L_n^1 &= L_n, \\ L_n^2 &= L_{2n} + 2(-1)^n, \\ &\vdots \\ L_n^8 &= L_{8n} + 8(-1)^n L_n^6 - 20.L_n^4 + 16(-1)^n L_n^2 - 2. \end{aligned}$$

Vajda [11] has reached many equations regarding Lucas Numbers. Some of these are
i. Let t odd prime, $L_{kt} = L_t^k + \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{k}{i}.(-1)^{i(t+1)}.L_t{}^{k-2i}\binom{k-i-1}{i-1}$,
ii. If $(s,t) = d$, than $(L_s, L_t) = L_d$,
iii. $L_{n+r} + (-1)^r.L_{n-r} = L_r.L_n$,
iv. $(L_n)^2 = L_{2n} + (-1)^n.2$,
v. If p is prime, than $L_p \equiv 1 (\mathrm{mod}\ p)$.

Koshy has done some studies on Lucas numbers. Some of these are given below:
i. [8] $L_{2m+2n} + L_{2m-2n} = L_{2m}.L_{2n}$ and $L_{n+r}^2 + L_{n-r}^2 = L_{2n}.L_{2r} + 4.(-1)^{n+r}$,
ii. [9] $L_k^2 + L_{k+1}^2 = L_{2k} + L_{2k+2}$, $L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}$, $L_{m+r}.L_{m+r+1} + L_{m-r}.L_{m-r+1} = L_{2m+2r+1} + L_{2m-2r+1} + 2.(-1)^{m+r}$ and $L_{m+r}.L_{m+r+1} + L_{m-r}.L_{m-r+1} = L_{2m+1} + L_{2r} + 2.(-1)^{m+r}$,
iii. [10] $L_{n+1}^2 - L_n^2 = L_{n-1}.L_{n+2}$, $L_{3n} = L_n.[L_{2n} - (-1)^n]$ and $L_{r-1}.L_{r+1} - L_r^2 = 5.(-1)^{r-1}$, $r \geq 1$.

In addition, Hoggatt and Bergum [6] examined the divisibility and congruance relationships in Lucas numbers. Keskin and Bitim [7] studied on Fibonacci and Lucas congruences and obtained divisibility properties of Fibonacci and Lucas numbers. In [4] authors obtained congruences including Fibonacci and Lucas numbers.

2. Some Divisibility Properties of Lucas Numbers

Theorem 2.1. *Let m is a naturel number, then*

$$L_{5m}/L_m = \begin{cases} (L_{3m}/L_m)^2 - (L_{3m}/L_m) - 1, & m \text{ is odd} \\ (L_{3m}/L_m)^2 + (L_{3m}/L_m) - 1, & m \text{ is even}. \end{cases}$$

Proof. Let m be odd and consider the Binet's formula $L_n = \alpha^n + \beta^n$, then

$$\begin{aligned} (L_{3m}/L_m)^2 - (L_{3m}/L_m) - 1 &= \left(\frac{\alpha^{3m} + \beta^{3m}}{\alpha^m + \beta^m}\right)^2 - \left(\frac{\alpha^{3m} + \beta^{3m}}{\alpha^m + \beta^m}\right) - 1 \\ &= \left(\left(\alpha^{2m} + \beta^{2m}\right) + 1\right)^2 - \left(\alpha^{2m} + \beta^{2m} + 1\right) - 1 \\ &= \alpha^{4m} - (\alpha.\beta)^m.\alpha^{2m} - (\alpha.\beta)^m.\beta^{2m} + \alpha^{2m}.\beta^{2m} + \beta^{4m} \\ &= \frac{(\alpha^m + \beta^m)\left(\alpha^{4m} - \alpha^{3m}.\beta^m + \alpha^{2m}.\beta^{2m} - \alpha^m.\beta^{3m} + \beta^{4m}\right)}{(\alpha^m + \beta^m)} \\ &= L_{5m}/L_m. \end{aligned}$$

The proof is similar for m is even. □

Theorem 2.2. *Let m, n are naturel numbers, for $n \geq 3$, then*

$$(L_{(2n-1)m}/L_m)^2 = \begin{cases} (L_{(2n-3)m}/L_m).(L_{(2n+1)m}/L_m) + (L_{3m}/L_m) + 1, & m \text{ is odd} \\ (L_{(2n-3)m}/L_m).(L_{(2n+1)m}/L_m) - (L_{3m}/L_m) + 1, & m \text{ is even}. \end{cases}$$

Proof. Let m be even and consider the Binet's formula $L_n = \alpha^n + \beta^n$, then

$$
\begin{aligned}
& (L_{(2n-3)m}/L_m) \,.\, (L_{(2n+1)m}/L_m) - (L_{3m}/L_m) + 1 \\
= & \left(\frac{\alpha^{(2n-3)m} + \beta^{(2n-3)m}}{\alpha^m + \beta^m}\right) . \left(\frac{\alpha^{(2n+1)m} + \beta^{(2n+1)m}}{\alpha^m + \beta^m}\right) - \left(\frac{\alpha^{3m} + \beta^{3m}}{\alpha^m + \beta^m}\right) + 1 \\
= & \frac{\alpha^{(2n-1)2m} + (\alpha.\beta)^{2mn} . \left(\alpha^{-3m}.\beta^m + \alpha^m.\beta^{-3m}\right) + \beta^{(2n-1)2m} - \alpha^{4m}}{(\alpha^m + \beta^m)(\alpha^m + \beta^m)} \\
& + \frac{-\alpha^m.\beta^{3m} - \beta^m.\alpha^{3m} - \beta^{4m} + \alpha^{2m} + 2.(\alpha.\beta)^m + \beta^{2m}}{(\alpha^m + \beta^m)(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{(2n-1)2m} + \beta^{4m} + \alpha^{4m} + \beta^{(2n-1)2m} - \alpha^{4m} - \beta^{2m} - \alpha^{2m} - \beta^{4m} + \alpha^{2m} + 2 + \beta^{2m}}{(\alpha^m + \beta^m)(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{(2n-1)2m} + 2 + \beta^{(2n-1)2m}}{(\alpha^m + \beta^m)(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{(2n-1)2m} + 2\,(\alpha.\beta)^{(2n-1)m} + \beta^{(2n-1)2m}}{(\alpha^m + \beta^{mn})(\alpha^m + \beta^m)} \\
= & \frac{\left(\alpha^{(2n-1)m} + \beta^{(2n-1)m}\right)^2}{(\alpha^m + \beta^m)^2} \\
= & (L_{(2n-1)m}/L_m)^2 .
\end{aligned}
$$

The proof is similar for m is odd. □

Theorem 2.3. *Let $m > 3$ and $n \geq 4$ are naturel numbers, then*

$$
L_{(2n-1)m}/L_m = \begin{cases} (L_{3m}/L_m) \,.\, [(L_{(2n-3)m}/L_m) - (L_{(2n-5)m}/L_m)] + (L_{(2n-7)m}/L_m), & m \textit{ is odd} \\ (L_{3m}/L_m) \,.\, [(L_{(2n-3)m}/L_m) + (L_{(2n-5)m}/L_m)] - (L_{(2n-7)m}/L_m), & m \textit{ is even.} \end{cases}
$$

Proof. Let m be odd and consider the Binet's formula $L_n = \alpha^n + \beta^n$, then

$$
\begin{aligned}
& (L_{3m}/L_m) \,.\, [(L_{(2n-3)m}/L_m) - (L_{(2n-5)m}/L_m)] + (L_{(2n-7)m}/L_m) \\
= & \left(\frac{\alpha^{3m} + \beta^{3m}}{\alpha^m + \beta^m}\right) . \left[\frac{\alpha^{(2n-3)m} + \beta^{(2n-3)m} - \alpha^{(2n-5)m} - \beta^{(2n-5)m}}{\alpha^m + \beta^m}\right] + \left(\frac{\alpha^{(2n-7)m} + \beta^{(2n-7)m}}{\alpha^m + \beta^m}\right) \\
= & \frac{\alpha^{2nm} + \alpha^{3m}.\beta^{(2n-3)m} - \alpha^{(2n-2)m} - \alpha^{3m}.\beta^{(2n-5)m} + \beta^{3m}.\alpha^{(2n-3)m} + \beta^{2nm}}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
& + \frac{-\beta^{3m}.\alpha^{(2n-5)m} - \beta^{(2n-2)m} + \alpha^{(2n-6)m} + \alpha^m.\beta^{(2n-7)m} + \beta^m.\alpha^{(2n-7)m} + \beta^{(2n-6)m}}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{2nm} + \beta^{2nm} + \alpha^{2nm}.\alpha^{-7m}.\left(\beta^{3m}.\alpha^{4m} - \beta^{3m}.\alpha^{2m} - \alpha^{5m} + \beta^m + \alpha^m\right)}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
& + \frac{\beta^{2nm}.\beta^{-7m}.\left(\alpha^{3m}.\beta^{4m} - \alpha^{3m}.\beta^{2m} - \beta^{5m} + \alpha^m + \beta^m\right)}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{2nm} + \beta^{2nm} + \alpha^{2nm}.\alpha^{-7m}.\left(-\alpha^{5m}\right) + \beta^{2nm}.\beta^{-7m}.\left(-\beta^{5m}\right)}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{2nm} + (\alpha.\beta)^m\,.\alpha^{2nm}.\alpha^{-2m} + \beta^{2nm} + (\alpha.\beta)^m\,.\beta^{2nm}.\beta^{-2m}}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{2nm}.\alpha^{-m}.(\alpha^m + \beta^m) + \beta^{2nm}.\beta^{-m}.(\alpha^m + \beta^m)}{(\alpha^m + \beta^m).(\alpha^m + \beta^m)} \\
= & \frac{\alpha^{(2n-1)m} + \beta^{(2n-1)m}}{\alpha^m + \beta^m} \\
= & L_{(2n-1)m}/L_m.
\end{aligned}
$$

The proof is similar for m is even. □

Theorem 2.4. *Let m, n are naturel number, $n \geq 3$, then*

$$L_{(2n-1)m} = \begin{cases} \left(L_m^2 + 2\right).L_{(2n-3)m} - L_{(2n-5)m}, & m \text{ is odd} \\ \left(L_m^2 - 2\right).L_{(2n-3)m} - L_{(2n-5)m}, & m \text{ is even.} \end{cases}$$

Proof. Let m be even and consider the Binet's formula $L_n = \alpha^n + \beta^n$, then

$$\begin{aligned} & \left(L_m^2 - 2\right).L_{(2n-3)m} - L_{(2n-5)m} \\ = \; & \left[(\alpha^m + \beta^m)^2 - 2\right].\left(\alpha^{(2n-3)m} + \beta^{(2n-3)m}\right) - \alpha^{(2n-5)m} - \beta^{(2n-5)m} \\ = \; & \alpha^{(2n-1)m} + \alpha^{2m}.\beta^{(2n-3)m} + \beta^{2m}.\alpha^{(2n-3)m} + \beta^{(2n-1)m} - \alpha^{(2n-5)m} - \beta^{(2n-5)m} \\ = \; & \alpha^{(2n-1)m} + \beta^{(2n-1)m} + \alpha^{2nm}.\alpha^{-5m}.\left[(\alpha.\beta)^{2m} - 1\right] + \beta^{2nm}.\beta^{-5m}.\left[(\alpha.\beta)^{2m} - 1\right] \\ = \; & \alpha^{(2n-1)m} + \beta^{(2n-1)m} = L_{(2n-1)m}. \end{aligned}$$

The proof is similar for m is odd. □

Theorem 2.5. *Let m, n are naturel number, then*

$$L_{(2n+1)m} = \begin{cases} \sum_{r=1}^{n+1} \frac{2n+1}{2n-2r+3}\binom{2n-r+1}{r-1}.L_m^{2n-2r+3}, & m \text{ is odd,} \\ \sum_{r=1}^{n+1} \frac{2n+1}{2n-2r+3}\binom{2n-r+1}{r-1}.L_m^{2n-2r+3}.(-1)^{r+1}, & m \text{ is even.} \end{cases}$$

Proof. Let m be odd.

The statement is true for $n = 1$ and $n = 2$.

Assume the correctness of the statement for $n = k - 1$ and for $n = k$, and prove the statement for $n = k + 1$.

By using Theorem 2.4 when m is odd and assumption, we obtain

$$\begin{aligned} L_{(2k+3)m} = \; & \left(L_m^2 + 2\right)\left[\sum_{r=1}^{k+1} \frac{2k+1}{2k-2r+3}\binom{2k-r+1}{r-1}.L_m^{2k-2r+3}\right] \\ & - \sum_{r=1}^{k} \frac{2k-1}{2k-2r+1}\binom{2k-r-1}{r-1}.L_m^{2k-2r+1} \\ = \; & \sum_{r=1}^{k+1} \frac{2k+1}{2k-2r+3}\binom{2k-r+1}{r-1}.L_m^{2k-2r+5} \\ & + \sum_{r=1}^{k+1} \frac{4k+2}{2k-2r+3}\binom{2k-r+1}{r-1}.L_m^{2k-2r+3} \\ & - \sum_{r=1}^{k} \frac{2k-1}{2k-2r+1}\binom{2k-r-1}{r-1}.L_m^{2k-2r+1} \\ = \; & L_m^{2k+3} + (2k+1).L_m^{2k+1} + \sum_{r=1}^{k-1} \frac{2k+1}{2k-2r-1}\binom{2k-r-1}{r+1}.L_m^{2k-2r+1} \\ & + 2.L_m^{2k+1} + (4k+2).L_m + \sum_{r=1}^{k-1} \frac{4k+2}{2k-2r+1}\binom{2k-r}{r}.L_m^{2k-2r+1} \\ & - (2k-1).L_m - \sum_{r=1}^{k-1} \frac{2k-1}{2k-2r+1}\binom{2k-r-1}{r-1}.L_m^{2k-2r+1} \end{aligned}$$

$$\begin{aligned}
&= L_m^{2k+3} + (2k+3)\,.L_m^{2k+1} + (2k+3)\,.L_m \\
&\quad + \sum_{r=1}^{k-1} \binom{2k-r+1}{r+1}.L_m^{2k-2r+1}.\left(\frac{2.(2k+1).(k-r)}{(2k-r).(2k-r+1)} + \frac{2.(2k+1).(r+1)}{(2k-2r+1).(2k-r+1)} - \frac{r.(2k-1).(r+1)}{(2k-r)(2k-2r+1)(2k-r+1)} \right) \\
&= L_m^{2k+3} + (2k+3)\,.L_m^{2k+1} + (2k+3)\,.L_m \\
&\quad + \sum_{r=1}^{k-1} \binom{2k-r+1}{r+1}.L_m^{2k-2r+1}.\frac{2k+3}{2k-2r+1} \\
&= \sum_{r=1}^{k+2} \frac{2k+3}{2k-2r+5}\binom{2k-r+3}{r-1}.L_m^{2k-2r+5}.
\end{aligned}$$

The proof is similar for m is even. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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