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NEW FUZZY DIFFERENTIAL SUBORDINATIONS

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ABSTRACT. In this paper, some new fuzzy differential subordinations obtained by using the integral operator $I_\gamma^m : A_n \to A_n$introduced in [13] are obtained.

1. Introduction and preliminaries

The notion of differential subordination was introduced by S.S. Miller and P.T. Mocanu in papers [6] and [7] and later developed in [8]. Many other authors have contributed to the development of this field of research. The notion of fuzzy subordination was recently introduced by G.I. Oros and Gh. Oros in paper [9] and the notion of fuzzy differential subordination was introduced by the same authors in [10]. After that, some papers related to fuzzy differential subordinations have been published by the same authors, [11], [12], and by other authors, such as [3], [4] and [15].

Similar results on fuzzy differential subordinations obtained by using operators were recently published in [1], [2].

We next give the notations used throughout the paper:

Let U denote the open disc in the complex plane, let $\overline{U}$ denote the closed unit disc in the complex plane and let $\partial U = \{z \in \mathbb{C} : \ |z| = 1\}$. Let $\mathcal{H}(U)$ denote the class of analytic functions in the unit disc U.

We denote the following classes of analytic functions:

$$A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U\}$$

with $A_1 = A$;

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

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for $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$;

$$S^* = \left\{ f \in A : \ \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in U \right\},$$

the class of starlike functions in U;

$$C = \left\{ f \in A : \ \exists \ g \in S^*, \ \operatorname{Re} \frac{zh'(z)}{g(z)} > 0, \ z \in U \right\}$$

the class of close-to-convex (univalent) functions.

Definition 1. *[8, Definition 2.2.b] We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exception set.

Lemma A. [8, Lemma 2.2.d] *Let $q \in Q$ with $q(0) = a$ and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U, with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$ *and*

(iii) $\operatorname{Re} \left(\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \right) \geq m \operatorname{Re} \left(\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right)$.

Definition 2. *[13, Definition 1] For $f \in A_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, let I_γ be the integral operator given by $I_\gamma : A_n \to A_n$,*

$$I_\gamma^0 f(z) = f(z)$$

$$I_\gamma^m f(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z I_\gamma^{m-1} f(z) \cdot t^{\gamma - 1} dt, \ z \in U.$$

By using Definition 2, we can prove the following property for this integral operator:

For $f \in A_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, we have

$$I_\gamma^m f(z) = z + \sum_{k=n+1}^{\infty} \left(\frac{\gamma + 1}{\gamma + k} \right)^m a_k z^k, \ z \in U.$$

Definition 3. *[8, p. 4], [14, p. 36] Let f and F be analytic functions. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.*

Definition 4. *[14] A function $L(z,t)$, $z\in U$, $t\geq 0$, is a subordination chain if $L(\cdot,t)$ is analytic and univalent in U for all $t\geq 0$, and $L(z,t_1)\prec L(z,t_2)$, when $0\leq t_1<t_2<\infty$.*

Lemma B. [8, p. 4], [14, p. 159] *The function*

$$L(z,t)=a_1(t)z+a_2(t)z^2+\dots,$$

with $a_1(t)\neq 0$ for $t\geq 0$ and $\lim_{t\to\infty}|a_1(t)|=\infty$ is a subordination chain if and only if there exist constant $r\in(0,1]$ and $M>0$ such that

(i) $L(z,t)$ is analytic in $|z|<r$ for each $t\geq 0$, locally absolutely continuous in $t\geq 0$ for each $|z|<r$, and satisfies

$$|L(z,t)|\leq M|a_1(t)|,\ \ \text{for } |z|<r \ \text{ and } t\geq 0;$$

(ii) there exists a function $p(z,t)$ analytic in U for all $t\in[0,\infty)$ and measurable in $[0,\infty)$ for each $z\in U$, such that $\operatorname{Re}p(z,t)>0$ for $z\in U$, $t\in[0,\infty)$ and

$$\frac{\partial L(z,t)}{\partial t}=\frac{z\cdot\partial L(z,t)}{\partial z}\cdot p(z,t)\quad \text{or}\quad \operatorname{Re}\frac{z\cdot\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}>0,\ \ z\in U,\ \ t\geq 0$$

for $|z|<r$ and for almost all $t\in[0,\infty)$.

Definition 5. *[8, p. 9] The function $f\in\mathcal{H}(U)$ is called close-to-convex if there exists a starlike function g such that*

$$\operatorname{Re}\frac{zf'(z)}{g(z)}>0,\ \ z\in U.$$

In order to use the concept of fuzzy differential subordination, we remember the following definitions.

Definition 6. *[5] A pair (A,F_A), where $F_A:X\to[0,1]$ and*

$$A=\{x\in X;\ 0\leq F_A(x)\leq 1\}$$

is called fuzzy subset of X. The set A is called the support of the fuzzy set (A,F_A) and F_A is called the membership function of the fuzzy set (A,F_A).

One can also denote $A=\operatorname{supp}(A,F_A)$.

If $A\subset X$, then

$$F_A(x)=\begin{cases}1 & \text{if} \quad x\in A\\ 0 & \text{if} \quad x\notin A.\end{cases} \tag{1}$$

A classical subset of X can be considered as the fuzzy set of X, with the membership function F_A defined as in (1).

Definition 7. *[9] Let $D\subset\mathbb{C}$ and let $z_0\in D$ be a fixed point. We take the functions $f,g\in\mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and we write $f\prec_F g$ or $f(z)\prec_F g(z)$, if there exists a function $F:\mathbb{C}\to[0,1]$, such that*

(i) $f(z_0)=g(z_0)$;

(ii) $F(f(z))\leq F(g(z))$ for all $z\in D$.

Remark 8. *Such function $F : \mathbb{C} \to [0,1]$ can be considered*

$$F(z) = \frac{|z|}{1+|z|},\ F(z) = \frac{1}{1+|z|},\ F(z) = \Big|\sin|z|\Big|,\ F(z) = \Big|\cos|z|\Big|.$$

Definition 9. *[10] Let $\psi : \mathbb{C}^3 \times \overline{U} - \mathbb{C}$, $a \in \mathbb{C}$, and let h be univalent in U, with $h(0) = \psi(a,0,0)$, q be univalent in U, with $q(0) = a$, and p be analytic in U, with $p(0) = a$. Also, $\psi(p(z), zp'^2p''(z); z)$ is analytic in U and $F : \mathbb{C} \to [0,1]$.*

If p is analytic in U and satisfies the (second-order) fuzzy differential subordination

$$\psi(p(z), zp'^2p''(z); z) \le F(h(z)), \tag{2}$$

i.e.

$$\psi(p(z), zp'^2p''(z); z) \prec_F h(z),\ z \in U,$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solution of the fuzzy differential subordination, or more simple a fuzzy dominant, if

$$p(z) \prec_F q(z),\ z \in U$$

for all p satisfying (2). A fuzzy dominant $\widetilde{q}$ that satisfies

$$\widetilde{q}(z) \prec_F q(z),\ z \in U,$$

for all fuzzy dominants q of (2) is said to be the fuzzy best dominant of (2). Note that the fuzzy best dominant is unique up to a rotation in U.

Remark 10. *The function $F : \mathbb{C} \to [0,1]$ can be a function of the form of functions shown in Remark 8.*

2. Main results

Theorem 11. *Let q be univalent in U and let θ and ϕ be analytic functions in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$.*

Let

$$F : U \to [0,1],\ F(z) = \frac{1}{1+|z|},\ z \in U.$$

Set

$$Q(z) = zq'(z) \cdot \phi[q(z)],\ h(z) = \theta[q(z)] + Q(z),$$

and suppose that we have

(i) Q is starlike;

(ii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re} \left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0,\ z \in U.$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$, then

$$\frac{1}{1+|\theta[p(z)] + zp'(z)\cdot\phi[p(z)]|} \le \frac{1}{1+|\theta[q(z)] + zq'(z)\cdot\phi[q(z)]|}, \tag{3}$$

that is

$$F(\theta[p(z)] + zp'(z) \cdot \phi[p(z)] \le F(\theta[q(z)] + zq'(z) \cdot \phi[q(z)])$$

implies

$$\frac{1}{1+|p(z)|} \le \frac{1}{1+|q(z)|}, \ z \in U, \tag{4}$$

that is

$$F(p(z)) \le F(q(z)), \ z \in U$$

and q is the fuzzy best dominant, through F.

Proof. From (i) we know that the function Q is starlike and from (ii) we know that the function h is close-to-convex.

Let the function:

$$\begin{aligned} L(z,t) &= a_1(t)z + a_2(t)z^2 + \dots \\ &= h(z) + tQ(z) = \theta[q(z)] + (1+t)Q(z) \\ &= \theta[q(z)] + (1+t)zq'(z) \cdot \phi[q(z)]. \end{aligned} \tag{5}$$

This function is analytic in U for all $t \ge 0$ and is continuously differentiable on $[0,\infty)$ for $z \in U$.

Differentiating (5) with respect to z we obtain

$$\begin{aligned} \frac{\partial L(z,t)}{\partial z} &= a_1(t) + 2a_2(t)z + \dots \\ &= \theta'[q(z)]q'(z) + (1+t)\{q'(z) \cdot \phi[q(z)] \\ &+ zq''(z) \cdot \phi[q(z)] + zq'(z) \cdot \phi'[q(z)]\}. \end{aligned}$$

For $z = 0$ we have

$$\begin{aligned} a_1(t) &= \theta'[q(0)] \cdot q'(0) + (1+t)q'(0) \cdot \phi[q(0)] \\ &= q'(0) \cdot \phi[q(0)] \cdot \left[\frac{\theta'[q(0)]}{\phi[q(0)]} + 1 + t\right] \ne 0. \end{aligned}$$

Differentiating (5) with respect to t we obtain

$$\frac{\partial L(z,t)}{\partial t} = Q(z) = zq'(z) \cdot \phi[q(z)].$$

We calculate:

$$\operatorname{Re} \frac{z \cdot \partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} = \operatorname{Re} \left[\frac{\theta'[q(z)]}{\phi[q(z)]} + (1+t)\frac{zQ'(z)}{Q(z)}\right].$$

From (i) and (ii), $t \ge 0$, we have

$$\operatorname{Re} \frac{z \cdot \partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} = \operatorname{Re} \left[\frac{\theta'[q(z)]}{\phi[q(z)]} + \frac{zQ'(z)}{Q(z)}\right] + t\operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0.$$

Hence $a_1(t) \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and

$$\mathrm{Re}\, \frac{z \cdot \partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} > 0,$$

for $z \in U$ and $t \geq 0$. Using Lemma B, $L(z,t)$ is a subordination chain which by Definition 4 implies

$$L(z,s) \prec L(z,t) \text{ for } 0 \leq s \leq t. \tag{6}$$

For $t = 0$, (5) becomes $L(z,0) = h(z)$, then (6) becomes

$$h(z) \prec L(z,t),\ t \geq 0,\ z \in U. \tag{7}$$

Using (7) and Definition 3, we have

$$L(\zeta,t) \notin h(U),\ |\zeta| = 1,\ t \geq 0. \tag{8}$$

Let the function $\psi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$,

$$\psi(r,s) = \theta(r) + s\phi(r).$$

For $r = p(z)$, $s = zp'(z)$, $z \in U$, we have

$$\psi(p(z), zp'(z)) = \theta[p(z)] + zp'(z) \cdot \phi[p(z)]$$

which is an analytic function since θ, ϕ and p are analytic functions.
For $r = q(z)$, $s = zq'(z)$, we have

$$\psi(q(z), zq'(z)) = \theta[q(z)] + zq'(z) \cdot \phi[q(z)],\ z \in U.$$

Then the fuzzy differential subordination (3) becomes

$$\frac{1}{1 + |\psi(p(z), zp'(z))|} \leq \frac{1}{1 + |\psi(q(z), zq'(z))|},\ \forall\ z \in \overline{U}. \tag{9}$$

In order to prove that (3) or (9) implies p is subordinate to function q, we apply Lemma A. For that we assume that the functions p, q and h satisfy the conditions in Lemma A in the unit disc $\overline{U}$.

Assume that function p is not subordinate to function q.

By Lemma A, there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and $m \geq n \geq 1$, that satisfy

$$p(z_0) = q(\zeta_0),\ z_0 p'(z_0) = m\zeta_0 q'(\zeta_0).$$

Then

$$\begin{aligned} \psi(p(z_0), z_0 p'(z_0)) &= \theta[p(z_0)] + z_0 p'(z_0) \cdot \phi[p(z_0)] \\ &= \theta[q(\zeta_0)] + m\zeta_0 q'(\zeta_0) \cdot \phi[q(\zeta_0)]. \end{aligned} \tag{10}$$

If in (5) we take $t = m - 1 \geq 0$, then

$$L(z, m-1) = \theta[q(z)] + mQ(z) = \theta[q(z)] + mzq'(z) \cdot \phi[q(z)]. \tag{11}$$

For $z = \zeta_0 \in \partial U \setminus E(q)$, (11) becomes

$$L(\zeta_0, m-1) = \theta[q(\zeta_0)] + m\zeta_0 q'(\zeta_0) \cdot \phi[q(\zeta_0)],\ |\zeta_0| = 1. \tag{12}$$

Using (10) and (12), we have

$$\psi(p(z_0), z_0p'(z_0)) = L(\zeta_0, m-1),\ z_0 \in U,\ \zeta_0 \in \partial U \setminus E(q). \tag{13}$$

From (8), relation (13) is equivalent to

$$\frac{1}{1+|\psi(p(z_0), z_0p(z_0))|} \geq \frac{1}{1+|\psi q(\zeta_0), m\zeta_0 q'(\zeta_0)|}. \tag{14}$$

Relation (14) contradicts (9), which proves that the assumption we made is false, hence p is subordinate to q, meaning

$$\frac{1}{1+|p(z)|} \leq \frac{1}{1+|q(z)|}.$$

Since q is the solution of the univalent equation

$$\theta[q(z)] + zq'(z)\phi[q(z)] = h(z),$$

we have that q is the best dominant. □

Theorem 12. *Let q be univalent in U, with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and let $\theta : \mathbb{C} \to \mathbb{C}$, $\theta(w) = w$ and $\phi : \mathbb{C} \to \mathbb{C}$, $\phi(w) = \dfrac{1}{w}$, $\phi(w) \neq 0$, $w \neq 0$. Let*

$$F : U \to [0,1],\ F(z) = \frac{1}{1+|z|},\ z \in U.$$

Set

$$Q(z) = zq'(z) \cdot \phi[q(z)],$$
$$h(z) = \theta[p(z)] + Q(z) = \theta[p(z)] + zq'(z) \cdot \phi[q(z)]$$

and suppose that we have

(j) $\operatorname{Re} \dfrac{\theta'[q(z)]}{\phi[q(z)]} > 0$;

(jj) $\operatorname{Re} \left[1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)}\right] > 0,\ z \in U.$

For $m \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$, the function R is analytic in U,

$$R(z) = \frac{z[I_\gamma^m f(z)]''}{[I_\gamma^m f(z)]'} + \frac{z[I_\gamma^m f(z)]'}{I_\gamma^m f(z)} - 1 + \frac{[I_\gamma^m f(z)]' \cdot I_\gamma^m f(z)}{z} \neq 0,\ z \in U.$$

Then

$$\frac{1}{1+\left|\dfrac{z[I_\gamma^m f(z)]''}{[I_\gamma^m f(z)]'} + \dfrac{z[I_\gamma^m f(z)]'}{I_\gamma^m f(z)} - 1 + \dfrac{[I_\gamma^m f(z)]' \cdot I_\gamma^m f(z)}{z}\right|} \tag{15}$$
$$\leq \frac{1}{1+|\theta[q(z)] + zq'(z)\phi[q(z)]|},$$

that is

$$F(R(z)) \leq F(h(z))$$

implies

$$\frac{1}{1+\left|\dfrac{[I_\gamma^m f(z)]'\cdot I_\gamma^m f(z)}{z}\right|}\le\frac{1}{1+|q(z)|},\ z\in U,$$

that is

$$F(p(z))\le F(q(z)),\ z\in U$$

and q *is the best dominant.*

Proof. We let

$$p(z)=\frac{[I_\gamma^m f(z)]'\cdot I_\gamma^m f(z)}{z},\ z\in U. \tag{16}$$

Using Property 1, in (16) we have

$$\begin{aligned}p(z)&=\frac{\left[z+\sum\limits_{k=n+1}^{\infty}\dfrac{(\gamma+1)^m}{(\gamma+k)^m}a_kz^k\right]'\left[z+\sum\limits_{k=n+1}^{\infty}\dfrac{(\gamma+1)^m}{(\gamma+k)^m}z_kz^k\right]}{z}\\&=\frac{\left[1+\sum\limits_{k=n+1}^{\infty}\dfrac{(\gamma+1)^m}{(\gamma+k)^m}a_kz^{k-1}k\right]\cdot z\cdot\left[1+\sum\limits_{k=n+1}^{\infty}\dfrac{(\gamma+1)^m}{(\gamma+k)^m}a_kz^{k-1}\right]}{z}\\&=\left(1+\sum_{k=n+1}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k\cdot k\cdot z^{k-1}\right)\left(1+\sum_{k=n+1}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}z_kz^{k-1}\right),\end{aligned}$$

and $p(0)=1$.

Differentiating (16) and after a short calculus, we obtain

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{z[I_\gamma^m f(z)]''}{[I_\gamma^m f(z)]'}+\frac{z[I_\gamma^m f(z)]'}{I_\gamma^m f(z)}-1+\frac{[I_\gamma^m f(z)]'\cdot I_\gamma^m f(z)}{z}. \tag{17}$$

We let the function

$$\psi:\mathbb{C}^2\times\overline{U}\to\mathbb{C},\ \psi(r,s)=r+\frac{s}{r}.$$

For $r=p(z)$, $s=zp'(z)$, we obtain

$$\psi(p(z),zp'(z))=p(z)+\frac{zp'(z)}{p(z)},\ z\in U. \tag{18}$$

Using (18) in (17), we have

$$\begin{aligned}\psi(p(z),zp'(z))=\frac{z[I_\gamma^m f(z)]''}{[I_\gamma^m f(z)]'}+\frac{z[I_\gamma^m f(z)]'}{I_\gamma^m f(z)}-1\\+\frac{[I_\gamma^m f(z)]'\cdot I_\gamma^m f(z)}{z}.\end{aligned} \tag{19}$$

Since $\theta(w) = w$, $\theta[q(z)] = q(z)$, $\phi(w) = \dfrac{1}{w}$, $\phi[q(z)] = \dfrac{1}{q(z)}$, $q(z) \neq 0$, we have

$$Q(z) = zq'(z) \cdot \frac{1}{q(z)} \tag{20}$$

and

$$h(z) = \theta[q(z)] + Q(z) = q(z) + \frac{zq'(z)}{q(z)}, \ z \in U. \tag{21}$$

Using (18) and (21), relation (15) becomes

$$\frac{1}{1 + \left| p(z) + \dfrac{zp'(z)}{p(z)} \right|} \leq \frac{1}{1 + \left| q(z) + \dfrac{zq'(z)}{q(z)} \right|}, \ z \in U. \tag{22}$$

In order to prove Theorem 12, we shall use Theorem 11. For that, we show that the necessary conditions are satisfied. Differentiating (20) and after a short calculus, we have

$$\frac{zQ'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}, \ z \in U. \tag{23}$$

Using (jj) in (23) we have

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, \ z \in U, \tag{24}$$

hence the function Q is starlike.

Differentiating (21) and using (j) and (24), and after a short calculus, we obtain

$$\begin{aligned} \operatorname{Re} \frac{zh'(z)}{Q(z)} &= \operatorname{Re} \left[\frac{z\phi'[q(z)] \cdot q'(z)}{zq'(z) \cdot \phi[q(z)]} + \frac{zQ'(z)}{Q(z)} \right] \\ &= \operatorname{Re} \frac{\theta'[q(z)]}{\phi[q(z)]} + \operatorname{Re} \frac{zQ'(z)}{Q(z)} > 0, \ z \in U. \end{aligned}$$

Since $\theta(w) = w$ and $\phi(w) = \dfrac{1}{w}$, we obtain

$$\theta[p(z)] + zp'(z) \cdot \phi[p(z)] = p(z) + \frac{zp'(z)}{p(z)}, \tag{25}$$

and

$$\theta[q(z)] + zq'(z) \cdot \phi[q(z)] = q(z) + \frac{zq'(z)}{q(z)}. \tag{26}$$

Using (25) and (26) in (22), it becomes

$$\frac{1}{1 + |\theta[p(z)] + zp'(z)\phi[p(z)]|} \leq \frac{1}{1 + |\theta[q(z)] + zq'(z) \cdot \phi[q(z)]|}.$$

Since the conditions from Theorem 11 are satisfied, by applying it, we obtain

$$\frac{1}{1 + |p(z)|} \leq \frac{1}{1 + |q(z)|}$$

i.e.

$$\frac{1}{1+\left|\frac{[I_\gamma^m f(z)]'\cdot I_\gamma^m f(z)}{z}\right|}\le\frac{1}{1+|q(z)|},\ z\in U.$$

Since q is the solution of the univalent equation

$$h(z)=q(z)+\frac{zq'(z)}{q(z)},$$

we have q is the best dominant of (15). □

3. Example

Let $q(z)=1+z$, be an univalent function in U, with $q(0)=1$ and let the functions $\theta:\mathbb{C}\to\mathbb{C}$, $\theta(w)=w$ and $\phi:\mathbb{C}\to\mathbb{C}$, $\phi(w)=\dfrac{1}{w}$, $w\neq 0$, $w\in q(U)$.

If $q(z)=w$, then

$$\theta[q(z)]=q(z)=1+z,\ \phi[q(z)]=\frac{1}{q(z)}=\frac{1}{1+z},\ z\in U.$$

We calculate:

(a) $\operatorname{Re}\dfrac{\theta'[q(z)]}{\phi[q(z)]}=\operatorname{Re}\dfrac{(1+z)'}{\frac{1}{1+z}}=\operatorname{Re}(1+z)>0,\ z\in U;$

(b) $\operatorname{Re}\left(1+\dfrac{zq''(z)}{q'(z)}-\dfrac{zq'(z)}{q(z)}\right)=\operatorname{Re}\left(1-\dfrac{z}{1+z}\right)$

$=\operatorname{Re}\dfrac{1}{1+z}>0,\ z\in U;$

(c) $h(z)=\theta[q(z)]+Q(z)=q(z)+\dfrac{zq'(z)}{q(z)}=1+z+\dfrac{z}{1+z},\ z\in U;$

(d) $p(0)=q(0)=1$.

For $f\in A$, $f(z)=z+\dfrac{4}{3}z^2$ and $m=1$, $\gamma=2$, we obtain

$$\begin{aligned}I_2^1f(z)&=I_2^1\left(z+\frac{4}{3}z^2\right)=\frac{\gamma+1}{z^\gamma}\int_0^z\left(t+\frac{4}{3}t^2\right)tdt\\&=\frac{3}{z^2}\int_0^z\left(t^2+\frac{4}{3}t^3\right)dt=\frac{3}{z^2}\left(\frac{z^3}{3}+\frac{4}{3}\cdot\frac{z^4}{4}\right)\\&=z+z^2.\end{aligned}$$

The function

$$\begin{aligned}&\frac{z[I_2^1f(z)]''}{[I_2^1f(z)]'}+\frac{z[I_2^1f(z)]'}{I_2^1f(z)}-1+\frac{[I_2^1f(z)]'\cdot I_2^1f(z)}{z}\\&\quad=\frac{3z+4z^2+(1+2z)^2(1+z)^2}{(1+z)(1+2z)}\end{aligned}$$

is analytic and

$$p(z) = \frac{I_\gamma^1[f(z)]' I_\gamma^1[f(z)]}{z} = (1+z)(1+2z)$$

is also an analytic function.
From Theorem 12, we have:

$$\frac{1}{1 + \left| \dfrac{3z + 4z^2 + (1+2z)^2(1+z)^2}{(1+z)(1+2z)} \right|} \le \frac{1}{1 + \left| 1 + z + \dfrac{z}{1+z} \right|},$$

that is

$$F\left(\frac{3z + 4z^2 + (1+2z)^2(1+z)^2}{(1+z)(1+2z)} \right) \le F\left(1 + z + \frac{z}{1+z} \right)$$

implies

$$\frac{1}{1 + |(1+2z)(1+z)|} \le \frac{1}{1 + |1+z|}, \ z \in U,$$

that is

$$F((1+z)(1+2z)) \le F(1+z), \ z \in U.$$

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