Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

ISSN: 2687-6531

Weighted Integral Transforms Involving Convolution With Some Subclasses of Analytic Functions

Syed Zakar Hussain Bukhari[1], Rehana Razzaq[2], Imtiaz Ahmed[3]

Keywords
Convolution, hypergeometric function weighted, integral transforms

Abstract — Let $\mathscr{A}$ represent the class of analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = f'(0) - 1 = 0$ and let $\mathscr{P}$ represent the well-known class of Carathéodory functions p such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0,\ z \in \mathbb{U}$. A functions p analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathscr{P}_k$ for $k \geq 2$, if and only if

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,\mathrm{d}\alpha(\theta) \qquad (z \in \mathbb{U}),$$

where $\alpha(\theta) : 0 \leq \theta \leq 2\pi$ is a function of bounded variation with $\int_0^{2\pi} d\alpha(\theta) = 2\pi$ and $\int_0^{2\pi} |\mathrm{d}\alpha(\theta)| \leq k\pi$. For some $\eta \in \mathbb{R}, \varsigma < 1, k \geq 2$ and $\gamma \geq 0$, let $\mathscr{R}_k^{\eta}(\gamma,\varsigma)$ denote the class of functions $f \in \mathscr{A}$ satisfying the condition: $e^{i\eta}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \varsigma\right) \in \mathscr{P}_k \quad (z \in \mathbb{U})$. For $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$, we define the integral transform $\Im_m(f)(z) = \int_0^1 m(t)\frac{f(tz)}{t}dt$, where m is a non-negative real-valued weight function with $\int_0^1 m(t)\,dt = 1$. The main objective of this paper is to study conditions for invariance of the integral transforms $\Im_m$ and other relevant properties in connection with functions in the class $\mathscr{R}_k^{\eta}(\gamma,\varsigma)$. Also by varying parameters, we encompass a large number of previously known results.

Subject Classification (2020): 30C45, 30C80.

1. Introduction and Definitions

Let $\mathscr{H}(\mathbb{U})$ represent the class of all analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, and for a positive integer n and $a \in \mathbb{C}$, let

$$\mathscr{H}[a,n] := \left\{f \in \mathscr{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots \qquad (z \in \mathbb{U})\right\}.$$

[1]fatmi@must.edu.pk (Corresponding Author); [2]rehana.geb@must.edu.pk; [3]drimtiaz.maths@must.edu.pk
[1,3]Department of Mathematics, Mirpur University of Science and Technology, Mirpur AJK,Pakistan
[2]Ghazi Ellahi Bakhsh PG College for Women, Mirpur-10250(AJK), Pakistan
Article History: Received: 22.08.2020 - Accepted: 26.04.2021 - Published: 21.06.2021

Also, the subclass $\mathscr{A}$ of the class $\mathscr{H}[a,n]$ is defined as:

$$\mathscr{A} := \left\{f \in \mathscr{H}[0,1] : f'(0) = 1\right\}. \tag{1.1}$$

The class of univalent functions is represented by $\mathscr{S}$ and it is a subclass of the class $\mathscr{A}$, whereas, $\mathscr{S}^*, \mathscr{C}, \mathscr{K}$ and $\mathscr{Q}$ are the well-known classes of starlike, convex, close-to-convex and quasi-convex functions respectively.

For $f,\ g \in \mathscr{A}$, we define the Hadamard product or convolution $f * g$ by

$$\left(f * g\right)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \qquad (z \in \mathbb{U}),$$

where f is defined by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}).$$

Let $\mathscr{P}$ denote the well-known class of *Carathéodory functions p* such that $p \in \mathscr{H}(\mathrm{U})$, with

$$p(0) = 1 \text{ and } \operatorname{Re} p(z) > 0 \qquad (z \in \mathbb{U}).$$

Also $\mathscr{P}(\varsigma)$ represents the class of *Carathéodory functions p* such that $p \in \mathscr{H}(\mathrm{U})$ with

$$p(0) = 1 \text{ and } \operatorname{Re} p(z) > \varsigma \qquad (0 \le \varsigma < 1, z \in \mathbb{U}).$$

For details of these classes, we refer [7]. The function $p \in \mathscr{P}_k$, if and only if it satisfies the conditions $p(0) = 1$ and

$$p(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \mathrm{d}\alpha(\theta) \qquad (z \in \mathbb{U}),$$

where $\alpha(\theta) : 0 \le \theta \le 2\pi$ is a function of bounded variation satisfies the conditions

$$\int_0^{2\pi} \mathrm{d}\alpha(\theta) = 2\pi \text{ and } \int_0^{2\pi} |\mathrm{d}\alpha(\theta)| \le k\pi.$$

or equivalently, $p \in \mathscr{P}_k$ if and only if there exist $p_1,\ p_2 \in \mathscr{P}$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (k \ge 2, z \in \mathbb{U}).$$

Let p be an analytic function defined in the open unit disk $\mathbb{U}$. Then $p \in \mathscr{P}_k(\varsigma)$, if and only if $p(0) = 1$ and

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (0 \le \varsigma < 1, k \ge 2, z \in \mathbb{U}),$$

where $p_1,\ p_2 \in \mathscr{P}(\varsigma)$. For detail of the classes $\mathscr{P}_k$ and $\mathscr{P}_k(\varsigma)$, see [17] and [18] respectively.

For some $\eta \in \mathbb{R}, \varsigma < 1, k \ge 2$ and $\gamma \ge 0$, let $\mathscr{R}_k^{\eta}(\gamma, \varsigma)$ denote the class of functions $f \in \mathscr{A}$ satisfying the condition:

$$e^{i\eta}\left(\left(1-\gamma\right)\frac{f(z)}{z} + \gamma f'(z) - \varsigma\right) \in \mathscr{P}_k \qquad (z \in \mathbb{U}). \tag{1.2}$$

where $\mathscr{A}$ is defined by (1.1). For various related classes, we refer [1, 5, 9, 11, 13, 14].

The well-known Gaussian hypergeometric function $\mathscr{F}$ is defined as:

$$\mathscr{F}(\alpha,\beta;\lambda;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\lambda)_n\, n!} z^n \qquad (z \in \mathbb{U}), \tag{1.3}$$

where $\alpha, \beta, \lambda \in \mathbb{C}$, $\lambda \notin \{0,-1,-2,...\}$. Here for $\alpha \neq 0$, we have

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), & n = 1,2,3.... \\ 1, & n = 0 \end{cases}$$

If $\operatorname{Re}\lambda > \operatorname{Re}\beta > 0$, then

$$\mathscr{F}(\alpha,\beta;\lambda;z) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\lambda-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\lambda-\beta-1}(1-tz)^{-\alpha}\,\mathrm{d}t \qquad (z \in \mathbb{U}).$$

Moreover, for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}(\lambda+1) > \operatorname{Re}(\alpha+\beta)$, we have

$$\mathscr{F}(\alpha,\beta;\lambda;z) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda-\alpha-\beta+1)} \int_0^1 \lambda_1(t) \frac{1}{(1-tz)}\mathrm{d}t \qquad (z \in \mathbb{U}),$$

where

$$\lambda_1(t) = t^{\beta-1}(1-t)^{\lambda-\alpha-\beta}\mathscr{F}(\lambda-\alpha, 1-\alpha; \lambda-\alpha-\beta+1; 1-t),$$

for detail, see [3, 8]. For special choices of parameters, $\mathscr{F}(\alpha,\beta;\lambda;z)$ contains Noor integral operator [12, 15], Ruscheweyh derivative [23] and others. For a function $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$, we define the integral transform

$$\Im_m(f)(z) = \int_0^1 m(t) \frac{f(tz)}{t}\mathrm{d}t, \tag{1.4}$$

where m is a non-negative real-valued integrable weight function such that $\int_0^1 m(t)\,\mathrm{d}t = 1$ and $f \in \mathscr{R}_k^{\eta}(\gamma,\varsigma)$ satisfies (1.2). The operator $\Im_m(f)$ contain Libera, Bernardi, and Komatu operators as special cases. For $f \in \mathscr{R}_2^{\eta}(\gamma,\varsigma)$, $\Im_m(f)$ has been investigated by various authors, for reference, see [3, 10, 19−−22].

2. A Set of Preliminary Results

To establish our main results, we will use the following lemmas.

Lemma 2.1. [20] Let $\varsigma_1, \varsigma_2 < 1$ and let the functions p and q be analytic in $\mathbb{U}$ with $p(0) = q(0) = 1$. Then the conditions

$$\operatorname{Re} p(z) > \varsigma_1 \qquad (z \in \mathbb{U}) \;\text{ and }\; \operatorname{Re} e^{i\eta} q(z) - \varsigma_2 > 0 \qquad (z \in \mathbb{U})$$

imply

$$\operatorname{Re}\left(e^{i\eta}(p * q)(z) - \delta\right) > 0 \qquad (z \in \mathbb{U}),$$

where $1-\delta = 2(1-\varsigma_1)(1-\varsigma_2)$.

Lemma 2.2. Let $\varsigma_1 < 1$, $\gamma \geq 1$ and $\varsigma = \varsigma(\varsigma_1, \gamma)$ be such that

$$\varsigma = 1 - \frac{1-\varsigma_1}{2}\{1 - \frac{1}{\gamma}\int_0^1 \frac{m(t)}{1+t}\mathrm{d}t + (\frac{1}{\gamma}-1)\int_0^1 m(t)(\int_0^1 \frac{\mathrm{d}u}{1+tu^{\gamma}})\mathrm{d}t\}^{-1}. \tag{2.1}$$

If $\mathscr{F}\left(\alpha,\beta;\lambda;z\right)=\mathscr{F}\left(2,\frac{1}{\gamma},1+\frac{1}{\gamma};z\right)$, then

$$\mathrm{Re}\int_0^1 m\left(t\right)\mathscr{F}\left(2,\frac{1}{\gamma},1+\frac{1}{\gamma};tz\right)\mathrm{d}t>1-\frac{1-\varsigma_1}{2\left(1-\varsigma\right)},$$

where m is a real-valued non-negative weight function with $\int_0^1 m\left(t\right)\mathrm{d}t=1$ and $\mathscr{F}\left(\alpha,\beta;\lambda;z\right)$ is defined by (1.3). The value of ς is sharp.

Lemma 2.3. Let $0<\alpha\le 1$ and $\beta<\lambda-\alpha\le\frac{1}{\alpha}$. Then

$$\mathrm{Re}\,M\left(z\right)=\mathrm{Re}\left\{\left(1-\alpha\right)\mathscr{F}\left(\alpha,\beta;\gamma;z\right)+\alpha\mathscr{F}\left(\alpha+1,\beta;\lambda;z\right)\right\}\ge M\left(-1\right)=\varsigma_1\qquad\left(z\in\mathbb{U}\right).$$

This result is sharp.

Lemma 2.4. Let $-1<\alpha<0$ and $\beta>\alpha$. Then for

$$M\left(z\right)=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1-tz}\mathrm{d}t, \text{ for } \beta\neq\alpha\\ (1+\alpha)^2\int\limits_0^1\frac{t^{\alpha}(1+\alpha\log t)}{1-tz}\mathrm{d}t, \text{ for } \beta=\alpha\end{cases}\qquad\left(z\in\mathbb{U}\right),$$

we have

$$\mathrm{Re}\,M\left(z\right)>M\left(-1\right)=\varsigma_1=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1+t}\mathrm{d}t, \text{ for } \beta\neq\alpha\\ (1+\alpha)^2\int\limits_0^1\frac{t^{\alpha}(1+\alpha\log t)}{1+t}\mathrm{d}t, \text{ for } \beta=\alpha\end{cases}\qquad\left(z\in\mathbb{U}\right).$$

These inequalities are sharp.

Lemma 2.5. Let $-1<\alpha\le 0$, $q>1$ and

$$M\left(z\right)=\frac{\left(1+\alpha\right)^q}{\Gamma\left(q\right)}\int\limits_0^1 t^{\alpha}\log\left(\frac{1}{t}\right)^{q-2}\frac{q-1-\alpha\log\left(\frac{1}{t}\right)}{1-tz}\mathrm{d}t\qquad\left(z\in\mathbb{U}\right).$$

Then

$$\mathrm{Re}\,M\left(z\right)\ge M\left(-1\right)=\varsigma_1=\frac{\left(1+\alpha\right)^q}{\Gamma\left(q\right)}\int\limits_0^1\log(\frac{1}{t})^{q-2}\left(q-1-\alpha\log(\frac{1}{t})\right)\frac{t^{\alpha}}{1+t}\mathrm{d}t.$$

For the proof of Lemma 2 to Lemma 5, we refer, [4].

3. Main Results

In the following theorem, we find the conditions such that $\Im_m(f)\in\mathscr{R}_k(1,\varsigma_1)$ whenever $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$.

Theorem 3.1. Let $\varsigma_1<1$, $\gamma\ge 1$, $k\ge 2$ and let $\varsigma=\varsigma\left(\varsigma_1,\gamma\right)$ be defined by (2.1). If $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$, then $\Im_m(f)$ defined by (1.4) also belongs to the class $\mathscr{R}_k(1,\varsigma_1)$. The value of ς is sharp.

Proof.

Let

$$\left(1-\gamma\right)\frac{f\left(z\right)}{z}+\gamma f'\left(z\right)=p(z)\qquad\left(z\in\mathbb{U}\right),\tag{3.1}$$

where $p\left(0\right)=1$. If $f\in\mathscr{R}_k^{\eta}(\gamma,\varsigma)$, then by (3.1), we have

$$p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1\left(z\right)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2\left(z\right)\in\mathscr{P}_k\left(\varsigma\right)\qquad\left(z\in\mathbb{U}\right),\tag{3.2}$$

$p_i \in \mathscr{P}(\varsigma), i = 1, 2$ and conversely. For $\gamma \neq 0$, from (3.2), we write

$$\left\{1 + \left(1+\gamma\right) z + \left(1+2\gamma\right) z^2 + ...\right\} * \frac{f(z)}{z} = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z).$$

On further simplification, we obtain

$$f'(z) = \left[\left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)\right] * \sum_{n=0}^{\infty} \frac{n+1}{1+n\gamma} z^n \qquad (z \in \mathbb{U}), \tag{3.3}$$

which is equivalent to

$$f'(z) = \left[\left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)\right] * \mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; z) \qquad (z \in \mathbb{U}) \tag{3.4}$$

where $\mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; z)$ is defined by (1.3). For $\gamma = 0$, we write

$$\begin{aligned} f'(z) &= \left(\frac{k}{4} + \frac{1}{2}\right) \left(z p_1(z)\right)' - \left(\frac{k}{4} - \frac{1}{2}\right) \left(z p_2(z)\right)' \\ &= \left[\left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)\right] * \mathscr{F}(2, 1; 1; z). \end{aligned}$$

This is the limiting case of (3.3) for $\gamma \longrightarrow 0$. Differentiating (1.4) and then simplifying, we have

$$\Im'_m(f)(z) = \frac{\mathrm{d}}{\mathrm{d}z} \int_0^1 m(t) \frac{f(tz)}{t} \mathrm{d}t = f'(z) * \int_0^1 \frac{m(t)}{1 - tz} \mathrm{d}t \qquad (z \in \mathbb{U}), \tag{3.5}$$

where m a non-negative real-valued weight function such that $\int_0^1 m(t) dt = 1$. Both (3.4) and (3.5) yield

$$\Im'_m(f)(z) = k_1 p_1(z) * \int_0^1 m(t) \mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; tz) \mathrm{d}t - k_2 p_2(z) * \int_0^1 m(t) \mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; tz) \mathrm{d}t. \tag{3.6}$$

For $\gamma = 0$, we have

$$\Im'_m(f)(z) = \left(\frac{k}{4} + \frac{1}{2}\right) \left[p_1(z) * \int_0^1 \frac{m(t)}{(1-tz)^2} \mathrm{d}t\right] - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) * \int_0^1 \frac{m(t)}{(1-tz)^2} \mathrm{d}t,$$

which is just the limiting case of (3.5) for $\gamma \longrightarrow 0$ and m a non-negative real-valued weight function such that $\int_0^1 m(t) dt = 1$. For $\gamma \geq 1$, using Lemma 2, we write

$$\operatorname{Re} \int_0^1 m(t) \mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; tz) \mathrm{d}t > \varsigma_1 = 1 - \frac{1-\rho}{2(1-\varsigma)}, \varsigma_1 < 1 \qquad (z \in \mathbb{U}),$$

where ς is given by $(2,1)$ the condition mentioned above in the statement of the theorem and m a non-negative real-valued weight function such that $\int_0^1 m(t)dt = 1$. Again using Lemma 1, we obtain

$$p_i(z) * \int_0^1 m(t)\mathscr{F}(2, \frac{1}{\gamma}; 1 + \frac{1}{\gamma}; tz)\mathrm{d}t \in \mathscr{P}(\varsigma_1) \text{ for } i = 1,2 \qquad (z \in \mathbb{U}). \tag{3.7}$$

From (3.5), (3.6) and (3.7), we obtain $\Im_m(f) \in \mathscr{R}_k(1,\rho)$. To prove the sharpness, we consider the function $f \in \mathscr{R}_k(\gamma,\varsigma)$ determined by the relation

$$\left(1-\gamma\right)\frac{f(z)}{z} + \gamma f'(z) = (1-\varsigma)\frac{1+kz+z^2}{1-z^2} + \varsigma \qquad (z \in \mathbb{U}).$$

On simplification, we obtain

$$f'(z) = 1 + (1-\varsigma)k\left\{\frac{2z}{1+\gamma} + \frac{4z^3}{1+3\gamma} + ...\right\} + (1-\varsigma)\left\{\frac{6z^2}{1+2\gamma} + \frac{10z^4}{1+4\gamma} + ...\right\}.$$

This implies that

$$f(z) = z + (1-\varsigma)\sum_{n=1}^{\infty}\left[\frac{k}{1+(2n-1)\gamma}z^{2n} + \frac{2}{1+2n\gamma}z^{2n+1}\right] \qquad (z \in \mathbb{U}). \tag{3.8}$$

Now, using (3.8) in (1.4), we have

$$\Im_m f(z) = z + k(1-\varsigma)\sum_{n=1}^{\infty}\frac{\mu_n}{1+(2n-1)\gamma}z^{2n} + 2(1-\varsigma)\sum_{n=1}^{\infty}\frac{\upsilon_n}{1+2n\gamma}z^{2n+1} \qquad (z \in \mathbb{U}), \tag{3.9}$$

where

$$\mu_n = \int_0^1 m(t)\, t^{2n-1}\mathrm{d}t \quad \text{and} \quad \upsilon_n = \int_0^1 m(t)\, t^{2n}\mathrm{d}t.$$

The function given in (3.9) is the required extremal function for the parameter ς.

Theorem 3.2. Let $0 < \alpha \leq 1,\ \beta < \lambda - \alpha \leq \frac{1}{\alpha}$ and let $\mathfrak{F}$ be the convolution operator defined as:

$$\mathfrak{F}(z) := f(z) * z\mathscr{F}\left(\alpha, \beta; \lambda; z\right) \qquad (z \in \mathbb{U}). \tag{3.10}$$

Suppose that $f \in \mathscr{R}_k(0,\varsigma)$. Then,

$$\mathfrak{F} \in \mathscr{R}_k(1, \gamma = 1 - 2(1-\varsigma)1 - \varsigma_1))$$

with

$$\varsigma_1 = M(-1) = (1-\alpha)\mathscr{F}\left(\alpha, \beta; \lambda; -1\right) + \alpha\mathscr{F}\left(\alpha+1, \beta; \lambda; -1\right).$$

In particular

(i) $e^{i\eta}\left(\frac{f(z)}{z} - \frac{1-2\varsigma_1}{2(1-\varsigma_1)}\right) \in \mathscr{P}_k$ implies that $e^{i\eta}\,\mathfrak{F}'(z) \in \mathscr{P}_k$

and

(ii) $e^{i\eta}\left(\frac{f(z)}{z} - \frac{1}{2}\right) \in \mathscr{P}_k$ yields$\left(e^{i\eta}\,\mathfrak{F}'(z) - \varsigma_1\right) \in \mathscr{P}_k$.

Proof.

Rewriting (3.10), we have $\mathfrak{F}(z) := f(z) * z\mathscr{F}(\alpha, \beta; \lambda; z)$, where $f \in \mathscr{R}_k(0, \varsigma)$. This implies that

$$\mathfrak{F}'(z) = \frac{f(z)}{z} * \left(z\mathscr{F}(\alpha, \beta; \lambda; z)\right)' = \frac{f(z)}{z} * M(z), \tag{3.11}$$

where $M(z) = \left(z\mathscr{F}(\alpha, \beta; \lambda; z)\right)'$. Now, taking derivative of hypergeometric function and using

$$\lambda\mathscr{F}(\alpha+1, \beta; \lambda; z) = \beta z\mathscr{F}(\alpha+1, \beta+1; \lambda+1; z) + \lambda\mathscr{F}(\alpha, \beta; \lambda; z),$$

we obtain

$$M(z) = (1-\alpha)\mathscr{F}(\alpha, \beta; \lambda; z) + \alpha\mathscr{F}(\alpha+1, \beta; \lambda; z) \qquad (z \in \mathbb{U}).$$

For $\lambda > \alpha + \beta$, we write

$$M(z) = \frac{\Gamma(\lambda)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda-\alpha-\beta)} \int_0^1 m_1(t) \frac{1}{1-tz} \mathrm{d}t \qquad (z \in \mathbb{U}),$$

where

$$\begin{aligned} m_1(t) = &\frac{(1-\alpha)\, t^{\beta-1}(1-t)^{\lambda-\alpha-\beta}}{\lambda-\alpha-\beta} \mathscr{F}(\lambda-\alpha, 1-\alpha; \lambda-\alpha-\beta+1; 1-t) \\ &+ t^{\beta-1}(1-t)^{\lambda-\alpha-\beta-1} \mathscr{F}(\lambda-\alpha-1, -\alpha; \lambda-\alpha-\beta; 1-t). \end{aligned}$$

For $\beta < \lambda - \alpha \leq 1$ and $\alpha \in (0, 1]$, using Lemma 3, we see that

$$\operatorname{Re} M(z) > M(-1) = \varsigma_1, \tag{3.12}$$

where

$$\varsigma_1 = (1-\alpha)\mathscr{F}(\alpha, \beta; \lambda; -1) + \alpha\mathscr{F}(\alpha+1, \beta; \lambda; -1).$$

For $f \in \mathscr{R}_k(0, \varsigma)$, we have

$$\frac{f(z)}{z} = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1, 2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) * M(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.13}$$

Using (3.12) and Lemma 1, we write

$$p_i * M \in \mathscr{P}(\gamma) \text{ for } i = 1, 2 \qquad (z \in \mathbb{U}), \tag{3.14}$$

where $\gamma = 1 - 2(1-\varsigma)1 - \varsigma_1)$. On combining (3.11), (3.13) and (3.14), we obtain

$$\mathfrak{F}'(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(\gamma).$$

This implies that $\mathfrak{F} \in \mathscr{R}_k(1, \gamma)$. Let $f \in \mathscr{R}_k(0, \varsigma)$. For the extremal function which gives the sharpness, consider

$$\frac{f(z)}{z}=\left(\frac{k}{4}+\frac{1}{2}\right)\left[1+2\left(1-\varsigma\right)\frac{z}{1-z}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[1-2\left(1-\varsigma\right)\frac{z}{1+z}\right]\qquad(z\in\mathbb{U})$$

and

$$M(z)=1+2\left(1-\varsigma_1\right)\frac{z}{1-z}.$$

Now

$$\frac{f(z)}{z}*M(z)=M(z)*\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left(1+\frac{2\left(1-\varsigma\right)z}{1-z}\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(1-\frac{2\left(1-\varsigma\right)z}{1+z}\right)\right\}$$

which on simplification yields

$$\frac{f(z)}{z}*M(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\left[1+\frac{4\left(1-\varsigma\right)\left(1-\varsigma_1\right)z}{1-z}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[1-\frac{4\left(1-\varsigma\right)\left(1-\varsigma_1\right)z}{1+z}\right].\tag{3.15}$$

Thus from (3.15), we obtain the required extremal function.

Theorem 3.3. Let $-1<\alpha<0,\ \beta>\alpha$ and $f\in\mathcal{R}_k\left(0,\varsigma\right)$. Then

$$\mathcal{G}\in\mathcal{R}_k\left(1,1-2\left(1-\varsigma\right)\left(1-\varsigma_1\right)\right),$$

where

$$\mathcal{G}(z)=\sum_{n=1}^{\infty}\frac{(1+\alpha)\left(1+\beta\right)}{(n+\alpha)\left(n+\beta\right)}z^n*f(z)=\mathcal{G}(f)(z)\qquad(z\in\mathbb{U}),\tag{3.16}$$

and

$$\varsigma_1=\begin{cases}\frac{(1+\alpha)(1+\beta)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1+t}\mathrm{d}t, \text{ for } \beta\neq\alpha,\\ (1+\alpha)^2\int\limits_0^1\frac{t^{\alpha}(1+\alpha\log t)}{1+t}\mathrm{d}t, \text{ for } \beta=\alpha.\end{cases}$$

This result is sharp.

Proof.

Let $\alpha\in(-1,0),\beta>\alpha$ and $\mathcal{G}$ be defined by (3.16). Then

$$\mathcal{G}'(z)=\frac{1}{(1-z)^2}*\sum_{n=0}^{\infty}\frac{(1+\alpha)\left(1+\beta\right)}{(n+\alpha)\left(n+\beta\right)}z^n*\frac{f(z)}{z}\qquad(z\in\mathbb{U})$$

or

$$\mathcal{G}'(z)=\sum_{n=0}^{\infty}\frac{(1+\alpha)\left(1+\beta\right)(n+1)}{(n+\alpha)\left(n+\beta\right)}z^n*\frac{f(z)}{z}=\frac{f(z)}{z}*M(z)\qquad(z\in\mathbb{U}),\tag{3.17}$$

where

$$M(z)=\frac{(1+\alpha)\left(1+\beta\right)}{\beta-\alpha}\left[-\alpha\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha+1)}+\beta\sum_{n=0}^{\infty}\frac{z^n}{\left(n+\beta+1\right)}\right]\qquad(z\in\mathbb{U}).$$

The function M can also be written as

$$M(z)=\frac{1}{\beta-\alpha}(1+\alpha)\left(1+\beta\right)\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1-tz}\mathrm{d}t\qquad(z\in\mathbb{U}).\tag{3.18}$$

Using Lemma 4, from (3.18), we have

$$\operatorname{Re}M(z)>M(-1)=\varsigma_1=\frac{(1+\alpha)\left(1+\beta\right)}{\beta-\alpha}\int\limits_0^1\frac{\beta t^{\beta}-\alpha t^{\alpha}}{1+t}\mathrm{d}t.\tag{3.19}$$

Now for $f \in \mathscr{R}_k(0,\varsigma)$, consider

$$\frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1,2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) * M(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.20}$$

Using (3.19) and Lemma 1, we write

$$p_i(z) * M(z) \in \mathscr{P}(1-2(1-\varsigma)1-\varsigma_1)) \text{ for } i = 1,2 \qquad (z \in \mathbb{U}). \tag{3.21}$$

On combining (3.17), (3.20) and (3.21), we obtain

$$\mathscr{G}'(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(1-2(1-\varsigma)1-\varsigma_1)) \qquad (z \in \mathbb{U}).$$

This implies that

$$\mathscr{G} \in \mathscr{R}_k(1, 1-2(1-\varsigma)1-\varsigma_1)).$$

For $\beta = \alpha$, the similar result for the conditions described in the theorem can be obtained by taking the limit $\beta \longrightarrow \alpha$ in the previous case $\alpha < \beta$. Sharpness can be obtained as in previous theorems.

Theorem 3.4. Let $-1 < \alpha \leq 0$, $q > 1$ and $f \in \mathscr{R}_k(0,\varsigma)$. Then the the operator $\mathfrak{F}_{\alpha,q}$ defined by

$$\mathfrak{F}_{\alpha,q}(z) = \mathfrak{F}_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{(1+\alpha)^q}{(n+\alpha)^q} z^n * f(z) \qquad (z \in \mathbb{U})$$

is in the class $\mathscr{R}_k(1, 1-2(1-\varsigma)(1-\varsigma_1))$ with

$$\varsigma_1 = M(-1) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-2} \left(q-1-\alpha\log\left(\frac{1}{t}\right)\right) \frac{t^\alpha}{1+t} \mathrm{d}t.$$

Proof.

For $q > 0$ and $\alpha > -1$, the operator $\mathfrak{F}_{\alpha,q}$ is defined as

$$\mathfrak{F}_{\alpha,q}(f)(z) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-1} t^{\alpha-1} f(tz)\,\mathrm{d}t \qquad (z \in \mathbb{U}).$$

Now for $-1 < \alpha \leq 0$, $q > 1$ and $f \in \mathscr{R}_k(0,\varsigma)$,

$$\mathfrak{F}_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{(1+\alpha)^q}{(n+\alpha)^q} z^n * f(z) \qquad (z \in \mathbb{U})$$

or

$$\mathfrak{F}'_{\alpha,q}(f)(z) = \sum_{n=1}^{\infty} \frac{n(1+\alpha)^q}{(n+\alpha)^q} z^{n-1} * \frac{f(z)}{z} = M(z) * \frac{f(z)}{z} \qquad (z \in \mathbb{U}). \tag{3.22}$$

By Lemma 5, we see that

$$\operatorname{Re} M(z) > M(-1) = \frac{(1+\alpha)^q}{\Gamma(q)} \int_0^1 \log\left(\frac{1}{t}\right)^{q-2} \left(q-1-\alpha\log\left(\frac{1}{t}\right)\right) \frac{t^\alpha}{1+t} \mathrm{d}t.$$

Now for $f \in \mathscr{R}_k(0,\varsigma)$, consider

$$\frac{f(z)}{z} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z \in \mathbb{U}),$$

where $p_i \in \mathscr{P}(\varsigma)$ for $i = 1,2$. This implies that

$$\frac{f(z)}{z} * M(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) * M(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) * M(z) \qquad (z \in \mathbb{U}). \tag{3.24}$$

Using (3.23) and Lemma 1, we write

$$p_i * M \in \mathscr{P}(1-2(1-\varsigma)1-\varsigma_1)) \text{ for } i = 1,2. \tag{3.25}$$

On combining (3.22), (3.24) and (3.25), we obtain

$$\mathfrak{F}'_{\alpha,q}(f)(z) = \frac{f(z)}{z} * M(z) \in \mathscr{P}_k(1-2(1-\varsigma)1-\varsigma_1)).$$

This implies that $\mathfrak{F}'_{\alpha,q} \in \mathscr{R}_k(1,\gamma)$. The sharpness of the above result is straight forward.

For special choices of parameter, we also refer [2, 6, 16, 24].

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