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Research Article

Controlled g-dual frames and their approximates in Hilbert spaces

Sayyed Mehrab Ramezani

Faculty of Technology and Mining, Yasouj University, Choram, Iran

Abstract

In this paper, we introduced and characterized the controlled g-duals of a frame in a separable Hilbert space $\mathcal{H}$. Afterwards, we obtained new C-controlled g-dual frames from the given C-controlled g-dual frames. In addition, the approximation for controlled g-dual frames was defined and some of their properties were investigated. Finally, we characterized the relationship between approximately C-controlled dual and C-controlled g-dual.

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1. Introduction

Frames for Hilbert space were formally defined by Duffin and Schaeffer [8] in 1952 while studying some problems in non-harmonic Fourier series. Recall that for a Hilbert space $\mathcal{H}$ and a countable index set J, a collection $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a frame for the Hilbert space $\mathcal{H}$ if there exist two positive constants c, d such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \le \sum_{j\in J} |\langle f, f_j\rangle|^2 \le d\|f\|^2; \tag{1.1}$$

c and d are called the lower and upper frame bounds, respectively. If only the right-hand inequality in (1.1) is satisfied, we call $\{f_j\}_{j\in J}$ a Bessel sequence for $\mathcal{H}$ with Bessel bound d.

The bounded linear operator T is defined by

$$T : \ell^2(J) \longrightarrow \mathcal{H}, \qquad T\{c_j\}_{j\in J} = \sum_{j\in J} c_j f_j,$$

which is called the synthesis operator of $\{f_j\}_{j\in J}$. Moreover, $T^*f = \{\langle f, f_j\rangle\}_{j\in J}$ for all $\{c_j\}_{j\in J} \in \ell^2(J)$. The map T^* is called the analysis operator of $\{f_j\}_{j\in J}$. The bounded linear operator S is also defined by

$$S = TT^* : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S(f) = \sum_{j\in J} \langle f, f_j\rangle f_j,$$

which is called the frame operator of $\{f_j\}_{j\in J}$. For more information about the frames see [5].

Email address: m.ramezani@yu.ac.ir
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Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are said to be duals for $\mathcal{H}$ if the following equalities hold

$$f = \sum_{j\in J}\langle f, f_j\rangle g_j = \sum_{j\in J}\langle f, g_j\rangle f_j, \text{ for all f} \in \mathcal{H}.$$

Note that because $S : \mathcal{H} \longrightarrow \mathcal{H}$ by $S(f) = \sum_{j\in J}\langle f, f_j\rangle f_j$ is bijective, self-adjoint and

$$f = S(S^{-1}f) = \sum_{j\in J}\langle S^{-1}f, f_j\rangle f_j = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j,$$

hence the sequence $\{S^{-1}f_j\}_{j\in J}$ is also a frame by Corollary 1.1.3 [5] and it is called the canonical dual of $\{f_j\}_{j\in J}$. Dual frames are important in reconstructing vectors (or signals) in terms of the frame elements.

Dehghan and Hasankhani Fard [7] introduced and characterized the g-duals of a frame in a separable Hilbert space and Ramezani and Nazari [9, 10] extended this concept for a generalized frame and a continuous frame. A frame $\{g_j\}_{j\in J}$ is called a g-dual frame of the frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that for all $f \in \mathcal{H}$

$$f = \sum_{j\in J}\langle Af, g_j\rangle f_j,$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all the bounded operators on $\mathcal{H}$. Thus, $\{g_j\}_{j\in J}$ is a g-dual frame for $\{f_j\}_{j\in J}$ associated to A if and only if $\{A^*g_j\}_{j\in J}$ is a dual frame for $\{f_j\}_{j\in J}$. They showed that also by applying g-duals, one can deduce further reconstruction formulas to obtain signals.

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [1]. By decreasing the ratio of the frame bounds, weighting improves the numerical efficiency of iterative algorithms, such as the "frame algorithm" [5] for the inversion of the frame operator. However, they have been employed earlier in [2] for spherical wavelets. Let $GL(\mathcal{H})$ be the set of all the bounded operators with a bounded inverse. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{f_j\}_{j\in J} \subseteq \mathcal{H}$, such that there exist two constants $A_c > 0$ and $B_c < \infty$, satisfying

$$A_c\|f\|^2 \leq \sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle \leq B_c\|f\|^2; \tag{1.2}$$

for every $f \in \mathcal{H}$, where $C \in GL(\mathcal{H})$. Every frame is an I-controlled frame. Hence the controlled frames are generalizations of frames. The controlled frame operator S_c is defined by

$$S_cf = \sum_{j\in J}\langle f, f_j\rangle Cf_j = CS, \qquad (f \in \mathcal{H}), \tag{1.3}$$

where S is the frame operator of $\{f_j\}_{j\in J}$. The synthesis operator for a C-controlled frame $\{f_j\}_{j\in J}$ is defined as follows

$$T_c(\{\alpha_j\}_{j\in J}) = \sum_{j\in J}\alpha_j Cf_j = CT,$$

where T is the synthesis operator of $\{f_j\}_{j\in J}$ and $S_c = T_cT^*$. C-Controlled frame $\{f_j\}_{j\in J}$ and Bessel sequences$\{g_j\}_{j\in J}$ are said to be the C-controlled duals for $\mathcal{H}$ if the following equality holds.

$$f = \sum_{j\in J}\langle f, g_j\rangle Cf_j, \text{ for all f} \in \mathcal{H}.$$

Through the exciting developments in the g-dual frames and controlled frames, we introduced the notion of controlled g-dual frames in Hilbert spaces and characterized all the controlled g-dual frames for a given controlled frame. We defined approximate controlled

g-duals for the controlled frames and using this concept and established a relationship between approximately controlled g-dual frames and controlled dual frames and controlled g-dual frames.

2. Controlled g-dual frames

In this section, we define the concept of controlled g-dual frame by extending the concept of controlled from dual to g-dual. We then show some properties of the dual g-dual frames.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space and $C \in GL(\mathcal{H})$. Suppose that $\{f_j\}_{j\in J}$ is a C-controlled frame and $\{g_j\}_{j\in J}$ is a Bessel sequence. Then, $\{g_j\}_{j\in J}$ is said to be a C-controlled g-dual of $\{f_j\}_{j\in J}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that for all $f \in \mathcal{H}$

$$f = \sum_{j\in J} \langle Af, g_j\rangle Cf_j.$$

When $A = I$, $\{g_j\}_{j\in J}$ is a C-controlled dual frame of $\{f_j\}_{j\in J}$ and if $A = C = I$, $\{g_j\}_{j\in J}$ is an ordinary dual frame of $\{g_j\}_{j\in J}$. Hence, the controlled g-duals are the generalizations of the duals. The following equivalent conditions for the Bessel mappings $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ can be proved straightforwardly from Definition 2.1.

Lemma 2.2. *For the Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ and $C \in GL(\mathcal{H})$, the following statements are equivalent:*

(i) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$f = \sum_{j\in J} \langle Af, g_j\rangle Cf_j, \text{ for all } f \in \mathcal{H};$$

(ii) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$f = \sum_{j\in J} \langle A^*f, Cf_j\rangle g_j, \text{ for all } f \in \mathcal{H};$$

(iii) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle f, g\rangle = \sum_{j\in J} \langle Af, g_j\rangle \langle Cf_j, g\rangle, \text{ for all } f, g \in \mathcal{H};$$

In case that the equivalent conditions are satisfied, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are the C-controlled g-dual frames.

Proof. Let (i) be satisfied and $f \in \mathcal{H}$. Then there exists $g \in \mathcal{H}$ such that $f = Ag$ and $g = \sum_{j\in J} \langle Ag, g_j\rangle Cf_j$. Therefore $f = Ag = \sum_{j\in J} \langle Ag, g_j\rangle ACf_j$. Since $\{ACf_j\}_{j\in J}$ is a Bessel sequence, by Lemma 5.6.2 from [5] we have

$$f = \sum_{j\in J} \langle Ag, g_j\rangle ACf_j = \sum_{j\in J} \langle Ag, ACf_j\rangle g_j = \sum_{j\in J} \langle A^*f, Cf_j\rangle g_j,$$

and hence (ii) holds. A similar argument reveals that (ii) implies (i). It is clear that (i) indicates (iii). To prove that (iii) implies (i), we fix $f \in \mathcal{H}$ and for all $g \in \mathcal{H}$, the assumption in (iii) demonstrates that

$$\begin{aligned}\langle f - \sum_{j\in J} \langle Af, g_j\rangle Cf_j, g\rangle &= \langle f, g\rangle - \sum_{j\in J} \langle Af, g_j\rangle \langle Cf_j, g\rangle \\ &= \langle f, g\rangle - \langle f, g\rangle \\ &= 0,\end{aligned}$$

and (i) follows.

Next, if the conditions $(i), (ii)$ are respectively satisfied for the Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ with the Bessel bounds B and D, then

$$\begin{aligned}\sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle &\leq \left(\sum_{j\in J}|\langle f, f_j\rangle|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{j\in J}|\langle Cf_j, f\rangle|^2\right)^{\frac{1}{2}}\\ &\leq B\|C\|\|f\|^2,\end{aligned}$$

on the other hand

$$\begin{aligned}\|f\|^4 = |\langle f, f\rangle|^2 &= |\sum_{j\in J}\langle A^*f, Cf_j\rangle\langle g_j, f\rangle|^2\\ &\leq \left(\sum_{j\in J}|\langle A^*f, Cf_j\rangle|^2\right)\cdot\left(\sum_{j\in J}|\langle g_j, f\rangle|^2\right)\\ &\leq D\|f\|^2\|A\|^2\sum_{j\in J}|\langle f, Cf_j\rangle|^2,\end{aligned}$$

and consequently,

$$\begin{aligned}\frac{1}{D\|A\|^2}\|f\|^2 &\leq \sum_{j\in J}|\langle f, Cf_j\rangle|^2\\ &\leq \|C\|\sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle,\end{aligned}$$

accordingly,

$$\frac{1}{D\|A\|^2\|C\|}\|f\|^2 \leq \sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle,$$

revealing that $\{f_j\}_{j\in J}$ is a C-controlled frame. Since (i) and (ii) are equivalent, $\{g_j\}_{j\in J}$ is also a C-controlled frame and $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames. □

The following propositions give a method to construct new C-controlled g-dual frames from given C-controlled g-dual frames.

Proposition 2.3. *Assume that $\{g_j\}_{j\in J}$ is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{H}$ with the invertible operator $A \in \mathcal{B}(\mathcal{H})$ and let α be a complex number. Then the sequence $\{h_j\}_{j\in J}$ is defined by $h_j = \alpha g_j + (1-\alpha)(A^{-1})^*(S_c^{-1})^*f_j$ which is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{H}$ with the invertible operator A.*

Proof. For all $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in J}\langle Af, h_j\rangle Cf_j &= \sum_{j\in J}\langle Af, \alpha g_j + (1-\alpha)(A^{-1})^*(S_c^{-1})^*f_j\rangle Cf_j\\ &= \overline{\alpha}\sum_{j\in J}\langle Af, g_j\rangle Cf_j + \overline{1-\alpha}\sum_{j\in J}\langle S_c^{-1}f, f_j\rangle Cf_j\\ &= f,\end{aligned}$$

as asserted. □

Proposition 2.4. *Assume that $\{g_j\}_{j\in J}$ and $\{h_j\}_{j\in J}$ are C-controlled g-dual frames for $\{f_j\}_{j\in J}$ with the invertible operators A and B, respectively. Then for any $\alpha \in \mathbb{C}$, $\alpha A^*g_j + (1-\alpha)B^*h_j$ is a C-controlled dual frame for the frame $\{f_j\}_{j\in J}$.*

Proof. By Lemma 2.2 we have

$$\begin{aligned}\sum_{j\in J}\langle f,\alpha A^*g_j+(1-\alpha)B^*h_j\rangle Cf_j &= \overline{\alpha}\sum_{j\in J}\langle f,A^*g_j\rangle Cf_j+\overline{(1-\alpha)}\sum_{j\in J}\langle f,B^*h_j\rangle Cf_j\\ &= \overline{\alpha}\sum_{j\in J}\langle Af,g_j\rangle Cf_j+\overline{(1-\alpha)}\sum_{j\in J}\langle Bf,h_j\rangle Cf_j\\ &= f.\end{aligned}$$

□

Proposition 2.5. *Let $\{f_j\}_{j\in J}$ be a C-controlled frame for $\mathcal{H}$ with the C-controlled frame operator S_c and let $\{g_j\}_{j\in J}$ be a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{V}=\overline{Range\{g_j\}_{j\in J}}$ with the invertible operator $B\in\mathcal{B}(\mathcal{V})$. Then the sequence $h_j=B^*g_j+(S_c^{-1})^*f_j$ is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{H}$.*

Proof. The operator B can be extended to the operator B_1 on $\mathcal{H}$ defined by $B_1=BP+Q$ where P and Q respectively are the orthogonal projections onto $\mathcal{V}$ and $\mathcal{V}^\perp$ of $\mathcal{H}$. By Proposition 2.3 from [7], $B_1(\mathcal{V}^\perp)\subseteq\mathcal{V}^\perp$ and $B_1^*=B^*$. Now, let $A=I-\frac{1}{2}P$, in which I denotes the identity operator on $\mathcal{H}$. Since $\|I-A\|\le 1$, the operator A is invertible and for $f\in\mathcal{H}$, there exist unique vectors $u\in\mathcal{V}$ and $v\in\mathcal{V}^\perp$ such that $f=u+v$. Therefore, we have

$$\begin{aligned}\sum_{j\in J}\langle Af,h_j\rangle Cf_j &= \sum_{j\in J}\langle\frac{1}{2}u+v,B^*g_j+(S_c^{-1})^*f_j\rangle Cf_j\\ &= \frac{1}{2}\sum_{j\in J}\langle Bu,g_j\rangle Cf_j+\sum_{j\in J}\langle Bv,g_j\rangle Cf_j+\sum_{j\in J}\langle S_c^{-1}(\frac{1}{2}u+v),f_j\rangle Cf_j\\ &= \frac{1}{2}u+0+(\frac{1}{2}u+v)\\ &= f,\end{aligned}$$

and this marks the end of the proof. □

Corollary 2.6. *Let $\{f_j\}_{j\in J}$ be a C-controlled frame for $\mathcal{H}$ with the C-controlled frame operator S_c and let $\{g_j\}_{j\in J}$ be a C-controlled dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{V}=\overline{Range\{g_j\}_{j\in J}}$. Subsequently, the mapping $h_j=g_j+(S_c^{-1})^*f_j$ is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$.*

Example 2.7. Let $\{g_j^i\}_{j\in J}$ be a C-controlled dual frame of $\{f_j\}_{j\in J}$ in $\mathcal{H}$ and put $h_j:=\sum_{i=1}^m g_j^i$. Then, $Af=\frac{1}{m}f$ defines a bounded invertible operator on $\mathcal{H}$ and

$$\begin{aligned}\sum_{j\in J}\langle\frac{1}{m}f,h_j\rangle Cf_j &= \sum_{j\in J}\langle\frac{1}{m}f,\sum_{i=1}^m g_j^i\rangle Cf_j\\ &= \frac{1}{m}\sum_{i=1}^m\sum_{j\in J}\langle f,g_j^i\rangle Cf_j\\ &= \frac{1}{m}\sum_{i=1}^m f\\ &= f,\end{aligned}$$

and therefore $\{h_j\}_{j\in J}$ is a C-controlled g-dual frame for $\{f_j\}_{j\in J}$ with the invertible operator $Af=\frac{1}{m}f$.

This example illustrates that the sum of many C-controlled dual frames can be a C-controlled g-dual frame. The following proposition states that the sum of two C-controlled g-dual frames is a C-controlled g-dual frame.

Proposition 2.8. *Let $\{g_j\}_{j\in J}$ and $\{h_j\}_{j\in J}$ be two C-controlled g-dual frames of $\{f_j\}_{j\in J}$ with corresponding invertible operators A and B, respectively. If $A^{-1}+B^{-1}$ is an invertible operator, then $\{(g_j+h_j)_j\}_{j\in J}$ is a C-controlled g-dual frame for $\{f_j\}_{j\in J}$.*

Proof. Let $\mathcal{T}\in\mathcal{B}(\mathcal{H})$ be the inverse operator of $A^{-1}+B^{-1}$. We have

$$\begin{aligned}\sum_{j\in J}\langle \mathcal{T}f,(g_j+h_j)\rangle Cf_j &= \sum_{j\in J}\langle \mathcal{T}f,g_j\rangle Cf_j+\sum_{j\in J}\langle \mathcal{T}f,h_j\rangle Cf_j\\ &= A^{-1}\mathcal{T}f+B^{-1}\mathcal{T}f\\ &= (A^{-1}+B^{-1})\mathcal{T}f\\ &= f,\end{aligned}$$

for all $f\in\mathcal{H}$. □

Proposition 2.9. *$\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are the D-controlled g-dual frames for $\mathcal{H}$ if and only if $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are CD-controlled g-dual frames for $\mathcal{H}$.*

Proof. Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be D-controlled g-dual frames for $\mathcal{H}$. Then there exists an invertible operator $A\in\mathcal{B}(\mathcal{H})$ such that $f=\sum_{j\in J}\langle Af,g_j\rangle Df_j$ for all $f\in\mathcal{H}$ and hence

$$f=CC^{-1}f=C\left(\sum_{j\in J}\langle AC^{-1}f,g_j\rangle Df_j\right)=\sum_{j\in J}\langle AC^{-1}f,g_j\rangle CDf_j.$$

This shows that $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are CD-controlled g-dual frames for $\mathcal{H}$ with the invertible operator $AC^{-1}\in\mathcal{B}(\mathcal{H})$. The converse is obtained by applying $f=C^{-1}Cf$. □

3. Approximately C-controlled g-dual frames

The stability of frames is of great importance in frame theory. That is, if $\{f_j\}_{j\in J}$ is a frame and $\{g_j\}_{j\in J}$ is in some sense "close" to $\{f_j\}_{j\in J}$, does it follow that $\{g_j\}_{j\in J}$ is also a frame? A classical result (Paley-Wiener theorem for frames) states that if $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$ with upper frame bound A, then a sequence $\{g_j\}_{j\in J}$ in $\mathcal{H}$ is also a frame if there exist constants $\lambda,\mu>0$ such that $\lambda+\frac{\mu}{\sqrt{A}}<1$ and

$$\|\sum c_j(f_j-g_j)\|\le\lambda\|\sum c_jf_j\|+\mu\left(\sum|c_j|^2\right)^{\frac{1}{2}}, \tag{3.1}$$

for all finite scalar sequences $\{c_j\}$. We can consider 3.1 as a condition on the operator

$$K:\ell^2(J)\longrightarrow\mathcal{H},\qquad K\{c_j\}_{j\in J}=\sum_{j\in J}c_j(f_j-g_j).$$

For this reason, K is called the perturbation operator [3–5,12]. In this section, we demonstrate that, under some conditions, approximately C-controlled g-dual frames are stable under some perturbations.

Definition 3.1 ([6])**.** Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ with synthesis operator T and U, respectively, are said to be approximately dual frames if $\|I-TU^*\|<1$ or $\|I-UT^*\|<1$

Definition 3.2 ([11])**.** The Bessel sequence $\{g_j\}_{j\in J}$ with synthesis operator U is called an approximate C-controlled dual of a C-controlled frame $\{f_j\}_{j\in J}$ with synthesis operator T_c whenever

$$\|f-\sum_{j\in J}\langle f,g_j\rangle Cf_j\|<\|f\|,\qquad (f\in\mathcal{H}),$$

in other words, $\|I-T_cU^*\|<1$.

The above definitions led us to define the following.

Definition 3.3. Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ with synthesis operator T and U, respectively, are approximately g-dual frames for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that $\|I - TU^*A\| < 1$ or $\|I - UT^*A\| < 1$.

Definition 3.4. The Bessel sequence $\{g_j\}_{j\in J}$ with synthesis operator U is called an approximate C-controlled g-dual of a C-controlled frame $\{f_j\}_{j\in J}$ with synthesis operator T_c if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that

$$\|f - \sum_{j\in J}\langle Af, g_j\rangle Cf_j\| < \|f\|, \qquad (f \in \mathcal{H}),$$

equivalently $\|I - T_cU^*A\| < 1$.

Theorem 3.5. *If two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames for $\mathcal{H}$, then $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames.*

Proof. Since $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames and $\|I - T_cU^*\| < 1$ where T_c and U are the synthesis operators of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, respectively. Hence, T_cU^* is an invertible operator. Then for all $f \in \mathcal{H}$ we have

$$f = T_cU^*(T_cU^*)^{-1}(f) = C\left(\sum_{j\in J}\langle (T_cU^*)^{-1}f, g_j\rangle f_j\right) = \sum_{j\in J}\langle Af, g_j\rangle Cf_j,$$

where $A = (T_cU^*)^{-1}$ is an invertible operator. Thus, for all $f \in \mathcal{H}$ we have

$$f = \sum_{j\in J}\langle Af, g_j\rangle Cf_j,$$

□

as claimed.

The following example illustrates that the set of approximately C-controlled duals of a frame is a proper subset of the set of its C-controlled g-duals.

Example 3.6. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. Set

(1) $\{f_j\}_{j\in J} = \{e_1, e_1, e_1, e_2, e_3, \cdots\}$

(2) $\{g_j\}_{j\in J} = \{\frac{1}{3}e_1, \frac{1}{3}e_1, \frac{1}{3}e_1, e_2, , e_3, \cdots\}$

and consider the operator $C : \mathcal{H} \longrightarrow \mathcal{H}$ given by $C(f) = \frac{1}{2}f$. Now we have

$$\sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle = \langle f, e_1\rangle\langle e_1, f\rangle + \frac{1}{2}\sum_{j\in J}\langle f, e_j\rangle\langle e_j, f\rangle.$$

Thus,

$$\frac{1}{2}\|f\|^2 \leq \sum_{j\in J}\langle f, f_j\rangle\langle Cf_j, f\rangle \leq \frac{3}{2}\|f\|^2.$$

Therefore, $\{f_j\}_{j\in J}$ is a C-controlled frame and $\{g_j\}_{j\in J}$ is a Bessel sequence and

$$\|f - \sum_{j\in J}\langle f, g_j\rangle Cf_j\| = \frac{1}{2}\|f\|, \text{ for all } f \in \mathcal{H}.$$

Hence, $\{g_j\}_{j\in J}$ is not an approximately C-controlled dual frame of $\{f_j\}_{j\in J}$ but a C-controlled g-dual frame for $\{f_j\}_{j\in J}$ with the invertible operator $A(f) = 2f$; because $\sum_{j\in J}\langle Af, g_j\rangle Cf_j = f$ for any $f \in \mathcal{H}$.

The following theorem shows under what conditions the opposite of Theorem 3.5 is established.

Theorem 3.7. *If $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames with invertible operator A such that $\|I - A^{-1}\| < 1$, then $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames.*

Proof. Since $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames; subsequently, for all $f \in \mathcal{H}$, $f = \sum_{j\in J}\langle Af, g_j\rangle Cf_j$. Hence,

$$\|f - \sum_{j\in J}\langle f, g_j\rangle Cf_j\| = \|f - \sum_{j\in J}\langle AA^{-1}f, g_j\rangle Cf_j\|$$
$$= \|f - A^{-1}f\| < \|f\|.$$

□

Corollary 3.8. *Let $\mathcal{H}$ be a Hilbert space and $C \in GL(\mathcal{H})$. Let also $\{f_j\}_{j\in J}$ be a C-controlled frame and $\{g_j\}_{j\in J}$ be an approximate C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator A. Then,*

(i) $\{(UT_c^*)^{-1}g_j\}_{j\in J}$ *is a C-controlled dual of $\{f_j\}_{j\in J}$ and*

$$(UT_c^*)^{-1}g_j = g_j + \sum_{n=1}^{+\infty}(I - UT_c^*)^n g_j.$$

(ii) $\{g_j\}_{j\in J}$ *is a C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator $(T_cU^*)^{-1}$.*

(iii) $\{g_j\}_{j\in J}$ *is an approximately C-controlled dual of $\{f_j\}_{j\in J}$.*

Proof. To prove (i) by the definition of an approximate C-controlled g-dual, we have

$$\|I - T_cU^*A\| < 1,$$

implying that T_cU^*A is an invertible operator. By assumption, A is an invertible operator, T_cU^* is thus an invertible operator. Therefore, similar to the argument in the proof of Theorem 3.2 from [11], it can be revealed that $\{(UT_c^*)^{-1}g_j\}_{j\in J}$ is a C-controlled dual of $\{f_j\}_{j\in J}$ and

$$(UT_c^*)^{-1}g_j = g_j + \sum_{n=1}^{+\infty}(I - UT_c^*)^n g_j.$$

Now we prove (ii). We have already seen in parts (i) that T_cU^* is an invertible operator, the remainder of proof (ii) follows immediately from the proof of Theorem 3.5.

Finally, to prove (iii), by part (ii), $\{g_j\}_{j\in J}$ is a C-controlled g-dual of $\{f_j\}_{j\in J}$ with invertible operator $(T_cU^*)^{-1}$. Additionally,

$$\|I - (T_cU^*A)^{-1})^{-1}\| = \|I - T_cU^*A\| < 1.$$

Now directly using Theorem 3.7, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are approximately C-controlled dual frames and this completes the proof. □

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