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THE TRIPLE ZERO GRAPH OF A COMMUTATIVE RING

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ABSTRACT. Let R be a commutative ring with non-zero identity. We define the set of triple zero elements of R by $TZ(R) = \{a \in Z(R)^* :$ there exist $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$. In this paper, we introduce and study some properties of the triple zero graph of R which is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$, and two vertices a and b are adjacent if and only if $ab \neq 0$ and there exists a non-zero element c of R such that $ac \neq 0$, $bc \neq 0$, and $abc = 0$. We investigate some properties of the triple zero graph of a general ZPI-ring R, we prove that $diam(TZ\Gamma(R)) \in \{0, 1, 2\}$ and $gr(TZ\Gamma(R)) \in \{3, \infty\}$.

1. Introduction

Throughout this paper, all rings are commutative with identity and $Z(R)$ denotes the set of zero-divisors of a ring R. The concept of the zero-divisor graph of a commutative ring was introduced by I. Beck [9]. He let all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. In [3], all elements of a commutative ring R are vertices, and distinct vertices a and b are adjacent if and only if $ab = 0$. This graph is denoted by $\Gamma_0(R)$. Then D.F. Anderson and P.S. Livingston [4] introduced a (induced) zero-divisor subgraph $\Gamma(R)$ of $\Gamma_0(R)$. The zero-divisor graph $\Gamma(R)$ introduced in [13] and [4] is as follows: Two distinct vertices $x, y \in Z(R)^* = Z(R)\backslash\{0\}$ are adjacent if and only if $xy = 0$. In [4], D.F. Anderson and P.S. Livingston have shown that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $gr((\Gamma(R)) \in \{3, 4, \infty\}$. The zero-divisor graph of a commutative ring in the sense of Anderson–Livingston has been studied extensively by several authors, [1], [2], [5], [6], [14], [15]. Since then, the concept of the zero-divisor graph of ring has been playing a vital role in its expansion.

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We define the set of the triple zero elements of R by $TZ(R) = \{a \in Z(R)^* :$ there exist $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0\}$. It is clear that every triple zero element of R is a zero-divisor of R, but the converse is not true in general. For example, the element 2 is a zero-divisor of $\mathbb{Z}_6$, but clearly it is not a triple zero element. In this paper, motivated from zero-divisor graphs, we introduce the triple zero graph of a commutative ring. Our starting point is the following definition: The triple zero graph of R is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$. If two distinct elements a and b are adjacent, then (a, b) is an edge and we will denote it by $a \sim b$. Two distinct vertices a and b are adjacent if and only if $ab \neq 0$ and there exists an element $c \in R\backslash\{0\}$ such that $ac \neq 0$, $bc \neq 0$ and $abc = 0$. The relation "$\sim$" is always symmetric, but neither reflexive nor transitive in general. For instance, let $R = \mathbb{Z}_{36}$. Then clearly 2, 3, $6 \in TZ(R)$ with $6 \nsim 6$, and also $2 \sim 3$, $2 \sim 9$, but $3 \nsim 9$.

Recall from [8] that I is said to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. As defined in [7], I is said to to be a weakly 2-absorbing ideal *of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$, $ac \in I$, or $bc \in I$. From these definitions, note that $\{0\}$ is always a weakly 2-absorbing ideal of R. If 0 is not a 2-absorbing ideal, then there are some triple zero elements of R. The concept of (weakly) 2-absorbing ideals and the zero-divisor graphs motivated us to define the triple zero divisor graph and also investigate the relations between triple zero graph of a ring R and 2-absorbing ideals of R.

Among many results in this paper, in Section 2, we justify some properties of the triple zero graph of commutative rings. In Theorem 1, we show that a proper ideal I of a ring R is 2-absorbing if and only if $TZ\Gamma(R/I) = \emptyset$. In Theorem 11, we characterize triangle free triple zero graphs of general ZPI-rings. In [11], the authors define 3-zero-divisor hypergraph regarding to an ideal with vertices $\{x \in R\backslash I : xyz \in I$ for some $y, z \in R\backslash I$ such that $xy \notin I$, $yz \notin I$, $xz \notin I\}$ where distinct vertices are adjacent if and only if $xyz \in I$, $xy \notin I$, $yz \notin I$ and $xz \notin I$. They conclude that diameter of this graph is at most 4. In Section 3, we study the triple zero graph of general ZPI-rings. The graph properties of the triple zero graph of general ZPI-rings such as diameter and girth are investigated. We obtain that the triple zero graph of a zero dimensional general ZPI-ring is always connected with diameter at most 2 and girth 3 if it is determined. (Corollary 12). Furthermore, we give some characterizations for the triple zero graph of $\mathbb{Z}_n$ where $n > 1$ and justify the diameter and girth of $TZ\Gamma(\mathbb{Z}_n)$. (Theorem 13, Theorem 14 and Corollary 15)

For the sake of completeness, we state some definitions and notation used throughout. Let G be a (undirected) graph. The order of G, denoted by $|G|$, is equal to the cardinality of the vertex set. The graph G is connected if there is a path between any two distinct vertices. For vertices a and b of G, we say that the distance between a and b, $d(a, b)$ is the length of a shortest path from a to b. If there is no path between a and b, then $d(a, b) = \infty$, and $d(a, a) = 0$. A graph G is said to be totally disconnected if it has no edges. The diameter of G is defined by $diam(G) = \sup\{d(a, b) : a$

and b are vertices of $G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. If G contains no cycles, then $gr(G) = \infty$. A cycle of length three is commonly called a triangle. A triangle-free graph is an undirected graph in which no three vertices form a triangle of edges. A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n. A complete bipartite graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K_{m,n}$ where A and B are partitions with $\mid A \mid= m$ and $\mid B \mid= n$. If one of the vertex sets is a singleton, then we call G a star graph. A star graph is clearly $K_{1,n}$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. For general background and terminology, the reader may consult [10].

2. Properties of The Triple Zero Graph

Theorem 1. *Let R be a commutative ring and I be a proper ideal of R. Then the following statements hold:*

1. $TZ\Gamma(R/I) = \emptyset$ if and only if I is a 2-absorbing ideal of R.
2. $TZ\Gamma(R) = \emptyset$ if and only if $\{0\}$ is a 2-absorbing ideal of R.
3. If (R, M) is a quasi-local ring with $M^2 = 0$, then $TZ\Gamma(R) = \emptyset$.

Proof. Suppose that I is not a 2-absorbing ideal of R. Then there exist some (not necessarily distinct) elements a, b, c of R with $abc \in I$ but neither $ab \in I$ nor $ac \in I$ nor $bc \in I$. Hence $(a + I)(b + I)(c + I) = I$ but neither $(a + I)(b + I) = I$ nor $(a + I)(c + I) = I$ nor $(b + I)(c + I) = I$. Thus $a, b, c \in TZ(R/I)$; and so $TZ\Gamma(R/I) \neq \emptyset$. Conversely, if $TZ\Gamma(R/I) \neq \emptyset$, then there are some (not necessarily distinct) elements $a + I, b + I, c + I$ of R/I satisfying $(a + I)(b + I)(c + I) = I$ but neither $(a + I)(b + I) = I$ nor $(a + I)(c + I) = I$ nor $(b + I)(c + I) = I$. It implies that $ab, ac, bc \notin I$ and $abc \in I$. Hence I is not a 2-absorbing ideal of R.

(2) It is clearly a particular case putting $I = 0$ in (1).

(3) Suppose that (R, M) is a quasi-local ring with $M^2 = 0$. Hence 0 is a 2-absorbing ideal of R by [7, Corollary 3.3]. Thus $TZ\Gamma(R) = \emptyset$ by (2). □

The following example shows that the converse of Theorem 1 (3) does not hold.

Example 2. *Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then clearly $TZ\Gamma(R) = \emptyset$ but since R has two maximal ideals $0 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times 0$, it is not a quasi-local ring.*

Let $R = \mathbb{Z}_p[X]/\langle X^n\rangle$, where p is prime and $n \geq 3$. We denote $a(X)$ as the congruence class of polynomials congruent to $a(X) \bmod \langle X^n\rangle$. It is well-known that an element of $\mathbb{Z}_p[X]/\langle X^n\rangle$ is of the form $a(X) = a_0 + a_1X + a_2X^2 + \cdots + a_kX^k$ of degree $k \leq n$ where $a_i \in \mathbb{Z}_p$ for $i \in \{1, 2, ..., k\}$. Now we determine the vertex set of the graph $TZ(\mathbb{Z}_p[X]/\langle X^n\rangle)$.

Theorem 3. *Let $a(X) = a_0 + a_1X + a_2X^2 + \cdots + a_kX^k \in \mathbb{Z}_p[X]/\langle X^n\rangle$ where $n \geq 3$. Then $a(X)$ is a vertex of the graph $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^n\rangle)$ if and only if $a_0 = 0(\operatorname{mod} p)$ and of the form in one of the following types:*

(1) $a_1 = a_2 = ... = a_{k-1} = 0$ and $k \leq n-2$.
(2) $a_i \neq 0$ for some $r = 1, 2, ..., k-1$ and $k \leq n-1$.

Proof. Let $a(X) \in TZ(\mathbb{Z}_p[X]/\langle X^n\rangle)$. Then there exists non-zero $b(X)$, $c(X) \in \mathbb{Z}_p[X]/\langle X^n\rangle$ such that $a(X)b(X)c(X) = 0 \operatorname{mod} \langle X^n\rangle$, $a(X)b(X) \neq 0 \operatorname{mod} \langle X^n\rangle$, $a(X)c(X) \neq 0 \operatorname{mod} \langle X^n\rangle$ and $b(X)c(X) \neq 0 \operatorname{mod} \langle X^n\rangle$. Let $b(X) = b_0 + b_1X + b_2X^2 + \cdots + b_tX^t$, $c(X) = c_0 + c_1X + c_2X^2 + \cdots + c_sX^s$ where b_j and c_r are the first non-zero (i.e., $b_j, c_r \neq 0(\operatorname{mod} p)$) coefficients in the polynomials $b(X)$ and $c(X)$, respectively. Then the coefficient of X_{j+r} in the product $a(X)b(X)c(X)$ is $a_0b_jc_r$. Since $a(X)b(X)c(X) = 0 \operatorname{mod} \langle X^n\rangle$ and $j, r < n$, we must have $a_0b_ic_j = 0(\operatorname{mod} p)$. Observe that since b_j, c_r are non-zero elements of $\mathbb{Z}_p$, we have $b_jc_r \neq 0$. Thus $a_0 = 0(\operatorname{mod} p)$.

Case I. Suppose that $a_1 = a_2 = ... = a_{k-1} = 0$. Then $a_kX^kb_jX^jc_rX^r = 0 \operatorname{mod} \langle X^n\rangle$ which implies that $k + j + r = n$. Since $j, r \geq 1$, we conclude that $k \leq n-2$.

Case II. Suppose that $a_i \neq 0$ for some $i = 1, 2, ..., k-1$. Then we show that k can be $n-1$. Assume that $\deg(a(X)) = k = n-1$. Then, clearly $a(X)\ X\ X = 0 \operatorname{mod} \langle X^n\rangle$ and $X\ X \neq 0 \operatorname{mod} \langle X^n\rangle$. Since $a_iX^iX \neq 0 \operatorname{mod} \langle X^n\rangle$ where $i = 1, 2, ..., k-1$, we conclude that $a(X)\ X \neq 0 \operatorname{mod} \langle X^n\rangle$.

Conversely, assume that $a_0 = 0\ (\operatorname{mod} p)$. If (1) holds, then $a(X) = a_kX^k$ and $k \leq n-2$. Then $a(X)\ X^j\ X^r = 0 \operatorname{mod} \langle X^n\rangle$ for all $j, r \geq 1$ such that $j + r = n - k$ but neither $a(X)\ X^j = 0 \operatorname{mod} \langle X^n\rangle$ nor $a(X)\ X^r = 0 \operatorname{mod} \langle X^n\rangle$ nor $X^j\ X^r = 0 \operatorname{mod} \langle X^n\rangle$. Hence $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n\rangle$. Suppose that (2) holds. We may assume that $a_1 \neq 0$ (mod p). Then $a(X)\ X^j\ X^r = 0 \operatorname{mod} \langle X^n\rangle$ for all $j, r \geq 1$ such that $j + r = n-1$. Since $a_1X\ X^j \neq 0 \operatorname{mod} \langle X^n\rangle$ and $a_1X\ X^r \neq 0 \operatorname{mod} \langle X^n\rangle$, we conclude that $a(X)\ X^j \neq 0 \operatorname{mod} \langle X^n\rangle$ and $a(X)\ X^r \neq 0 \operatorname{mod} \langle X^n\rangle$. Thus $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n\rangle$. □

Theorem 4. *Let $R = \mathbb{Z}_p[X]/\langle X^3\rangle$. Then $TZ\Gamma(R)$ is a complete graph with $p^2 - p$ vertices, i.e., $TZ\Gamma(R) \cong K_{p^2-p}$. In particular, if $p = 2$, then $TZ\Gamma(R) \cong K_2$.*

Proof. From Theorem 3, the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3\rangle)$ of the type $nX + mX^2$, where n, m are integers with $1 \leq n \leq p$ and $0 \leq m \leq p$. Hence, the number of the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3\rangle)$ is $p^2 - p$. Observe that all vertices of this graph are adjacent, thus it is the complete graph K_{p^2-p}. For $p = 3$, this graph is illustrated by Figure 2. In the particular case, since $X\ X\ (X + X^2) = 0$ but $X\ X \neq 0$ and $X(X + X^2) \neq 0$, X and $(X + X^2)$ are the only distinct adjacent vertices of $TZ\Gamma(\mathbb{Z}_2[X]/\langle X^3\rangle)$. □

We are unable to answer the following question which may be inspiring for the possible other work:

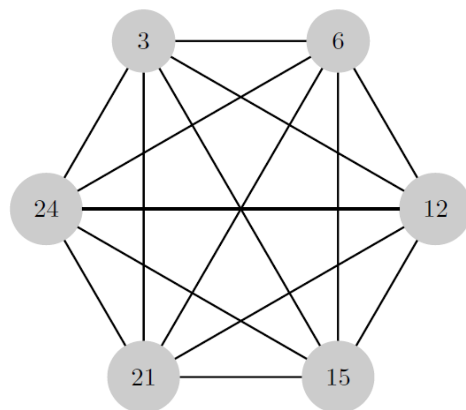

FIGURE 1. $TZ\Gamma(\mathbb{Z}_{27})$

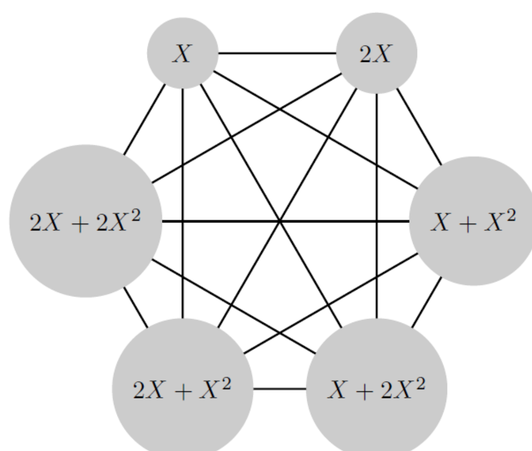

FIGURE 2. $TZ\Gamma(\mathbb{Z}_3[X]/\langle X^3 \rangle)$

Question. Let $R = \mathbb{Z}_p[X]/\langle X^n \rangle$ where p is a prime number and $n \geq 3$. Can we have a general characterization for the triple zero graph of R?

We recall that an n-gon is a regular polygon with n sides. In the next example, we show that there are triple zero graphs with cycles of arbitrary specified length.

Example 5. *Let T be an integral domain and $n \geq 3$ is an integer. Consider $R = T[X_1, X_2, \cdots, X_n]/(X_1X_2X_3, X_3X_4X_5, \cdots, X_{n-1}X_nX_1)$. Then $TZ\Gamma(R)$ is a connected graph which has an n-gon, an $n/2$-gon and has triangles more than n.*

Proof. Observe that $X_1 \sim X_2 \sim X_3$, $X_3 \sim X_4 \sim X_5, \cdots, X_{n-1} \sim X_n \sim X_1$ are some of the triangles, and it is easy to see that $(X_k + X_kX_{k+1}) \sim (X_{k+1} + X_kX_{k+1}) \sim X_{k+2}$ is another triangle for each k, where k is odd, and $k < n-2$. Also $X_1 \sim X_3 \sim \cdots \sim X_{n-1} \sim X_1$ is an $n/2 - gon$ and $X_1 \sim X_2 \sim \cdots \sim X_{n-1} \sim X_n \sim X_1$ is an n-gon. □

3. Triple Zero Graph of General ZPI-rings

A ring is called a general ZPI-ring (resp. ZPI-ring) if each ideal (resp. each nonzero ideal) I of R is uniquely expressible as product of prime ideals of R. Dedekind domains are indecomposable general ZPI-rings. For a general background, the reader may refer to [12]. In this section, we study the graph theoretical properties of the triple zero graph for general ZPI-rings. First we need to prove the following lemma which is a generalization of [8, Theorem 3.15].

Lemma 6. *Let R be a zero dimensional Noetherian ring which is not a field. Then the following statements are equivalent:*

(1) R is a general ZPI-ring.
(2) If I is a 2-absorbing ideal of R, then I is a maximal ideal of R or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R.
(3) If I is a 2-absorbing ideal of R, then I is a prime ideal of R or $I = P^2$ for some prime ideal P of R or $I = PQ$ for some prime ideals P, Q of R.

Proof. (1)⇒(2) Let I be a 2-absorbing ideal of R. Since maximal ideals coincide with prime ideals, $\sqrt{I} = M$ for some maximal ideal M of R with $M^2 \subseteq I$ or $\sqrt{I} = M \cap M' = MM'$ for some maximal ideals M, M' of R with $MM' \subseteq I$ by [8, Theorem 2.4]. Thus, we have either $I = M$ is maximal or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R.

(2)⇒(3) is straightforward.

(3)⇒(1) Suppose that (3) holds. Assume that there is an ideal J of R which satisfies $M^2 \subseteq I \subseteq M$. Then I is an M-primary ideal of R; so I is a 2-absorbing ideal by [8, Theorem 3.1]. Hence $I = M$ or $I = M^2$ from our assumption (3). Thus there are no ideals properly between M and M^2. From [12, (39.2) Theorem], R is a general ZPI-ring. □

Theorem 7. *Let R be a zero dimensional general ZPI-ring. Then $TZ\Gamma(R) = \emptyset$ if and ony if either R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R.*

Proof. If R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R, then it is easy to verify that there is no triple zero elements of R; so $TZ\Gamma(R) = \emptyset$. Conversely, suppose that $TZ\Gamma(R) = \emptyset$. Then 0 is a 2-absorbing ideal of R by Theorem 1. From Lemma 6, either 0 is prime, $0 = P^2$ for some prime ideal P or $0 = PQ$ for some prime ideals P,Q of R, so we are done. □

We recall that a special primary is an indecomposable general ZPI-ring which is a local ring with maximal ideal M such that each proper ideal of R is a power of M.

Lemma 8. *[12] An indecomposable general ZPI-ring with identity is either a Dedekind domain or a special primary ring.*

Theorem 9. *Let R be a general ZPI-ring and $0 = P^3$ where P is a prime ideal of R such that $P^2 \neq 0$. Then $TZ\Gamma(R)$ is a complete graph on $|P| - \left|P^2\right|$ vertices; i.e. $TZ\Gamma(R) \cong K_{|P|-|P^2|}$*

Proof. Suppose that $0 = P^3$ where P is a prime ideal of R. It is well-known that a ring R is indecomposable if and only if 1 is the only non-zero idempotent element of R. Let $0 \neq a \in R$ and $a^2 = a$. Hence $a - a^2 = a(1 - a) = 0 \in P$ implies $a \in P$ or $(1 - a) \in P$. If $a \in P$, then we get $0 = a^3 = a^2 = a$, a contradiction. Thus $(1 - a) \in P$. It follows $0 = (1 - a)^3 = 1 - 2a^2 + 2a - a^3 = 1 - a$, and so $a = 1$. Therefore, R is a indecomposable ring which is clearly not a domain as $0 = P^3$ and P is nonzero. Hence, we conclude from Lemma 8 that R is a special primary ring. Let M be the unique maximal ideal of R. Since every ideal, in particular, the zero ideal is a power of M, we have $M \subseteq \sqrt{0}$. Since $0 = P^3$, clearly we have $P = \sqrt{0} = M$.

Now, we show that a is a vertex of $TZ\Gamma(R)$ if and only if $a \in P \backslash P^2$. Let a be a vertex of $TZ\Gamma(R)$. Then, there exist $b, c \in R\backslash\{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$. If $a \notin P$, then a is unit and $bc = 0$ which is a contradiction. Thus $TZ(R) \subseteq P$. If $a \in P^2$, then since $b \in TZ(R) \subseteq P$, we conclude $ab \in P^3 = 0$, a contradiction. Therefore, $a \in P\backslash P^2$. Conversely, if $a \in P\backslash P^2$, then the claim follows from $a^3 = 0$ and $a^2 \in P^2 \neq 0$. Suppose a and b are any two distinct vertices. Since $a^2b = ab^2 = 0$ and ab, a^2, b^2 are nonzero, a and b are adjacent. Thus, $TZ\Gamma(R)$ is a complete graph on $|P| - \left|P^2\right|$ vertices. □

Theorem 10. *Let $0 = P^2Q$ where P and Q are prime ideals of a general ZPI-ring R. Then $TZ\Gamma(R)$ is a connected graph with diameter 2 and girth 3.*

Proof. Suppose that $0 = P^2Q$. Let a be a vertex of $TZ\Gamma(R)$. We show that $a \in Q\backslash P$ or $a \in P\backslash(P^2 \cup Q)$. Since $a \in TZ(R)$, there exist $b, c \in R\backslash P^2Q$ such that $abc \in P^2Q$ and $ab, bc, ac \notin P^2Q$. Hence, we have either $a \in P$ or $b \in P$ or $c \in P$, and $a \in Q$ or $b \in Q$ or $c \in Q$.

Case I. Let $a \in P \cap Q$. If $a \in P^2$, then $a \in P^2 \cap Q = P^2Q = 0$ as P^2 and Q are coprime, a contradiction. So, assume that $a \in (P\backslash P^2) \cap Q$. If $b \in P$ or $c \in P$, then

$ab = 0$ or $ac = 0$, a contradiction. If $b \in Q\backslash P$ and $c \in Q\backslash P$, then we get $abc \notin P^2Q$ which is again a contradiction. Thus, $TZ(R) \subseteq (P\backslash Q) \cup (Q\backslash P)$.

Case II. Let $a \in P\backslash Q$. Suppose that $a \in P^2$. If $b \in Q\backslash P$ or $c \in Q\backslash P$, then we have either $ab = 0$ or $ac = 0$, a contradiction. If $b, c \in P\backslash Q$, then $abc \notin Q$, and so $abc \notin P^2Q$, a contradiction.

Therefore, we conclude that $a \in P\backslash(P^2 \cup Q)$ or $a \in Q\backslash P$.

Observe that all pairs are adjacent except for the elements of $Q\backslash P$. In fact, if an element $x \in TZ(R)$ satisfies $a_1b_1x = 0$, where $a_1, a_2 \in Q\backslash P$, we conclude that $x \in P^2$, a contradiction. Thus $TZ\Gamma(R)$ is a connected graph with $diam(TZ\Gamma(R)) = 2$ and $gr(TZ\Gamma(R)) = 3$. □

In the next theorem, we give a necessary and sufficient conditions for $TZ\Gamma(R)$ to be triangle free.

Theorem 11. *Let R be a zero dimensional general ZPI-ring. $TZ\Gamma(R)$ is triangle free if and only if one of the following statements is hold:*

(1) *R is an integral domain.*
(2) *$0 = PQ$ for some distinct prime ideals P and Q of R.*
(3) *$0 = P^2$ for some prime ideal P of R.*
(4) *$0 = P^3$ for some prime ideal P of R such that $|P| = 4$ and $\left|P^2\right| = 2$.*

Proof. $(\Rightarrow)$. We investigate the following cases separately.

Case I. Suppose that 0 is divisible by at least three prime ideals of R, say P, Q and T. Then $p \sim q \sim t$ where $p \in P$, $q \in Q$, $t \in T$ forms a triangle.

Case II. If 0 is divisible by P^2 and Q, where P and Q are distinct prime ideals of R, then we obtain the triangle $p \sim q \sim kp$, where $p \in P$, $q \in Q$ and $1 \neq k \in R\backslash Q$.

Case III. Suppose that $0 = P^n$, where P is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $0 = P^n$ $(n \geq 4)$, then $p \sim p^2 \sim kp$, where $p \in P$ and $1 \neq k \in R\backslash P$ forms a triangle.

$(\Leftarrow)$. If (1), (2) or (3) holds, then $TZ\Gamma(R) = \emptyset$ by Theorem 7. If (4) holds, then there are the only two vertices connected by an edge by Theorem 9; so $TZ\Gamma(R) \cong K_2$. □

So we conclude the following result.

Corollary 12. *The diameter of the triple zero graph of a zero dimensional general ZPI-ring R is an element of $\{0, 1, 2\}$ and the girth of the triple zero graph of R is 3 or undefined.*

In the following result, we characterize the triple zero graph of $\mathbb{Z}_n$ and calculate $|TZ\Gamma(\mathbb{Z}_n)|$ cardinality of the vertex set for some particular cases.

Theorem 13. *Let $R = \mathbb{Z}_n$ where n is a positive integer. Then the following statements hold:*

(1) *If $n = p$ or $n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$.*

(2) *If $n = p^3$ where p is prime, then $TZ\Gamma(\mathbb{Z}_n)$ is a complete graph on $p^2 - p$ vertices.*

(3) *If $n = p^2 q$ where p and q are distinct prime integers, then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph with diameter 2 and girth 3.*

Proof. (1) is clear by Theorem 7.

(2) The vertices of $TZ\Gamma(\mathbb{Z}_n)$ are kp, where $k \in \mathbb{Z}^*_{p^2} = \{k \in \mathbb{Z} : (k, p^2) = 1, k < p^2\}$. So the number of vertices can be calculated by Euler's function $\phi(p^2) = p(p-1)$. Since $(kp)(mp)(tp) = 0$ for all $k, m, t \in \mathbb{Z}^*_{p^2}$ and neither $(kp)(mp) = 0$ nor $(kp)(tp) = 0$ nor $(mp)(tp) = 0$, there is an edge between all vertices. Thus the graph is complete; so it is K_{p^2-p}.

(3) Suppose that $n = p^2 q$. Then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph with diameter 2 and girth 3 by Theorem 10. Observe that the vertices of this graph are of the form kq where $k \in \mathbb{Z}^*_{p^2} = \{k \in \mathbb{Z} : (k, p^2) = 1,\ k < p^2\}$ and of the form sp where $s \in \Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$. So the number of vertices is $|\Omega| + \phi(p^2) = |\Omega| + p^2 - p$. Moreover, the number of edges can be calculated as $\binom{|\Omega|}{2} + (p^2 - p)\,|\Omega|$. □

Theorem 14. *Let $n > 0$ and $R = \mathbb{Z}_n$. Then the following statements are equivalent:*

(1) *$TZ\Gamma(\mathbb{Z}_n)$ is triangle free.*

(2) *Either $n = p$, $n = p^2$, $n = pq$, or $n = 8$, where p and q are distinct prime integers.*

Proof. We investigate the following cases separately.

Case I. Suppose that n is divisible by at least three primes, say p, q, and r. Then $p \sim q \sim (n/pq)$ forms a triangle.

Case II. If n is divisible by p^2 and q, where p and q are distinct prime integers, then we obtain the triangle $p \sim q \sim kp$, where $(k, q) = 1$ and $k < pq$.

Case III. Suppose that $n = p^n$, where p is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $n = p^n$, where $n \geq 3$, except from $p = 2$, then $p \sim p^2 \sim kp$, where $(k, p) = 1$, $k < p^{n-3}$ forms a triangle. Thus, $n = p$, $n = p^2$, $n = pq$, or $n = 8$.

Conversely, if $n = p$, $n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$ by Theorem 7. If $n = 8$, then 2 and 6 are the only vertices connected by an edge; and so the claim is clear. □

So we conclude the following result which shows that $TZ\Gamma(\mathbb{Z}_n)$ is connected with diameter at most 2.

Corollary 15. *The diameter of the triple zero graph of $\mathbb{Z}_n$ is an element of $\{0, 1, 2\}$ and the girth of the triple zero graph of $\mathbb{Z}_n$ is 3 or undefined.*

Now we can summarize these results by the table below. Let p and q be distinct prime integers and $\Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$.

TABLE 1. $TZ\Gamma(\mathbb{Z}_n)$ Summary Table

n	Number of vertices	Number of edges	Diam	Girth	Remarks
p or p^2or pq	0	0	0	∞	$TZ\Gamma(\mathbb{Z}_n) = \emptyset$
8	2	1	1	∞	2∼6
$p^3(p \geq 3)$	p^2-p	$\binom{p^2-p}{2}$	2	3	K_{p^2-p}
p^2q	$\lvert\Omega\rvert + p^2 - p$	$\binom{\lvert\Omega\rvert}{2} + (p^2-p)\lvert\Omega\rvert$	2	3	Connected
All others			2	3	Connected

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