Sufficient condition for $q$-starlike and $q$-convex functions associated with generalized confluent hypergeometric function

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Abstract

The main object of this paper is to investigate and determine a sufficient condition for $q$-starlike and $q$-convex functions which are associated with generalized confluent hypergeometric function.

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1. Introduction and Preliminary

Let $A$ denote the class of all functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathbb{U},$$

which are analytic in open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Let $S$ be the subclass of $A$ consists of univalent functions in $\mathbb{U}$. Further suppose that $S^*$ is subclass of functions of $A$ which are starlike in $\mathbb{U}$, that is $f$ satisfy the subsequent conditions:

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad \forall z \in \mathbb{U}$$

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Let $C^*$ be subclass of functions of $A$ which are convex in $U$, that is $f$ satisfy the following conditions:

$$
\text{Re}\left\{\frac{(zf'(z))^'}{f'(z)}\right\} = \text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad \forall z \in U
$$

For analytic functions $f$ and $g$ in $U$ we say that the function $f$ is subordinate to the function $g$ and written as

$$
f(z) < g(z)
$$

If there exists a Schwarz function $w$ which is analytic in $U$ and $w(0) = 0$, $|w(z)| < 1$, such that $f(z) = g(w(z))$

Further, if $g$ is the function which is univalent in $U$, then it becomes

$$
f(z) < g(z), \quad z \in U \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U)
$$

Now we define $P$ the class of analytic function with positive real part which is given as

$$
p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m; \quad \text{Re}(p(z)) > 0, \quad z \in U.
$$

**Definition 1.1.** An analytic function $h$ with $h(0) = 1$ belongs to the class $P[M, N]$, with $-1 \leq N < M \leq 1$, if and only if

$$
h(z) < \frac{1 + Mz}{1 + Nz}.
$$

The class $P[M, N]$ of analytic functions was introduced and studied by Janowski [6], who showed that $h \in P[M, N]$ if and only if there exists a function $p \in P$, such that

$$
h(z) = \frac{(M + 1)p(z) - (M - 1)}{(N + 1)p(z) - (N - 1)}, \quad z \in U.
$$

**Definition 1.2.** [6] (i) A function $f \in A$ is in the class $S^*[M, N]$, with $-1 \leq N < M \leq 1$, if and only if

$$
\frac{zf'(z)}{f(z)} < \frac{1 + Mz}{1 + Nz}.
$$

(ii) A function $f \in A$ is in the class $C^*[M, N]$, with $-1 \leq N < M \leq 1$, if and only if

$$
1 + \frac{f''(z)}{f'(z)} < \frac{1 + Mz}{1 + Nz}.
$$

**Definition 1.3.** The $q$-number $[m]_q$ defined in [15] for $q \in (0, 1)$, is given by

$$
[m]_q := \begin{cases} 
1 - q^m, & \text{if } m \in \mathbb{C}, \\
\frac{1 - q}{1 - q^m}, & \sum_{k=0}^{m-1} q^k = 1 + q + q^2 + \cdots + q^{m-1}, \quad \text{if } m \in \mathbb{N} := \{1, 2, \ldots\}.
\end{cases}
$$

**Definition 1.4.** [16] The $q$-derivative $D_q f$ of a function $f$ is defined as

$$
D_q f(z) := \begin{cases} 
\frac{f(z) - f(qz)}{(1 - q)z}, & \text{if } z \in \mathbb{C} \setminus \{0\}, \\
f'(0), & \text{if } z = 0,
\end{cases}
$$

provided that $f'(0)$ exists, and $0 < q < 1$. 
From the Definition 1.4 it follows immediately that
\[
\lim_{q \to 1} D_q f(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} = f'(z).
\]

For a function \( f \in A \) which has the power expansion series of the form (1.1), it is easy to check that
\[
D_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1}, \quad z \in U,
\]
as it was previously defined by Srivastava and Bansal [14], although the \( q \)-derivative operator \( D_q \) was presumably first applied by Ismail et al. [5] to study a \( q \)-extension of the class \( S^* \) of starlike functions in \( U \) (see [5], [3], [13]).

**Definition 1.5.** A function \( f \in A \) is in the class \( S_q^* \) if and only if
\[
\left| \frac{z}{f(z)} D_q f(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad z \in U.
\]

It is observed that, as \( q \to 1^- \) the closed disk
\[
\left| w - \frac{1}{1-q} \right| < \frac{1}{1-q}
\]
becomes the right-half plane and the class \( S_q^* \) of \( q \)-starlike functions diminishes to the acquainted class \( S^* \).

Consistently, by with the principle of subordination among analytic functions, we can rewrite the inequality (5) as
\[
\frac{z}{f(z)} D_q f(z) \prec \phi(z),
\]
where
\[
\phi(z) := \left( M + 1 \right) z + 2 + \left( M - 1 \right) qz
\]
which by using the definition of the subordination can be written as follows:
\[
\frac{z D_q f(z)}{f(z)} < \phi(z),
\]
where
\[
\phi(z) := \frac{(M + 1)z + 2 + (M - 1)qz}{(N + 1)z + 2 + (N - 1)qz}, \quad -1 \leq N < M \leq 1, \quad q \in (0,1).
\]

**Remark 1.1.** (i) It is easy to check that
\[
\lim_{q \to 1^-} S_q^*[M,N] = S^*[M,N].
\]

Also, \( S_q^*[1,-1] = S_q^* \), where \( S_q^* \) is the class of functions introduced and studied by Ismail et al. [5].
Lemma 1.2. A function \( f \) from the definition of \( Q \) we get that
\[
\left| Q(z) - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad z \in U,
\]
and from (7) it follows
\[
Q(z) = \frac{(N - 1)zD_qf(z) - (M - 1)}{(N + 1)zD_qf(z) - (M + 1)}, \quad z \in U.
\]

From these computations we conclude that a function \( f \in A \) is in the class \( S_q^*[M, N] \), if and only if
\[
\left| \frac{(N - 1)zD_qf(z) - (M - 1)}{(N + 1)zD_qf(z) - (M + 1)} - \frac{1}{1-q} \right| < \frac{1}{1-q}, \quad z \in U.
\]

In itâ€™s special case when \( M = 1 - 2\beta \) and \( N = -1 \), with \( 0 \leq \beta < 1 \), the function class \( S_q^*[M, N] \) reduces to the function class \( S_q^*(\beta) \) which was presented and deliberated by Agrawal and Sahoo [1].

(iii) By means of the well-known Alexanderâ€™s theorem [2], the class \( C_q^*[M, N] \) of \( q \)-convex functions can be defined in the following way:
\[
f \in C_q^*[M, N] \iff zD_qf(z) \in S_q^*[M, N].
\]

The confluent hypergeometric function in the series form is given by
\[
F(\xi; \eta; z) = \sum_{m=0}^{\infty} \frac{(\xi)_m z^m}{(\eta)_m m!}; \quad \forall z \in \mathbb{C},
\]
where \( \eta \) is neither zero nor a negative integer and the series is convergent for \( \xi, \eta \). Now the generalized confluent hypergeometric function (normalized function) is defined as
\[
zF(\xi; \eta; z) = \sum_{m=0}^{\infty} \frac{(\xi)_m z^{m+1}}{(\eta)_m m!} \quad \text{(By using convolution of two functions)}
\]
\[
= z + \sum_{m=2}^{\infty} \frac{(\xi)_{m-1} z^m}{(\eta)_{m-1} (m-1)!},
\]
where \( (\beta)_m \) is the Pochhammer symbol defined as
\[
(\beta)_m = \begin{cases} 
1 & \text{if } m = 0 \\
\beta(\beta+1)(\beta+2)\cdots(\beta+m-1) & \text{if } m \in \mathbb{N} \\
\Gamma(\beta+m)/\Gamma\beta & \text{if } m \geq 0
\end{cases}
\]
and
\[
(\beta)_{m+k} = (\beta)_m (\beta+m)_k = (\beta)_k (\beta+k)_m.
\]

In this paper we determine sufficient conditions for \( q \)-starlike functions and \( q \)-convex functions associated with confluent hypergeometric function by using following sufficient conditions obtained by Srivastava [15]:

**Lemma 1.1.** [15] A function \( f \in A \) is in the class \( S_q^*[M, N] \), if it satisfying the following condition
\[
\sum_{m=2}^{\infty} \frac{(2q[m-1]_q + [(N+1)[m]_q - (M+1)])|a_m|}{|N-M|} < 1
\]

**Lemma 1.2.** [15] A function \( f \in A \) is in the class \( C_q^*[M, N] \), if it satisfying the following condition
\[
\sum_{m=2}^{\infty} \frac{|m|_q(2q[m-1]_q + [(N+1)[m]_q - (M+1)])|a_m|}{|N-M|} < 1
\]
2. Main Results

**Theorem 2.1.** Let $E_j$, $j \in \{1, 2\}$, be defined as follows:

(i) If $\xi > 0$ and $\gamma > 0$, then $E_1$ is given by

$$E_1(\xi, \gamma, q) := \frac{1}{1-q}\left\{ (q + N + 2 + M(1-q))F(\xi; \gamma, 1) - (N + 3)qF(\xi; \gamma, q) - (M + N + 2)(1-q) \right\}.$$

(ii) If $\xi, \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $E_2$ is given by

$$E_2(\xi, \gamma, q) := \frac{1}{1-q}\left\{ (q + N + 2 + M(1-q))F(\xi; \gamma, 1) - (N + 3)qF(\xi; \gamma, q) - (M + N + 2)(1-q) \right\}.$$

If for any $j \in \{1, 2\}$ the inequality

$$E_j(\xi, \gamma, q) < |N - M|$$

holds, then function $zF(\xi; \gamma, z)$ belongs to the class $S^*_q[M, N]$.

**Proof.** Since

$$zF(\xi; \gamma, z) = z + \sum_{m=2}^{\infty} \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}z^m, z \in U,$$

according to Lemma 1.1 any function $f \in A$ is in the class $S^*_q[M, N]$ if it satisfies the inequality (9). Then, for $f(z) := zF(\xi; \gamma, z)$ it is sufficient to show that (9) holds, for

$$a_m = \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}, \quad \text{and} \quad [m]_q = \frac{1 - q^m}{1 - q}.$$

Using the triangle’s inequality we get

$$\sum_{m=2}^{\infty} \left( 2q[m-1]_q + |(N + 1)[m]_q - (M + 1)| \right) |a_m|$$

$$\leq \sum_{m=2}^{\infty} 2q \frac{1 - q^{m-1}}{1-q} |a_m| + \sum_{m=2}^{\infty} (N + 1) \frac{1 - q^m}{1-q} |a_m| + \sum_{m=2}^{\infty} (M + 1) |a_m|$$

$$= \sum_{m=2}^{\infty} \left( \frac{2q + (N + 1)}{1 - q} + (M + 1) \right) |a_m| - \sum_{m=2}^{\infty} \frac{(N + 3)q^m}{1-q} |a_m|. \quad (11)$$

Case (i) If $\xi > 0$ and $\gamma > 0$, from (11) we get

$$\sum_{m=2}^{\infty} \left( 2q[m-1]_q + |(N + 1)[m]_q - (M + 1)| \right) |a_m|$$

$$\leq \left( \frac{2q + (N + 1)}{1 - q} + (M + 1) \right) \sum_{m=2}^{\infty} \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!} - \frac{N + 3}{1-q} \sum_{m=2}^{\infty} \frac{(\xi)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!}$$

$$= \frac{1}{1-q} \left\{ (q + N + 2 + M(1-q))F(\xi; \gamma, 1) - (N + 3)qF(\xi, \eta; \gamma, q) - (M + N + 2)(1-q) \right\}$$

$$= \frac{1}{1-q} \left\{ (q + N + 2 + M(1-q))F(\xi; \gamma, 1) - (N + 3)qF(\xi; \gamma, q) - (M + N + 2)(1-q) \right\} =: E_1(\xi, \gamma, q),$$
and the assumption of the theorem implies (9), that is \( zF(\xi; \gamma; z) \in S_q^*[M, N] \).

**Case (ii)** If \( \xi \in \mathbb{C} \setminus \{0\}, \gamma > 0 \), from (11) we have

\[
\sum_{m=2}^{\infty} (2q[m-1]_q + |(N+1)[m]_q - (M+1)|) |a_m| \\
\leq \left( \frac{2q + (N+1)}{1-q} + (M+1) \right) \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} \frac{N+3}{1-q} \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|q^m}{(\gamma)_{m-1}(m-1)!} \\
= \left( \frac{2q + (N+1)}{1-q} + (M+1) \right) \sum_{m=1}^{\infty} \frac{|(\xi)_m|}{(\gamma)_m m!} \frac{N+3}{1-q} \sum_{m=1}^{\infty} \frac{|(\xi)_m|q^m}{(\gamma)_m m!}.
\]

Since \( |(a)_m| \leq |(a)|_m \), from (12), we deduce that

\[
\sum_{m=2}^{\infty} (2q[m-1]_q + |(N+1)[m]_q - (M+1)|) |a_m| \\
\leq \left( \frac{2q + (N+1)}{1-q} + (M+1) \right) \sum_{m=1}^{\infty} \frac{|(\xi)_m|}{(\gamma)_m m!} \frac{N+3}{1-q} \sum_{m=1}^{\infty} \frac{|(\xi)_m|q^m}{(\gamma)_m m!} \\
= \frac{1}{1-q} \left\{ (q + N + 2 + M(1-q)) \left( F(|\xi|; \gamma; 1) - 1 \right) \\
- (N+3)qF(|\xi|; \gamma; q) - (M + N + 2)(1-q) \right\} =: E_2(\xi, \eta, \gamma, q).
\]

and the assumption of the theorem implies (9), that is \( zF(\xi; \gamma; z) \in S_q^*[M, N] \).

For the special case \( M = 1 - 2\beta \), \( 0 \leq \beta < 1 \), and \( N = -1 \), we have \( S_q^*[1 - 2\beta, -1] =: S_q^*(\beta) \) and Theorem 2.1 reduces to the following result:

**Corollary 2.1.** Let \( E_j^*, j \in \{1, 2\} \), be defined as follows:

(i) If \( \xi > 0 \) and \( \gamma > 0 \), then \( E_1^* \) is given by

\[
E_1^*(\xi, \eta, \gamma, q) := \frac{1}{1-q} \left\{ 2(1 - \beta(1-q)) F(\xi; \gamma; 1) \\
- 2qF(\xi; \gamma; q) - 2(1 - \beta)(1-q) \right\}.
\]

(ii) If \( \xi \in \mathbb{C} \setminus \{0\} \) and \( \gamma > 0 \), then \( E_2^* \) is given by

\[
E_2^*(\xi, \eta, \gamma, q) := \frac{1}{1-q} \left\{ 2(1 - \beta(1-q)) F(|\xi|; \gamma; 1) \\
- 2qF(|\xi|; \gamma; q) - 2(1 - \beta)(1-q) \right\}.
\]

If for any \( j \in \{1, 2\} \) the inequality

\[
E_j^*(\xi, \gamma, q) < 2(1 - \beta)
\]

holds for \( 0 \leq \beta < 1 \), then function \( zF(\xi; \gamma; z) \) belongs to the class \( S_q^*(\beta) \).
For $\beta = 0$ the above corollary gives us the next special case:

**Example 2.1.** Let $\tilde{E}_j$, $j \in \{1, 2\}$, be defined as follows:

(i) If $\xi > 0$ and $\gamma > 0$, then $\tilde{E}_1$ is given by

$$
\tilde{E}_1(\xi, \gamma, q) = \frac{1}{1-q} \{2F(\xi; \gamma; 1) - 2qF(\xi; \gamma; q) - 2(1-q)\}.
$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $\tilde{E}_2$ is given by

$$
\tilde{E}_2(\xi, \gamma, q) := \frac{1}{1-q} \{2F(|\xi|; \gamma; 1) - 2qF(|\xi|; \gamma; q) - 2(1-q)\}.
$$

If for any $j \in \{1, 2\}$ the inequality

$$
\tilde{E}_j(\xi, \gamma, q) < 2
$$

holds, then function $zF(\xi; \gamma; z)$ belongs to the class $S^*_q(0)$.

**Theorem 2.2.** Let $G_j$, $j \in \{1, 2\}$, be defined as follows:

(i) If $\xi > 0$ and $\gamma > 0$, then $G_1$ is given by

$$
G_1(\xi, \gamma, q) := \frac{1}{(1-q)^2} \left\{ (N+2+q+M(1-q))F(\xi; \gamma; 1) - (M(1-q) + 2N + 5 + q)qF(\xi; \gamma; q) + (N + 3)q^2F(\xi; \gamma; q^2) - (M + N + 2)(1-q)^2 \right\}.
$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $G_2$ is given by

$$
G_2(\xi, \gamma, q) := \frac{1}{(1-q)^2} \left\{ (N+2+q+M(1-q))F(|\xi|; \gamma; 1) - (M(1-q) + 2N + 5 + q)qF(|\xi|; \gamma; q) + (N + 3)q^2F(|\xi|; \gamma; q^2) - (M + N + 2)(1-q)^2 \right\}.
$$

If for any $j \in \{1, 2\}$ the inequality

$$
G_j(\xi, \gamma, q) < |N - M|
$$

holds, then function $zF(\xi; \gamma; z)$ belongs to the class $C^*_q[M, N]$.

**Proof.** Since, according to Lemma 1.2 any function $f \in A$ belongs to the class $C^*_q[M, N]$ if it satisfies the inequality (10) for

$$
a_m = \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}, \quad \text{and} \quad [m]_q = \frac{1 - q^m}{1 - q}.
$$
Using first the triangle’s inequality, we have

$$\sum_{m=2}^{\infty} |m|_q (2q|m-1|_q + |(N+1)|m_q - (M+1)) |a_m|$$

$$\leq \sum_{m=2}^{\infty} 2q|m|_q |m|_q |a_m| + \sum_{m=2}^{\infty} (N+1)|m|_q |m|_q |a_m| + \sum_{m=2}^{\infty} (M+1)|m|_q |a_m|$$

$$= \sum_{m=2}^{\infty} 2q \frac{1-q^m}{1-q} |a_m| + \sum_{m=2}^{\infty} (N+1) \frac{1-q^m}{1-q} |a_m|$$

$$+ \sum_{m=2}^{\infty} (M+1) \frac{1-q^m}{1-q} |a_m|$$

$$= \sum_{m=2}^{\infty} \left( 2q + (N+1) + (M+1)(1-q) \right) (1-q)^2 |a_m|$$

$$- \sum_{m=2}^{\infty} \left( (M+1)(1-q) + 2(N+1) + 2q + 2 \right) (1-q)^2 q^m |a_m| + \sum_{m=2}^{\infty} \left( \frac{2 + (N+1)}{(1-q)^2} \right) q^{2m} |a_m|$$

$$= \frac{q + N + 2 + M(1-q)}{(1-q)^2} \sum_{m=2}^{\infty} |a_m| - \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} \sum_{m=2}^{\infty} q^m |a_m|$$

$$+ \frac{N + 3}{(1-q)^2} \sum_{m=2}^{\infty} q^{2m} |a_m|.$$  \hspace{1cm} (13)

**Case (i)** If $\xi > 0$ and $\gamma > 0$, from (13) we obtain

$$\sum_{m=2}^{\infty} |m|_q (2q|m-1|_q + |(N+1)|m_q - (M+1)) |a_m|$$

$$\leq \frac{q + N + 2 + M(1-q)}{(1-q)^2} \sum_{m=2}^{\infty} \frac{\langle \xi \rangle_{m-1}}{\langle \gamma \rangle_{m-1} (m-1)!} + \frac{N + 3}{(1-q)^2} \sum_{m=2}^{\infty} \frac{\langle \xi \rangle_{m-1}}{\langle \gamma \rangle_{m-1} (m-1)!} q^{2m}$$

$$- \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} \sum_{m=2}^{\infty} \frac{\langle \xi \rangle_{m-1}}{\langle \gamma \rangle_{m-1} (m-1)!} q^m$$

$$= \frac{q + N + 2 + M(1-q)}{(1-q)^2} \sum_{m=1}^{\infty} \frac{\langle \xi \rangle}{\langle \gamma \rangle m!} + \frac{N + 3}{(1-q)^2} \sum_{m=1}^{\infty} \frac{\langle \xi \rangle}{\langle \gamma \rangle m!} q^{2m}$$

$$- \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} q \sum_{m=1}^{\infty} \frac{\langle \xi \rangle}{\langle \gamma \rangle m!} q^m$$

$$= \frac{q + N + 2 + M(1-q)}{(1-q)^2} (F(\xi; \gamma; 1) - 1) + \frac{(N + 3)q^2}{(1-q)^2} (F(\xi; \gamma; q) - 1)$$

$$- \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} q(F(\xi; \gamma; q) - 1)$$

$$= \frac{1}{(1-q)^2} \left\{ (N + 2 + q + M(1-q)) F(\xi; \gamma; 1) \right.$$}

$$- (M(1-q) + 2N + 5 + q) q F(\xi; \gamma; q) + (N + 3)q^2 F(\xi; \gamma; q^2)$$

$$- (M + N + 2)(1-q)^2 \right\} =: G_1(\xi, \gamma, q).$$
Therefore, the assumption of the theorem implies (10), hence \( zF(\xi; \gamma; z) \in C_q^*[M, N] \).

Case (ii) If \( \xi \in \mathbb{C} \setminus \{0\} \), \( \gamma > 0 \), then the inequality (13) leads to

\[
\sum_{m=2}^{\infty} [m]_q (2q[m-1]_q + [(N+1)[m]_q - (M+1)]) |a_m| \\
\leq \frac{q + N + 2 + M(1-q)}{(1-q)^2} \sum_{m=1}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} + \frac{N + 3}{(1-q)^2} \sum_{m=1}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} q^{2m} \\
- \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} q^m \\
= \frac{q + N + 2 + M(1-q)}{(1-q)^2} (F(|\xi|; \gamma, 1) - 1) + \frac{(N + 3)q^2}{(1-q)^2} (F(|\xi|; \gamma; q^2) - 1) \\
- \frac{M(1-q) + 2N + 5 + q}{(1-q)^2} qF(|\xi|; \gamma; q) \\
= \frac{1}{(1-q)^2} \left\{ (N + 2 + q + M(1-q))F(|\xi|; \gamma; 1) + (N + 3)q^2F(|\xi|; \gamma; q^2) \\
- (M(1-q) + 2N + 5 + q)qF(|\xi|; \gamma; q) - (M + N + 2)(1-q)^2 \right\} =: G_2(\xi, \gamma, q).
\]

It follows that the assumption of the theorem implies (10), hence \( zF(\xi; \gamma; z) \in C_q^*[M, N] \). \( \square \)

For the special case \( M = 1 - 2\beta \), \( 0 \leq \beta < 1 \) and \( N = -1 \), we have \( C_q^*[1 - 2\beta, -1] =: C_q^*(\beta) \), and Theorem 2.2 reduces to the following result:

**Corollary 2.2.** Let \( G_j^* \), \( j \in \{1, 2\} \), be defined as follows:

(i) If \( \xi > 0 \) and \( \gamma > 0 \), then \( G_1^* \) is given by

\[
G_1^*(\xi, \eta, \gamma, q) := \frac{1}{(1-q)^2} \left\{ 2(1 - \beta(1-q))F(\xi; \gamma; 1) \\
- 2(2 - \beta(1-q))qF(\xi; \gamma; q) + 2q^2F(\xi; \gamma; q^2) - 2(1 - \beta)(1-q)^2 \right\}.
\]
(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $G^*_2$ is given by

$$
G^*_2(\xi, \gamma, q) := \frac{1}{(1-q)^2} \left\{ 2(1-\beta(1-q)) F(|\xi|; \gamma; 1) \\
- 2(2-\beta(1-q))q F(|\xi|; \gamma; q) + 2q^2 F(|\xi|; \gamma; q^2) \\
- 2(1-\beta)(1-q)^2 \right\}.
$$

If for any $j \in \{1, 2\}$ the inequality

$$
G^*_j(\xi, \gamma, q) < 2(1-\beta)
$$

holds for $0 \leq \beta < 1$, then function $z F(\xi; \gamma; z)$ belongs to the class $C^*_q(\beta)$.

For $\beta = 0$ the above corollary gives us the next example:

**Example 2.2.** Let $\tilde{G}_j$, $j \in \{1, 2\}$, be defined as follows:

(i) If $\xi > 0$ and $\gamma > 0$, then $\tilde{G}_1$ is given by

$$
\tilde{G}_1(\xi, \gamma, q) := \frac{1}{(1-q)^2} \left\{ 2 F(\xi; \gamma; 1) - 4q F(\xi; \gamma; q) \\
+ 2q^2 F(\xi; \gamma; q^2) - 2(1-q)^2 \right\}.
$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $\tilde{G}_2$ is given by

$$
\tilde{G}_2(\xi, \gamma, q) := \frac{1}{(1-q)^2} \left\{ 2 F(|\xi|; \gamma; 1) - 4q F(|\xi|; \gamma; q) \\
+ 2q^2 F(|\xi|; \gamma; q^2) - 2(1-q)^2 \right\}.
$$

If for any $j \in \{1, 2\}$ the inequality

$$
\tilde{G}_j(\xi, \gamma, q) < 2
$$

holds, then function $z F(\xi; \gamma; z)$ belongs to the class $C^*_q(0)$.

**References**


