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Sufficient condition for q-starlike and q-convex functions associated with generalized confluent hypergeometric function

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Abstract

The main object of this paper is to investigate and determine a sufficient condition for q-starlike and q-convex functions which are associated with generalized confluent hypergeometric function.

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1. Introduction and Preliminary

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \ z \in \mathbb{U}, \tag{1}$$

which are analytic in open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ be the subclass of A consists of univalent functions in $\mathbb{U}$. Further suppose that $\mathcal{S}^*$ is subclass of functions of A which are starlike in $\mathbb{U}$, that is f satisfy the subsequent conditions:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad \forall z \in \mathbb{U} \tag{2}$$

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Let $\mathcal{C}^*$ is subclass of functions of A which are convex in $\mathbb{U}$, that is f satisfy the following conditions:

$$Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} = Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \qquad \forall z \in \mathbb{U} \tag{3}$$

For analytic functions f and g in $\mathbb{U}$ we say that the function f is subordinate to the function g and written as

$$f(z) \prec g(z)$$

If there exists a Schwarz function w which is analytic in $\mathbb{U}$ and $w(0) = 0$, $|w(z)| < 1$, such that $f(z) = g(w(z))$ Further, if g is the function which is univalent in $\mathbb{U}$, then it becomes

$$f(z) \prec g(z);\ z \in \mathbb{U} \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U})$$

Now we define P the class of analytic function with positive real part which is given as

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m;\ Re(p(z)) > 0,\ z \in \mathbb{U}.$$

Definition 1.1. *An analytic function h with $h(0) = 1$ belongs to the class $P[M, N]$, with $-1 \leq N < M \leq 1$, if and only of*

$$h(z) \prec \frac{1+Mz}{1+Nz}.$$

The class $P[M, N]$ of analytic functions was introduced and studied by Janowski [6], who showed that $h \in P[M, N]$ if and only if there exists a function $p \in P$, such that

$$h(z) = \frac{(M+1)p(z)-(M-1)}{(N+1)p(z)-(N-1)},\ z \in \mathbb{U}.$$

Definition 1.2. *[6] (i) A function $f \in A$ is in the class $\mathcal{S}^*[M, N]$, with $-1 \leq N < M \leq 1$, if and only if*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Mz}{1+Nz}. \tag{4}$$

(ii) A function $f \in A$ is in the class $\mathcal{C}^*[M, N]$, with $-1 \leq N < M \leq 1$, if and only if

$$1+\frac{f''(z)}{f'(z)} \prec \frac{1+Mz}{1+Nz}.$$

Definition 1.3. *The q-number $[m]_q$ defined in [15] for $q \in (0, 1)$, is given by*

$$[m]_q := \begin{cases} \dfrac{1-q^m}{1-q}, & \text{if} \quad m \in \mathbb{C}, \\ \sum\limits_{k=0}^{m-1} q^k = 1+q+q^2+\cdots+q^{m-1}, & \text{if} \quad m \in \mathbb{N} := \{1, 2, \dots\}. \end{cases}$$

Definition 1.4. *[15] The q-derivative $D_q f$ of a function f is defined as*

$$D_q f(z) := \begin{cases} \dfrac{f(z)-f(qz)}{(1-q)z}, & \text{if} \quad z \in \mathbb{C}\setminus\{0\}, \\ f'(0), & \text{if} \quad z = 0, \end{cases}$$

provided that $f'(0)$ exists, and $0 < q < 1$.

From the Definition 1.4 it follows immediately that

$$\lim_{q\to 1} D_q f(z) = \lim_{q\to 1} \frac{f(z)-f(qz)}{(1-q)z} = f'(z).$$

For a function $f \in A$ which has the power expansion series of the form (1.1), it is easy to check that

$$D_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1},\ z \in \mathbb{U},$$

as it was previously defined by Srivastava and Bansal [14], although the q-derivative operator D_q was presumably first applied by Ismail et. al. [5] to study a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$ (see [5], [3], [13]).

Definition 1.5. *A function $f \in A$ is in the class $\mathcal{S}_q^*$ if and and only if*

$$\left| \frac{z}{f(z)} D_q f(z) - \frac{1}{1-q} \right| < \frac{1}{1-q},\ z \in \mathbb{U}. \tag{5}$$

It is observed that, as $q \to 1^-$ the closed disk

$$\left| w - \frac{1}{1-q} \right| < \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ of q-starlike functions diminishes to the acquainted class $\mathcal{S}^*$. Consistently, by with the principle of subordination among analytic functions, we can rewrite the inequality (5) as

$$\frac{z}{f(z)} D_q f(z) \prec \frac{1+z}{1-qz}. \tag{6}$$

One way to generalize the class $\mathcal{S}^*[M,N]$ of Definition 1.2 is to replace in (4) the function $(1+Mz)/(1+Nz)$ by the function $(1+z)/(1-qz)$ which is involved in (6). The appropriate definition of the corresponding q-extension $\mathcal{S}_q^*[M,N]$ is specified below.

Definition 1.6. *A function $f \in A$ is said to be in the class $\mathcal{S}_q^*[M,N]$ if and only if*

$$\frac{zD_q f(z)}{f(z)} = \frac{(M+1)Q(z)-(M-1)}{(N+1)Q(z)-(N-1)},\ z \in \mathbb{U}, \tag{7}$$

where

$$Q(z) = \frac{1+z}{1-qz}$$

which by using the definition of the subordination can be written as follows:

$$\frac{zD_q f(z)}{f(z)} \prec \phi(z),$$

where

$$\phi(z) := \frac{(M+1)z+2+(M-1)qz}{(N+1)z+2+(N-1)qz},\ -1 \le N < M \le 1,\ q \in (0,1).$$

Remark 1.1. *(i) It is easy to check that*

$$\lim_{q\to 1^-} \mathcal{S}_q^*[M,N] = \mathcal{S}^*[M,N].$$

Also, $\mathcal{S}_q^*[1,-1] =: \mathcal{S}_q^*$, where $\mathcal{S}_q^*$ is the class of functions introduced and studied by Ismail et. al [5].

(ii) If w is a Schwarz function, from the definition of Q we get that

$$\left|Q(z) - \frac{1}{1-q}\right| < \frac{1}{1-q}, \ z \in \mathbb{U},$$

and from (7) *it follows*

$$Q(z) = \frac{(N-1)\frac{zD_q f(z)}{f(z)} - (M-1)}{(N+1)\frac{zD_q f(z)}{f(z)} - (M+1)}, \ z \in \mathbb{U}.$$

From these computations we conclude that a function $f \in A$ is in the class $\mathcal{S}_q^*[M,N]$, if and only if

$$\left|\frac{(N-1)\frac{zD_q f(z)}{f(z)} - (M-1)}{(N+1)\frac{zD_q f(z)}{f(z)} - (M+1)} - \frac{1}{1-q}\right| < \frac{1}{1-q}, \ z \in \mathbb{U}.$$

In itâĂŹs special case when $M = 1-2\beta$ and $N = -1$, with $0 \leq \beta < 1$, the function class $\mathcal{S}_q^*[M,N]$ reduces to the function class $\mathcal{S}_q^*(\beta)$ which was presented and deliberated by Agrawal and Sahoo [1].

(iii) By means of the well-known AlexanderâĂŹs theorem [2], the class $\mathcal{C}_q^*[M,N]$ of q-convex functions can be defined in the following way:

$$f \in \mathcal{C}_q^*[M,N] \Leftrightarrow zD_q f(z) \in \mathcal{S}_q^*[M,N].$$

The confluent hypergeometric function in the series form is given by

$$F(\xi;\eta;z) = \sum_{m=0}^{\infty} \frac{(\xi)_m z^m}{(\eta)_m m!}; \ \forall z \in \mathbb{C},$$

where η is neither zero nor a negative integer and the series is convergent for ξ, η. Now the generalized confluent hypergeometric function (normalized function) is defined as

$$\begin{aligned} zF(\xi;\eta;z) &= \sum_{m=0}^{\infty} \frac{(\xi)_m z^{m+1}}{(\eta)_m m!} \ \text{(By using convolution of two functions)} \\ &= z + \sum_{m=2}^{\infty} \frac{(\xi)_{m-1} z^m}{(\eta)_{m-1}(m-1)!}, \end{aligned} \tag{8}$$

where $(\beta)_m$ is the Pochhammer symbol defined as

$$\begin{aligned} (\beta)_m &= \begin{cases} 1 & \text{if } m = 0 \\ \beta(\beta+1)(\beta+2)\ldots(\beta+m-1) & \text{if } m \in \mathbb{N} \end{cases} \\ &\equiv \frac{\Gamma(\beta+m)}{\Gamma\beta} \end{aligned}$$

and

$$(\beta)_{m+k} = (\beta)_m(\beta+m)_k = (\beta)_k(\beta+k)_m.$$

In this paper we determine sufficient conditions for q-starlike functions and q-convex functions associated with confluent hypergeometric function by using following sufficient conditions obtained by Srivastava [15]:

Lemma 1.1. *[15] A function $f \in A$ is in the class $\mathcal{S}_q^*[M,N]$, if it satisfying the following condition*

$$\sum_{m=2}^{\infty} (2q[m-1]_q + |(N+1)[m]_q - (M+1)\,|)|a_m| < |N-M\,| \tag{9}$$

Lemma 1.2. *[15] A function $f \in A$ is in the class $\mathcal{C}_q^*[M,N]$, if it satisfying the following condition*

$$\sum_{m=2}^{\infty} [m]_q (2q[m-1]_q + |(N+1)[m]_q - (M+1)\,|)|a_m| < |N-M\,| \tag{10}$$

2. Main Results

Theorem 2.1. *Let E_j, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then E_1 is given by

$$E_1(\xi,\gamma,q) := \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)F(\xi;\gamma;1) - (N+3)qF(\xi;\gamma;q) - (M+N+2)(1-q)\Big\}.$$

(ii) If $\xi, \in \mathbb{C}\setminus\{0\}$ and $\gamma > 0$, then E_2 is given by

$$E_2(\xi,\gamma,q) := \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)F(|\xi|;\gamma;1) - (N+3)qF(|\xi|;\gamma;q) - (M+N+2)(1-q)\Big\}.$$

If for any $j \in \{1,2\}$ the inequality

$$E_j(\xi,\gamma,q) < |N-M|$$

holds, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{S}_q^*[M,N]$.

Proof. Since

$$zF(\xi;\gamma;z) = z + \sum_{m=2}^{\infty}\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}z^m,\ z\in\mathbb{U},$$

according to Lemma 1.1, any function $f \in A$ is in the class $\mathcal{S}_q^*[M,N]$ if it satisfies the inequality (9). Then, for $f(z) := zF(\xi;\gamma;z)$ it is sufficient to show that (9) holds, for

$$a_m = \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}, \quad \text{and} \quad [m]_q = \frac{1-q^m}{1-q}.$$

Using the triangle's inequality we get

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m| \\
&\le \sum_{m=2}^{\infty}2q\frac{1-q^{m-1}}{1-q}|a_m| + \sum_{m=2}^{\infty}(N+1)\frac{1-q^m}{1-q}|a_m| + \sum_{m=2}^{\infty}(M+1)|a_m| \\
&= \sum_{m=2}^{\infty}\left(\frac{2q+(N+1)}{1-q}+(M+1)\right)|a_m| - \sum_{m=2}^{\infty}\frac{(N+3)q^m}{1-q}|a_m|.
\end{aligned} \tag{11}$$

Case (i) If $\xi > 0$ and $\gamma > 0$, from (11) we get

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m| \\
&\le \left(\frac{2q+(N+1)}{1-q}+(M+1)\right)\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!} - \frac{N+3}{1-q}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!} \\
&= \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)\big(F(\xi;\gamma;1)-1\big) - (N+3)q\big(F(\xi,\eta;\gamma;q)-1\big)\Big\} \\
&= \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)F(\xi;\gamma;1) - (N+3)qF(\xi;\gamma;q) \\
&- (M+N+2)(1-q)\Big\} =: E_1(\xi,\gamma,q),
\end{aligned}$$

and the assumption of the theorem implies (9), that is $zF(\xi;\gamma;z) \in \mathcal{S}_q^*[M,N]$.
Case (ii) If $\xi \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$, from (11) we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m| \\
&\le \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right| - \frac{N+3}{1-q}\sum_{m=2}^{\infty}\left|\frac{(\xi)_{m-1}q^m}{(\gamma)_{m-1}(m-1)!}\right| \\
&= \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=1}^{\infty}\frac{|(\xi)_m|}{(\gamma)_m m!} - \frac{N+3}{1-q}q\sum_{m=1}^{\infty}\frac{|(\xi)_m|q^m}{(\gamma)_m m!}. \qquad (12)
\end{aligned}$$

Since $|(a)_n| \le (|a|)_n$, from (12), we deduce that

$$\begin{aligned}
&\sum_{m=2}^{\infty}(2q[m-1]_q + |(N+1)[m]_q - (M+1)|)|a_m| \\
&\le \left(\frac{2q+(N+1)}{1-q} + (M+1)\right)\sum_{m=1}^{\infty}\frac{(|\xi|)_m}{(\gamma)_m m!} - \frac{(N+3)q}{1-q}\sum_{m=1}^{\infty}\frac{(|\xi|)_m q^m}{(\gamma)_m m!} \\
&= \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)\big(F(|\xi|;\gamma;1)-1\big) \\
&\quad - (N+3)q\big(F(|\xi|;\gamma;q)-1\big)\Big\} \\
&= \frac{1}{1-q}\Big\{\big(q+N+2+M(1-q)\big)F(|\xi|;\gamma;1) \\
&\quad - (N+3)qF(|\xi|;\gamma;q) - (M+N+2)(1-q)\Big\} =: E_2(\xi,\eta,\gamma,q).
\end{aligned}$$

and the assumption of the theorem implies (9), that is $zF(\xi;\gamma;z) \in \mathcal{S}_q^*[M,N]$. □

For the special case $M = 1-2\beta$, $0 \le \beta < 1$, and $N = -1$, we have $\mathcal{S}_q^*[1-2\beta,-1] =: \mathcal{S}_q^*(\beta)$ and Theorem 2.1 reduces to the following result:

Corollary 2.1. *Let E_j^*, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then E_1^* is given by

$$\begin{aligned}
E_1^*(\xi,\eta,\gamma,q) := \frac{1}{1-q}\Big\{&2\big(1-\beta(1-q)\big)F(\xi;\gamma;1) \\
&- 2qF(\xi;\gamma;q) - 2(1-\beta)(1-q)\Big\}.
\end{aligned}$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then E_2^* is given by

$$\begin{aligned}
E_2^*(\xi,\eta,\gamma,q) := \frac{1}{1-q}\Big\{&2\big(1-\beta(1-q)\big)F(|\xi|;\gamma;1) \\
&- 2qF(|\xi|;\gamma;q) - 2(1-\beta)(1-q)\Big\}.
\end{aligned}$$

If for any $j \in \{1,2\}$ the inequality

$$E_j^*(\xi,\gamma,q) < 2(1-\beta)$$

holds for $0 \le \beta < 1$, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{S}_q^*(\beta)$.

For $\beta = 0$ the above corollary gives us the next special case:

Example 2.1. *Let $\widetilde{E}_j$, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then $\widetilde{E}_1$ is given by

$$\widetilde{E}_1(\xi,\gamma,q) = \frac{1}{1-q}\left\{2F(\xi;\gamma;1) - 2qF(\xi;\gamma;q) - 2(1-q)\right\}.$$

(ii) If $\xi \in \mathbb{C}\setminus\{0\}$ and $\gamma > 0$, then $\widetilde{E}_2$ is given by

$$\widetilde{E}_2(\xi,\gamma,q) := \frac{1}{1-q}\left\{2F(|\xi|;\gamma;1) - 2qF(|\xi|;\gamma;q) - 2(1-q)\right\}.$$

If for any $j \in \{1,2\}$ the inequality

$$\widetilde{E}_j(\xi,\gamma,q) < 2$$

holds, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{S}_q^*(0)$.

Theorem 2.2. *Let G_j, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then G_1 is given by

$$\begin{aligned} G_1(\xi,\gamma,q) := \frac{1}{(1-q)^2}\Big\{&\big(N+2+q+M(1-q)\big)F(\xi;\gamma;1) \\ &- \big(M(1-q)+2N+5+q\big)qF(\xi;\gamma;q) + (N+3)q^2F(\xi;\gamma;q^2) \\ &- (M+N+2)(1-q)^2\Big\}. \end{aligned}$$

(ii) If $\xi \in \mathbb{C}\setminus\{0\}$ and $\gamma > 0$, then G_2 is given by

$$\begin{aligned} G_2(\xi,\gamma,q) := \frac{1}{(1-q)^2}\Big\{&\big(N+2+q+M(1-q)\big)F(|\xi|;\gamma;1) \\ &- \big(M(1-q)+2N+5+q\big)qF(|\xi|;\gamma;q) + (N+3)q^2F(|\xi|;\gamma;q^2) \\ &- (M+N+2)(1-q)^2\Big\}. \end{aligned}$$

If for any $j \in \{1,2\}$ the inequality

$$G_j(\xi,\gamma,q) < |N-M|$$

holds, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{C}_q^*[M,N]$.

Proof. Since, according to Lemma 1.2 any function $f \in A$ belongs to the class $\mathcal{C}_q^*[M,N]$ if it satisfies the inequality (10) for

$$a_m = \frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}, \quad \text{and} \quad [m]_q = \frac{1-q^m}{1-q}.$$

Using first the triangle's inequality, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\big(2q[m-1]_q+\big|(N+1)[m]_q-(M+1)\big|\big)|a_m| \\
&\le \sum_{m=2}^{\infty} 2q[m]_q[m-1]_q|a_m| + \sum_{m=2}^{\infty}(N+1)[m]_q|[m]_q|a_m| + \sum_{m=2}^{\infty}(M+1)[m]_q|a_m| \\
&= \sum_{m=2}^{\infty} 2q\,\frac{1-q^m}{1-q}\,\frac{1-q^{m-1}}{1-q}\,|a_m| + \sum_{m=2}^{\infty}(N+1)\,\frac{1-q^m}{1-q}\,\frac{1-q^m}{1-q}\,|a_m| \\
&+ \sum_{m=2}^{\infty}(M+1)\,\frac{1-q^m}{1-q}\,|a_m| \\
&= \sum_{m=2}^{\infty}\left(\frac{2q+(N+1)+(M+1)(1-q)}{(1-q)^2}\right)|a_m| \\
&- \sum_{m=2}^{\infty}\left(\frac{(M+1)(1-q)+2(N+1)+2q+2}{(1-q)^2}\right)q^m|a_m| + \sum_{m=2}^{\infty}\left(\frac{2+(N+1)}{(1-q)^2}\right)q^{2m}|a_m| \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}|a_m| - \frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}q^m|a_m| \\
&+ \frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}q^{2m}|a_m|.
\end{aligned} \tag{13}$$

Case (i) If $\xi>0$ and $\gamma>0$, from (13) we obtain

$$\begin{aligned}
&\sum_{m=2}^{\infty}[m]_q\big(2q[m-1]_q+\big|(N+1)[m]_q-(M+1)\big|\big)|a_m| \\
&\le \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!} + \frac{N+3}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^{2m} \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2}\sum_{m=2}^{\infty}\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}q^m \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2}\sum_{m=1}^{\infty}\frac{(\xi)_m}{(\gamma)_m m!} + \frac{N+3}{(1-q)^2}q^2\sum_{m=1}^{\infty}\frac{(\xi)_m}{(\gamma)_m m!}q^{2m} \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2}q\sum_{m=1}^{\infty}\frac{(\xi)_m}{(\gamma)_m m!}q^m \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2}(F(\xi;\gamma;1)-1) + \frac{(N+3)q^2}{(1-q)^2}(F(\xi;\gamma;q^2)-1) \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2}q(F(\xi;\gamma;q)-1)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(1-q)^2}\Big\{\big(N+2+q+M(1-q)\big)F(\xi;\gamma;1) \\
&- \big(M(1-q)+2N+5+q\big)qF(\xi;\gamma;q) + (N+3)q^2F(\xi;\gamma;q^2) \\
&- (M+N+2)(1-q)^2\Big\} =: G_1(\xi,\gamma,q).
\end{aligned}$$

Therefore, the assumption of the theorem implies (10), hence $zF(\xi;\gamma;z) \in \mathcal{C}_q^*[M,N]$.
Case (ii) If $\xi \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$, then the inequality (13) leads to

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q \big(2q[m-1]_q + \big|(N+1)[m]_q - (M+1)\big|\big)|a_m| \\
&\leq \frac{q+N+2+M(1-q)}{(1-q)^2} \sum_{m=2}^{\infty} \left|\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!}\right| + \frac{N+3}{(1-q)^2} \sum_{m=2}^{\infty} \left|\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!} q^{2m}\right| \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2} \sum_{m=2}^{\infty} \left|\frac{(\xi)_{m-1}}{(\gamma)_{m-1}(m-1)!} q^{m}\right| \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2} \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} + \frac{N+3}{(1-q)^2} \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} q^{2m} \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2} \sum_{m=2}^{\infty} \frac{|(\xi)_{m-1}|}{(\gamma)_{m-1}(m-1)!} q^{m}.
\end{aligned}$$

Since $(a)_n| \leq (|a|)_n$, the above inequality implies

$$\begin{aligned}
&\sum_{m=2}^{\infty} [m]_q \big(2q[m-1]_q + \big|(N+1)[m]_q - (M+1)\big|\big)|a_m| \\
&\leq \frac{q+N+2+M(1-q)}{(1-q)^2} \sum_{m=1}^{\infty} \frac{(|\xi|)_m}{(\gamma)_m m!} + \frac{N+3}{(1-q)^2} q^2 \sum_{m=1}^{\infty} \frac{(|\xi|)_m}{(\gamma)_m m!} q^{2m} \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2} q \sum_{m=1}^{\infty} \frac{(|\xi|)_m}{(\gamma)_m m!} q^{m} \\
&= \frac{q+N+2+M(1-q)}{(1-q)^2} (F(|\xi|;\gamma,1)-1) + \frac{(N+3)q^2}{(1-q)^2} (F(|\xi|;\gamma;q^2)-1) \\
&- \frac{M(1-q)+2N+5+q}{(1-q)^2} q(F(|\xi|;\gamma;q)-1) \\
&= \frac{1}{(1-q)^2} \Big\{ \big(N+2+q+M(1-q)\big) F(|\xi|;\gamma;1) + (N+3)q^2 F(|\xi|;\gamma;q^2) \\
&- \big(M(1-q)+2N+5+q\big) q F(|\xi|;\gamma;q) - (M+N+2)(1-q)^2 \Big\} =: G_2(\xi,\gamma,q).
\end{aligned}$$

It follows that the assumption of the theorem implies (10), hence $zF(\xi;\gamma;z) \in \mathcal{C}_q^*[M,N]$. □

For the special case $M = 1-2\beta$, $0 \leq \beta < 1$ and $N = -1$, we have $\mathcal{C}_q^*[1-2\beta,-1] =: \mathcal{C}_q^*(\beta)$, and Theorem 2.2 reduces to the following result:

Corollary 2.2. *Let G_j^*, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then G_1^* is given by

$$\begin{aligned}
G_1^*(\xi,\eta,\gamma,q) := \frac{1}{(1-q)^2} \Big\{ &2\big(1-\beta(1-q)\big) F(\xi;\gamma;1) \\
&- 2\big(2-\beta(1-q)\big) q F(\xi;\gamma;q) + 2q^2 F(\xi;\gamma;q^2) - 2(1-\beta)(1-q)^2 \Big\}.
\end{aligned}$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then G_2^* is given by

$$G_2^*(\xi,\gamma,q) := \frac{1}{(1-q)^2}\Big\{2\big(1-\beta(1-q)\big)F(|\xi|;\gamma;1) - 2\big(2-\beta(1-q)\big)qF(|\xi|;\gamma;q) + 2q^2F(|\xi|;\gamma;q^2) - 2(1-\beta)(1-q)^2\Big\}.$$

If for any $j \in \{1,2\}$ the inequality

$$G_j^*(\xi,\gamma,q) < 2(1-\beta)$$

holds for $0 \le \beta < 1$, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{C}_q^*(\beta)$.

For $\beta = 0$ the above corollary gives us the next example:

Example 2.2. *Let $\widetilde{G}_j$, $j \in \{1,2\}$, be defined as follows:*
(i) If $\xi > 0$ and $\gamma > 0$, then $\widetilde{G}_1$ is given by

$$\widetilde{G}_1(\xi,\gamma,q) := \frac{1}{(1-q)^2}\Big\{2F(\xi;\gamma;1) - 4qF(\xi;\gamma;q) + 2q^2F(\xi;\gamma;q^2) - 2(1-q)^2\Big\}.$$

(ii) If $\xi \in \mathbb{C} \setminus \{0\}$ and $\gamma > 0$, then $\widetilde{G}_2$ is given by

$$\widetilde{G}_2(\xi,\gamma,q) := \frac{1}{(1-q)^2}\Big\{2F(|\xi|;\gamma;1) - 4qF(|\xi|;\gamma;q) + 2q^2F(|\xi|;\gamma;q^2) - 2(1-q)^2\Big\}.$$

If for any $j \in \{1,2\}$ the inequality

$$\widetilde{G}_j(\xi,\gamma,q) < 2$$

holds, then function $zF(\xi;\gamma;z)$ belongs to the class $\mathcal{C}_q^*(0)$.

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