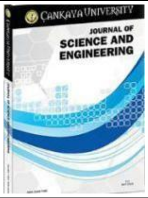
e-ISSN: 2564-7954 CUJSE 17(2): 108-117 (2020) Research Article



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A Comparative Performance Analysis of HLLC and AUSM+-up Riemann Solvers

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Keywords

Riemann Solvers
AUSM+-up
HLLC
Shock Tube Problem

Abstract

An in-depth comparative performance evaluation of the HLLC (Harten-Lax-van Leer-Contact) and the latest version of the AUSM (Advection Upstream Splitting Method), the AUSM+-up, numerical schemes is carried out with the help of the one-dimensional shock tube problem. The efficiency of schemes is assessed on the basis of the accuracy in capturing of the shock, contact discontinuity, and the expansion fan in the solution. Numerical schemes viz., the upwind difference, the Godunov, the MacCormack, and the basic AUSM scheme are also investigated for their performance while solving the same problem to do a wider comparison. Numerical results from each method are compared against the exact solution to the problem. The HLLC numerical scheme is found to be the most efficient followed by AUSM+-up, which is marginally inferior with respect to the shock capturing accuracy.

1. Introduction

The HLLC (Harten-Lax-van Leer-Contact) by Toro [1] and AUSM+-up by Chang & Liou [2] are advanced approximate-Riemann solvers and are widely used numerical schemes in solving hyperbolic conservation laws. The authors, in the present study, have carried out a detailed comparison on the performance of the two schemes with the help of the one-dimensional shock tube problem. The classical problem of the one-dimensional shock tube, or more commonly the Sod's shock tube problem [3] serve as one of the standard bench-mark problems in compressible fluid dynamics. The applicability, robustness, convergence, and other aspects of numerical schemes are easily brought out by the Sod's shock tube problem. It remains one of the most used benchmark problems to test the shock capturing capabilities, including resolutions of contact discontinuity and the expansion fan, of various Riemann solvers [4-9]. The Sod's shock tube problem is also an apt choice for the study as it has an analytical solution with which the numerical results can be directly compared. The present study mainly focuses on the HLLC and AUSM+-up Riemann solvers and the solvers like the Upwind difference, MacCormack, Godunov, and the Advection upstream splitting method are also used for comparing the performance on a wider scale. The section 2 presents the problem definition and governing equations for the test case problem. Section 3 details on the computational process and numerical schemes used in the study. The results of numerical experiments are further discussed and useful conclusions are drawn in the section 4.

2. Problem definition and Governing equations

The physical problem of the Sod's shock tube [1] involves the calculation of fluid properties in different zones formed, after the rupture of a diaphragm which separates two initial zones of gas at two different states inside a

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Received: August 28, 2020, Accepted: September 20, 2020

sealed tube. On the rupture of the diaphragm, the pressurized gas on the high-pressure side is expected to move towards low pressure side forming a compression front. In the meantime, an expansion fan propagates into the compressed region. The successive compression waves coalesce to form a shock which moves further right at a supersonic speed into the low-pressure side. Across the contact surface formed, the density and temperature are discontinuous. The waves traversing through the fluid results in the formation of different zones and the extent of each region varies with time. The motive is to predict the transient phenomena prior to shock/expansion wave reflection at the boundaries. The transient problem thus evolved for an ideal gas, with few waves to be captured and their relative effect on the fluid properties defines the one-dimensional shock-tube problem.

The fluid chosen for the problem is air which is treated as an ideal gas and the flow is assumed to be inviscid and non-heat conducting. A set of nonlinear hyperbolic conservation laws, the one-dimensional equations of gas dynamics forms the governing equation for the problem. In conservation form, as vector quantities, the system of governing equations is given by

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0 \tag{1}$$

where, U is the vector of conserved variables and F(U) is the flux vector given by

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho e\end{bmatrix} \quad \text{and} \quad F(U)=\begin{bmatrix}\rho u\\ \rho u^2+P\\ U(\rho e+P)\end{bmatrix}$$

where, ρ is the density, u the velocity, p the pressure, and e the total energy per unit mass of the fluid. The governing equations form a coupled nonlinear hyperbolic system. The system of equations can be solved analytically for the shock tube problem as the complete analytical solution procedure is available in [10].

3. Computational Method and Numerical Schemes

In the present study, numerical solution to the shock tube problem is obtained with the basic first-order upwind method, the standard MacCormack method, the first-order Godunov method, the basic AUSM method proposed by Liou & Steffen [11], the generalized AUSM+-up by Liou [12], and the HLLC method proposed by Toro [1]. Depending upon the numerical scheme employed, either a finite difference or a finite volume framework are used for the spatial discretization. Algorithm for each numerical scheme used in the study and their implementation are discussed briefly below.

3.1. The First-Order Upwind Scheme

The upwind scheme used by Sod in [3] with first order accuracy in space and time is an explicit single step method. An adaptive finite difference stencil is used to capture the physics of convection phenomena. The pressure term in the momentum equation is handled separately. The pressure term extracted into a separate flux term is provided with a more suitable central differencing as pressure is an isotropic property.

3.2. MacCormack Scheme

The MacCormack scheme is a second order accurate two-step predictor-corrector method. In the predictor step, provisional values of the conserved variables at an intermediate time level is obtained from the property values at the previous time level and the corresponding fluxes are evaluated. In the subsequent corrector step, the property values computed at the intermediate time level are used to obtain the corrected values of conserved variables and fluxes at the new updated time level. In [3], Sod employed MacCormack scheme with artificial viscosity to obtain the results. The introduction of artificial viscosity can provide smoothened solution profiles by controlling the oscillations observed in the solution if any. The present work uses the MacCormack algorithm excluding artificial viscosity in order to make a more sensible evaluation of the scheme. The scheme is implemented on a finite difference grid.

3.3. First-Order Godunov Scheme

Godunov scheme as given by Sod in [3] is first order accurate in space and time. The two-step algorithm for the scheme is implemented on a finite difference grid. In the first step, the conserved variables at the cell interfaces are calculated, which is used in computing the interface flux vector. Conserved variables at the main grid points for the new updated time level are calculated with this updated interfacial flux in the second step. As with the case of MacCormack scheme, Sod employed artificial viscosity with Godunov in obtaining the results. The present work, however, focuses on implementing the algorithm for the numerical scheme without the use of artificial viscosity.

3.4. The Advection Upstream Splitting Method

Advection Upstream Splitting Method (AUSM) by Liou & Steen [11] is a flux splitting method as the name suggests. This method has undergone many modifications as reported by the works [13-15], and complete evolution of this method is available in [16]. The method used in the study is the basic first-order accurate version. The inviscid flux vector F is split into two physically distinct parts, the convective part and the pressure part. The method takes into account the physical nature of convection and acoustic propagation process while treating them numerically. The convective flux term is treated as a quantity getting advected at the speed of velocity computed at the interface, whereas the pressure flux is governed by the acoustic speed in the medium. More details on the method including the complete algorithm is available in [11], where the authors focused only on steady state problems and suggested the limitation of AUSM in handling unsteady problems like the shock-tube problem. However, the basic AUSM scheme is also employed in this work in solving the Sod's problem. The scheme is implemented on a finite-volume stencil.

3.5. The AUSM+-up Method

The AUSM scheme has a notable drawback in the form of pressure oscillations along the grid direction with very small velocity component. This is primarily due to the lack of dissipative mechanism in the pressure field. The AUSM+-up is proposed by Chang & Liou in [2] and modified by Liou in [12], and by Chang & Liou in [17], mitigates this problem to some extent and is designed to provide high accuracy results at all flow speeds. As in the case of basic AUSM scheme, the inviscid flux vector is split into two physically distinct parts; the convective part and the pressure part. However, the flux terms are redefined in this method to suit for all flow speeds. The detailed algorithm of the AUSM+-up scheme with the complete set of relations is available in [2].

3.6. The HLLC Method

Harten, Lax, and van Leer in 1983 developed an approximate Riemann solver for computing flows with discontinuities, the HLL solver. The solver has a two-wave model which restricts its ability to capture the intermediate waves. Following the philosophy of HLL scheme, Toro et al. in 1992 developed a solver based on a three-wave model called the HLLC scheme [1,18,19]. The new scheme provides better resolution to the intermediate waves. The HLLC scheme initially computes the slowest and fastest signal speeds respectively, using which it assesses the middle wave speed. The middle wave speed further divides the central star-region of the Riemann solution profile into two sub star-regions, which are the left and right star regions. The flux computation follows an algorithm and the ultimate choice of flux at each cell interface is based on the sign of computed signal velocities. The HLLC flux is used with Godunov's first-order method for the computation. The procedure for computing and selecting the appropriate fluxes at the interfaces using the HLLC three-wave model is available in [1].

4. Results and Discussion

In this section, the results obtained with each of the six numerical schemes for the one-dimensional shock tube problem is discussed in detail. A comparative evaluation of the methods is carried out by assessing the behavior of solution profiles from each method against the exact solution to the problem. The solution profiles used for the analysis are those evaluated at 0.2 seconds since the rupture of diaphragm. None of the enlisted methods incorporated artificial viscosity or artificial compression into their numerical algorithm. The first three schemes viz., the Upwind, the MacCormack, and the Godunov schemes use finite difference stencils whereas a finite volume framework is used by the basic AUSM, AUSM+-up and HLLC schemes. Results are obtained over a wide range of grid resolution ranging from 100 to 3200 grid divisions.

Fig. 1 displays the velocity, density, pressure, and specific internal energy profiles of the fluid domain from each of the six numerical schemes, obtained with a resolution of 400 grid divisions. Property values are plotted against their respective location in the numerical domain. The first-order upwind is a naive procedure that does not impose the entropy condition to compute the correct wave speed. The results from the upwind scheme are displayed on the top row of Fig.1. It is observed that the upwind scheme induces diffusive errors near the contact discontinuity and the shock even for a resolution of 400 divisions. This is a typical observation for any first-order method. Near the expansion fan, however, the upwind method fails to capture the physics completely as an unexpected jump in the property values is observed. With grid refinement, the dissipative errors decrease to a good extent whereas are still high near the contact discontinuity. The refinement shows no improvement on the property jump reported across the expansion wave, which is still prominent. The non-enforcement of the applicable entropy conditions as detailed in [20, 21], is held responsible for the same. As the upwind method does not provide an entropy satisfying solution, the method and the results with scheme are excluded from further discussion. The remaining schemes with their results are examined thoroughly for their performance evaluation by concentrating on the behavior of the solution profiles near the key features such as the expansion fan, the contact discontinuity, and the shock.

The MacCormack scheme is second-order accurate in space and time and is implemented on a finite difference stencil. Artificial viscosity or compression is not attached to the algorithm of the original scheme. A Courant number of 1.0 is found to be optimal for the problem. The solution profiles obtained are displayed on the second row from the top in Fig. 1. The profiles obtained are comparatively sharp with a negligible diffusive error even on a coarse grid, typically attributed to the second order accuracy. The dispersive error inherent in the scheme is reflected as oscillations in the vicinity of sharp gradients in the solution profile. Fig. 2 gives a zoomed-in view of the profiles near the expansion fan.

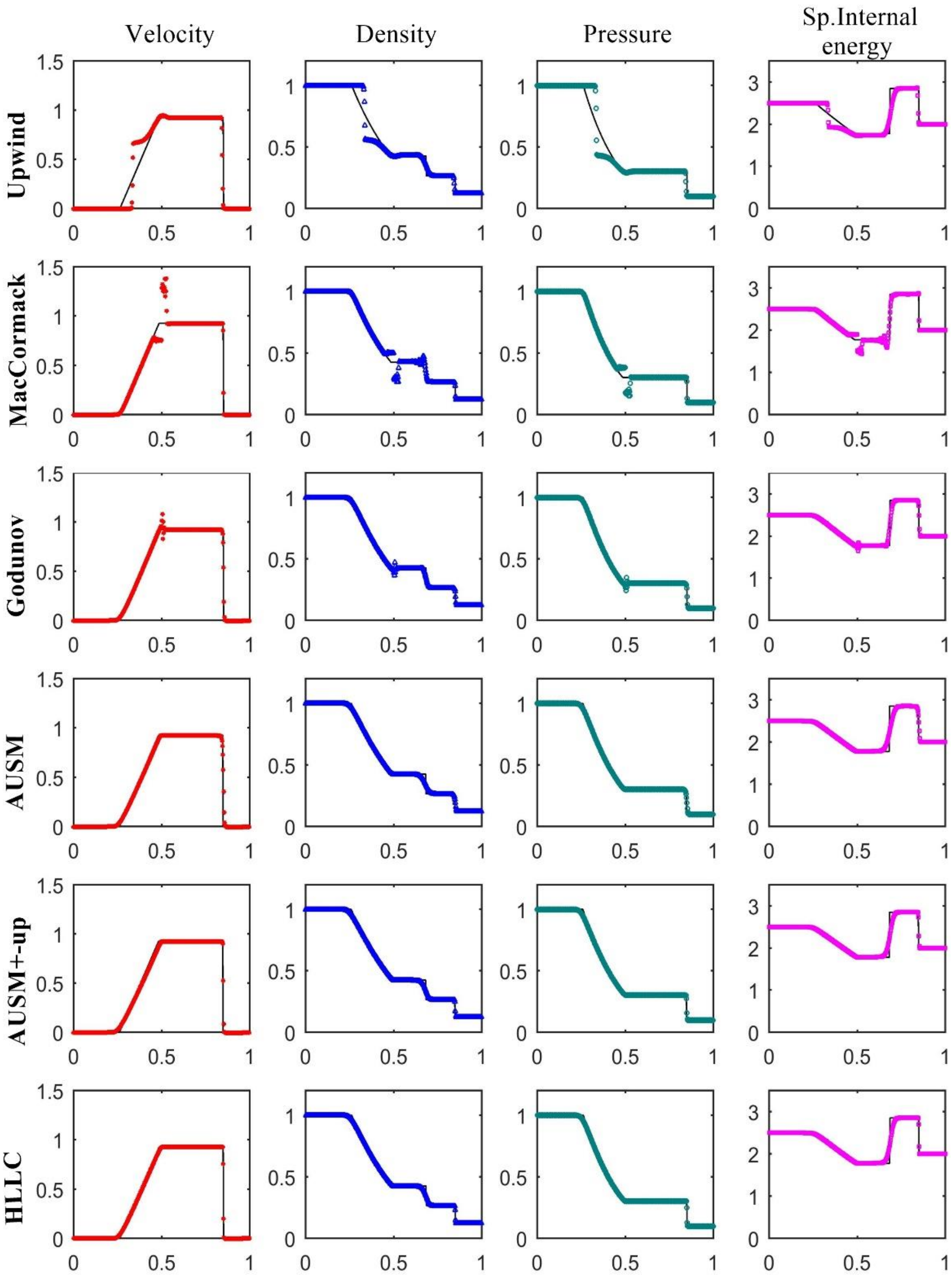

Figure 1. Results on computational domain with 400 divisions.

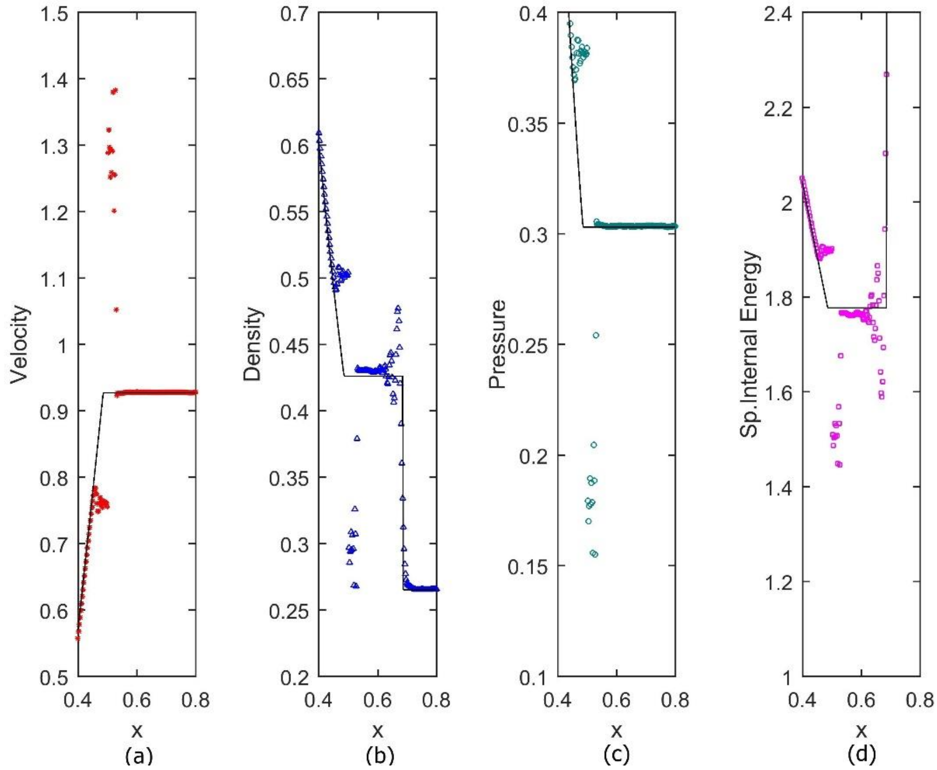

Figure 2. Oscillations in the expansion fan region with MacCormack scheme

The profiles very clearly reveal the severity of oscillations near the expansion fan and are observed with all the four fluid properties plotted. Regions near the contact discontinuity and shock are also not free from oscillations. The amplitude and spread of the fluctuations are relatively small at the contact discontinuity in comparison to that observed near the expansion fan, and near the shock, it becomes negligibly small. Refinement of the grid is helpful in narrowing down the spread of oscillations and have no effect on their amplitude.

The Godunov solver used is a two-step method implemented on a finite difference framework and the results obtained are displayed along the third row from top in Fig. 1. The optimized Courant number for the problem with the scheme is 0.5. A close examination of the results reveals that the Godunov method even with a relatively simple algorithm captures certain features of the solution with reasonable accuracy. As an example, of all the methods used, the contact discontinuity captured is the sharpest with the Godunov scheme as demonstrated in Fig.3. The eminence of the method in terms of sharpness provided, however, is evaded by a small amplitude oscillation observed near the contact discontinuity. The major drawback of the method, however, is that it produces oscillation near the tail end of the expansion fan as shown in Fig.4. The small amount of diffusion observed in the vicinity of the shock is typically attributed to the first-order accuracy of numerical scheme. Grid refinement sharpens the shock and contact surface while the fluctuation observed near the expansion fan is persistent.

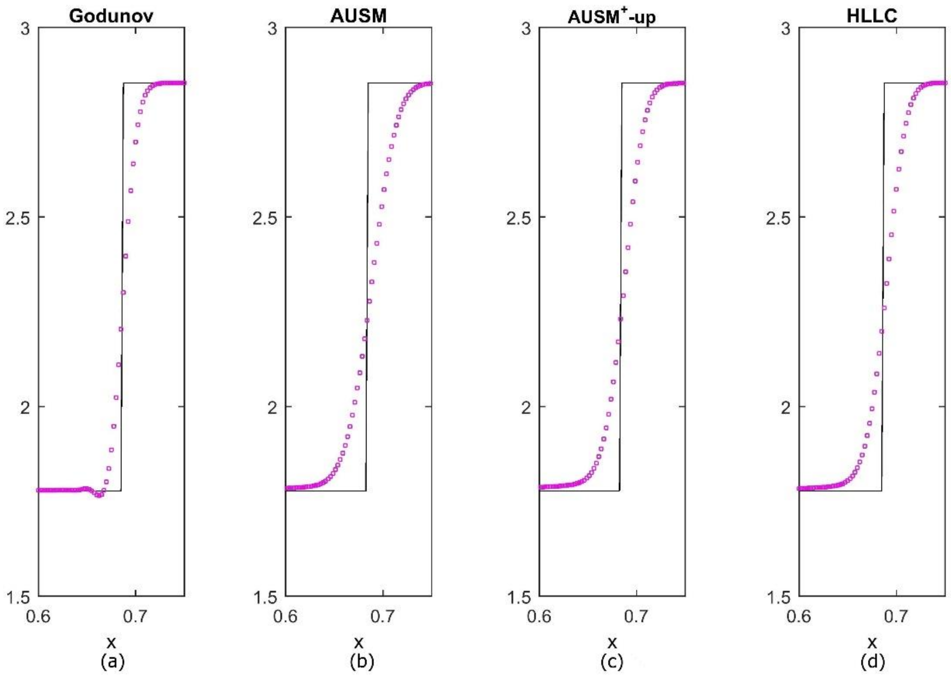

Figure 3. Solution at the contact discontinuity using different numerical schemes.

The results using Advection Upstream Splitting Method [11, 13] are displayed along the fourth row from top in Fig. 1. The first-order accurate method is implemented on a finite volume grid. Liou & Steffen in their introductory paper on AUSM [11], mentions that the algorithm is primarily developed to solve steady state problem with discontinuities in the solution. In spite of this apparent limitation of AUSM, this method is used to solve unsteady, one-dimensional shock tube problem. An optimized Courant number of 0.3 is used for the computation. The results reveal that the method is able to capture the expansion fan, despite the solution profiles suffering from dissipative effects at the contact discontinuity and at the shock for low grid resolution. It is observed that the diffusive error at the shock decreases with grid refinement, however, the diffusion near the contact discontinuity is persistent even on finer grid.

The AUSM+-up [2, 12] is an improved version of the basic AUSM scheme. The Courant number optimized for the computation is 0.95. The results obtained using AUSM+-up solver is shown on the fifth row from top in Fig. 1. Improvement in the results is observed on both coarse as well as fine grids. On a coarse grid, the higher dissipation observed near the expansion region in comparison to the basic AUSM scheme, is well paid off by the sharpness of the shock captured (see Fig. 5).

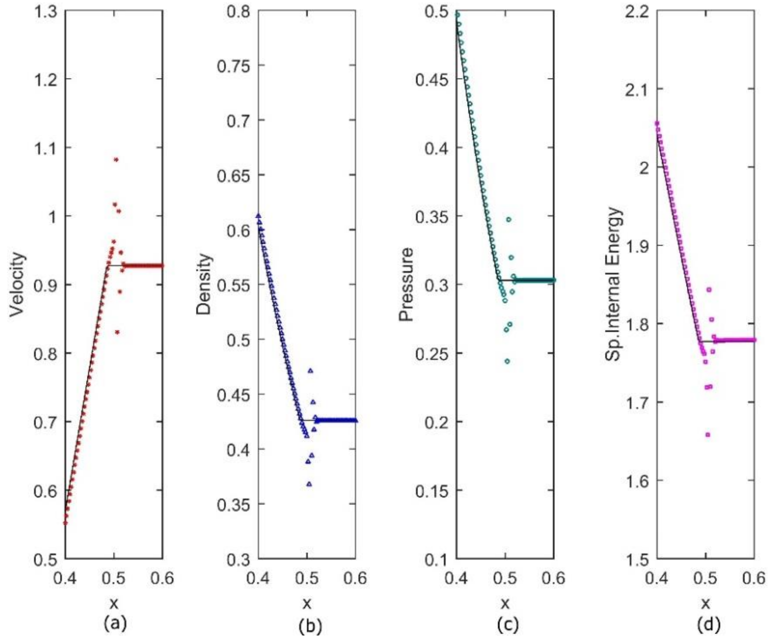

Figure 4. Solution near the expansion fan using Godunov scheme.

Except for the expansion fan, the results from the AUSM+-up for the rest of the domain is superior to that from the basic AUSM scheme. On a finer grid, both these methods perform nearly at par, with the AUSM+-up having a slight advantage over the basic scheme, in being less dissipative at the contact discontinuity as revealed in Fig. 3. It may be noted that the results obtained with the speed of sound at cell interface taken as the averaged sound speed and as the critical sound speed calculated from isoenergetic equation show no variation.

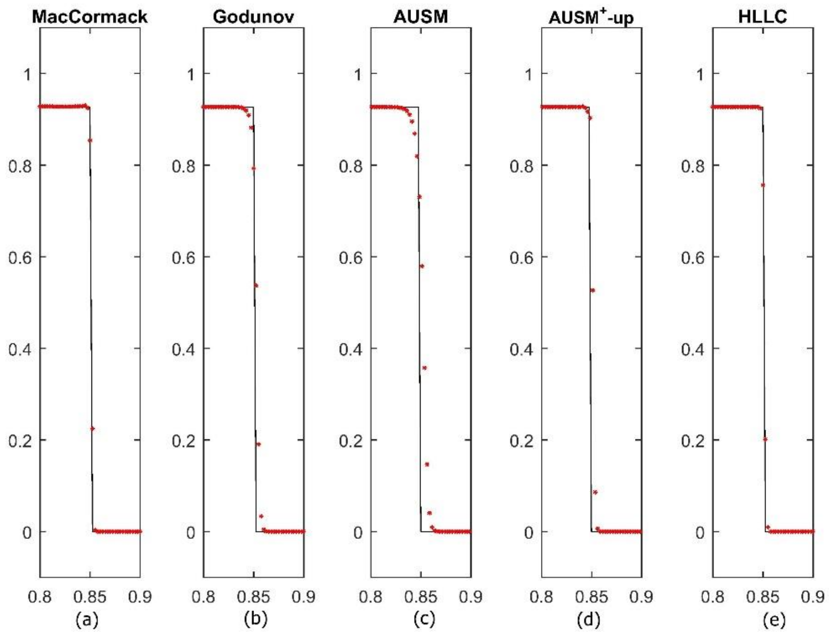

Figure 5. Comparison of solutions at the shock using different numerical schemes.

Results from the HLLC method are given along the bottom row in Fig. 1. With the HLLC solver, a Courant number of 1.0 is found optimal for the problem. The solution obtained with the solver clearly suggests its supremacy over the already discussed methods. The results obtained with the scheme are superior to that from the AUSM+-up, especially at the shock and contact discontinuity (see figures Fig. 3 and Fig. 5). The contact surface capture is only marginally improved with the HLLC over AUSM+-up, whereas the shock capture shows considerable improvement in sharpness with the HLLC scheme.

The expansion fan capture is unsatisfactory with the MacCormack and Godunov schemes as both these methods are prone to oscillations in the region unless used in conjunction with artificial viscosity or artificial compression. The methods viz., basic AUSM, AUSM+-up and HLLC captures the expansion fan with reasonable accuracy. Comparing the performance of the schemes at the contact discontinuity reveals the severity of oscillations with the MacCormack scheme and hence the applicability of the same for solving the shock tube problem is ruled out. Results from the remaining four schemes (see Fig. 3) reveals the high dissipation with the basic AUSM scheme and an oscillation affected profile for the Godunov scheme. The AUSM+-up and HLLC, however, are able to capture the contact surface with reasonable accuracy.

The shock capturing ability of the methods is judged with the aid of figure Fig. 5 to arrive at the best solver. More distinct shock interfaces are provided by the MacCormack, AUSM+-up, and the HLLC compared to the diffused ones from the Godunov and the AUSM schemes. The MacCormack scheme, however, with the spurious oscillations exhibited near the expansion fan and contact discontinuity cannot be considered as the best method. The HLLC scheme is observed superior to AUSM+-up in capturing the shock and the contact discontinuity and is hence concluded to be the best scheme out of the enlisted methods to model the one-dimensional shock tube problem. The number of grid points involved in capturing the shock is a standard way to assess its sharpness. The studies done on much finer grid followed the same trend and the conclusions are the same as reported here.

5. Conclusion

Six different numerical schemes are used to solve the one-dimensional Sod's shock tube problem and the numerical results are compared for their relative ability to capture the shock, the contact discontinuity, and the expansion fan. Some methods exhibited excessive diffusion or dispersion in the vicinity of discontinuous property jumps. The upwind scheme is inferred as an inefficient method for the solution of the problem, which fails to capture the physics of expansion fan as a result of non-enforcement of entropy conditions. Two schemes, viz., the MacCormack and the Godunov produces oscillations at the trailing edge of the expansion fan and is more severe with the former. The AUSM, the AUSM+-up, and the HLLC schemes reveal their ability in replicating the expansion fan with very small diffusion. With regard to the capability of capturing the shock with minimum diffusion and dispersion, the AUSM+-up and the HLLC schemes outperform the basic AUSM scheme. Though the results provided by the AUSM+-up and the HLLC schemes are very close to each other, the solution from the HLLC is supreme on the grounds that the shock captured is much sharper and the diffusion at the contact discontinuity is less from that observed with the AUSM+-up. The work finally concludes by stating that, the HLLC solver is the best of the six numerical schemes considered in solving the one-dimensional Sod's shock tube problem. The method proved its superiority over the other methods by producing the sharpest and least diffusive solution profile. The AUSM+-up scheme is observed only to be marginally inferior to the HLLC solver. These two methods are capable of producing numerical solutions which are very close to the exact solution available for the problem. The results obtained with a coarse grid is what is presented in the work, and on a sufficiently refined grid, the results show excellent match with the analytical solution.

Declaration of Competing Interest

No conflict of interest was declared by the authors.

Author Creditship Statement

Jishnu Chandran Ramachandran: Conceptualization, Methodology, Investigation, Visualization, Typing, and Software.
Abdussamad Salih: Supervision, Methodology, Investigation, Reviewing and Editing.

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