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Comparing a Three-Term Perturbation Solution of the Nonlinear ODE of the Jacobi Elliptic SN Function to Its Approximation into Circular Functions

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Abstract

In this paper, the nonlinear differential equation of the elliptic sn function is solved analytically using the Lindstedt-Poincaré perturbation method. This differential equation has a cubic nonlinearity and a constant known as the modulus of elliptic integral. This constant takes any value from zero to one and the square of its value is used as a small parameter. Fortunately, there is an exact solution to this differential equation known as the Jacobi sn elliptic function. When the modulus approaches zero, the differential equation becomes linear with the circular sine function as exact solution. The Lindstedt-Poincaré technique is used to render the perturbation solution uniformly valid at larger values of the independent variable and a three-term perturbation solution is obtained. This solution is compared analytically with the approximate expansion of the elliptic function into circular functions in case of a small modulus. Then, it is compared with the exact, numerically calculated, sn elliptic function. The relative percentage error is calculated at certain values of the modulus and for all values of the independent variable. The relative error is reasonably small but increases at larger values of the modulus. In addition, the approximate expansion of the exact solution gives smaller relative error than that of the perturbation solution including the same order of the modulus.

1. Introduction

In some nonlinear problems a perturbation solution may be obtained when a small parameter exists [1]. The obtained perturbation solution depends mainly on the existence of an unperturbed solution i.e. the solution of the same problem when the small parameter vanishes. Through an iterative like procedures, the solution is getting closer to the exact one by adding terms of order of magnitudes less than the base or unperturbed solution. Difficulties arise when a singularity exists in the solution. In this case, it will be non-uniformly valid and some techniques such as those established by Lindstedt-Poincaré or Lighthill can be used to eliminate the non-uniformity in the solution [1, 2, 3]. When applying these techniques the analytical iterations become more difficult as more terms are included in each iteration step. The differential equation of the Jacobi elliptic sn function is an example of a nonlinear ordinary differential equation which includes a cubic nonlinearity and a small parameter. This small parameter has the property of deforming the solution from an initial function to a final one as it goes from zero to unity. The nonlinear differential equation has an exact solution known as Jacobi elliptic sine function or sn function. The value of this function can be obtained from tables or using scientific software such as Matlab or Maple. But when the value of the modulus is close to zero the sn function can be approximated as series expansion of circular functions with different harmonics. This approximation can be calculated without special software [4, 5] and its explicit analytical nature makes it useful in analytical comparison with the perturbation solution.
In this paper the Lindstedt-Poincaré technique will be used to obtain a uniformly valid three-term perturbation solution to the differential equation of the Jacobi elliptic function. In the second section, the perturbation solution is derived and the effect of the modulus on its

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behavior is analytically indicated. In the third section, an approximate series expansion to the exact solution is reviewed. The approximation is derived so that it includes the same order of the small parameter as the perturbation solution. In the fourth section, the perturbation solution is compared with the exact solution and its series expansion in case of small modulus. Solutions in addition to relative errors are tabulated and represented graphically at different values of the modulus. Detailed analysis of behavior of solutions and errors are introduced. Finally, conclusions are drawn in the fifth section.

2. Perturbation solution

Consider the nonlinear differential equation [6]

$$d^2y/dx^2 + (1+k^2)y - 2k^2y^3 = 0, 0 \le k \le 1 \tag{1}$$

where k is constant known as the modulus of elliptic integral and $k \in [0,1]$. When the modulus $k \to 0$, (1) reduces to a simple harmonic oscillator whose solution is a circular function. But when k takes any small positive value less than one i.e. $k \in (0,1)$, a cubic nonlinearity exists. Let define another small parameter $\varepsilon = k^2$, where $\varepsilon < k$ for $k \in (0,1)$. Existence of the small parameter allows using the perturbation technique to solve the above problem. Furthermore, to apply Lindstedt-Poincaré technique, let transform the independent variable from x to u through the following transformation

$$u = \omega x, \omega = \sum_{i=0}^{\infty} \varepsilon^i \omega_i = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3) \tag{2}$$

substituting (2) into (1) gives

$$\omega^2 y'' + (1+\varepsilon)y - 2\varepsilon y^3 = 0, \tag{3}$$

where $(.)'$ denotes differentiation with respect to u. The next step is expand the dependent variable as series in the small parameter ε

$$y(u;\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i y_i = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3). \tag{4}$$

When substituting (4) into (3) then collecting and equating coefficients of equal powers of ε one obtains the following set of linear differential equations

$$y_0'' + y_0 = 0 \tag{5}$$
$$y_1'' + y_1 = (2\omega_1 - 1)y_0 + 2{y_0}^3 \tag{6}$$
$$y_2'' + y_2 = (2\omega_2 + 2\omega_1 - 3{\omega_1}^2)y_0 + (2\omega_1 - 1)y_1 - 4\omega_1{y_0}^3 + 6y_1{y_0}^2. \tag{7}$$

The solution to (5) is the unperturbed solution $y_0(u) = A_0 \sin u$. Knowing from the initial conditions of the sn function and its derivative, i.e. $y(x=0) = sn(0;k) = 0$, $\frac{dy}{dx}(x=0) = cn(0;k)dn(0;k) = 1$, and using the relation in (2), the initial conditions for (5) are $y_0(u=0) = 0, y_0'(u=0) = 1$. After applying these initial conditions, the unperturbed solution takes the form

$$y_0(u) = \sin(u). \tag{8}$$

When substituting (8) into (6) one can solve for $y_1(u)$. But, to make sure that the solution of $y_1(u)$ converges at larger values of u, it is necessary to set $\omega_1 = -1/4$. In this case, the solution to (6) reads

$$y_1(u) = -\frac{3}{16}\sin(u) + \frac{1}{16}\sin(3u). \tag{9}$$

Similarly by substituting $y_0(u), y_1(u)$ from (8), (9) respectively into (7) one can solve for $y_2(u)$. To enforce convergence of $y_2(u)$, it is necessary to set $\omega_2 = 19/64$. Then, by integrating (7) one obtains

$$y_2(u) = \frac{7}{256}\sin(u) - \frac{1}{64}\sin(3u) + \frac{1}{256}\sin(5u). \tag{10}$$

Substituting (8),(9), and (10) into (4), a three term perturbation solution reads

$$y_p(u) = \sin(u) + \varepsilon\left(-\frac{3}{16}\sin(u) + \frac{1}{16}\sin(3u)\right) + \varepsilon^2\left(\frac{7}{256}\sin(u) - \frac{1}{64}\sin(3u) + \frac{1}{256}\sin(5u)\right) + O(\varepsilon^3) \tag{11}$$

or

$$y_p(u) = \left(1 - \frac{3}{16}\varepsilon + \frac{7}{256}\varepsilon^2\right)\sin(u) + \left(\frac{1}{16}\varepsilon - \frac{1}{64}\varepsilon^2\right)\sin(3u) + \frac{1}{256}\varepsilon^2\sin(5u) + O(\varepsilon^3)$$

Rewriting (11) after substituting $\varepsilon = k^2$ gives

$$y_p(u) = \sin(u) + k^2\left(-\frac{3}{16}\sin(u) + \frac{1}{16}\sin(3u)\right) + k^4\left(\frac{7}{256}\sin(u) - \frac{1}{64}\sin(3u) + \frac{1}{256}\sin(5u)\right) + O(k^6)$$

or

$$y_p(u) = \left(1 - \frac{3}{16}k^2 + \frac{7}{256}k^4\right)\sin(u) + \left(\frac{1}{16}k^2 - \frac{1}{64}k^4\right)\sin(3u) + \frac{1}{256}k^4\sin(5u) + O(k^6) \tag{12}$$

where

$$u = \left(1 - \frac{1}{4}k^2 + \frac{19}{64}k^4 + O(k^6)\right)x.$$

2.1. Behavior of the perturbation solution

Equation (12) shows that any term in the obtained solution takes the form of a circular function multiplied by a finite quantity. In addition for the solution to converge the following condition should be satisfied

$$\lim_{n\to\infty} |\frac{\varepsilon^{n+1}y_{n+1}}{\varepsilon^n y_n}| = \varepsilon |\left(\frac{y_{n+1}}{y_n}\right)| = o(1).$$

knowing that $|\left(\frac{y_{n+1}}{y_n}\right)| = O(1)$, the condition of convergence, then, reduces to the condition $\varepsilon = o(1)$, which is known by the definition $\varepsilon = k^2$, where $k \in (0,1)$ as indicated in the second section.

3. Approximate series solution

The Jacobi elliptic function $\mathrm{sn}(x;k)$ is the solution to the differential equation in (1) [6]. For small values of the modulus k this function can be expressed in terms of the circular sine and cosine functions. The derivation of this approximation can be started from knowing that the independent variable in (1), which is the argument of the sn function, is the incomplete elliptic integral of the first kind $F(\phi;k)$;

$$x = F(\phi;k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}}$$

or

$$x = F(\chi;k) = \int_0^{\chi=sin\phi} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

where ϕ is known as the amplitude and θ, t are dummy variables. For small values of the modulus k, the sn function can be written as follows [4, 5]

$$sn(x;k) = \sin x - \frac{k^2}{4}\cos x(x - \sin x\cos x).$$

As we derived our perturbation solution to include k^4, it may be reasonable if we compare with approximation of the sn function including k^4 as well. Thus, with some efforts we could derive the following approximation

$$sn(x;k)_{approx} = \sin x - \frac{k^2}{4}\cos x g(x;k) + \frac{k^4}{32}\left(2\cos x g(x;k) - \sin x g^2(x;k)\right), \tag{13}$$

where

$$g(x;k) = (x - \sin x\cos x). \tag{14}$$

4. Results and discussion

The perturbation solution is compared with the exact solution, i.e. the elliptic sn function, and its approximate series expansion in (13). Equations (12), (13) show that when $k \to 0$ the two solutions there reduce to the same base solution $\sin(x)$. Also, the exact sn function reduces to the same solution when $k \to 0$. In this specific case there is no need to compare these solutions numerically. For other values of the modulus, the perturbation solution is expected to be different than the other ones. The three solutions are listed in Table 1 to Table 4 for values of the modulus $k = 0.2, 0.4, 0.6, 0.8$, respectively. In addition, the following relative percentage error form is used to show how close are the explicit perturbation solution and sn_{approx} to the exact, numerically calculated, sn solution.

$$E_{pert} = \frac{sn_{exact} - y_p}{sn_{exact}} \times 100 \tag{15}$$

$$E_{approx} = \frac{sn_{exact} - sn_{app}}{sn_{exact}} \times 100. \tag{16}$$

The perturbation solution, the approximate expansion, and the exact solution at values of the modulus $k = 0.2, 0.4, 0.6, 0.8$ are graphically represented in Figure 1 to Figure 4 respectively, for $x \in [0, K(k)]$, where $K(k)$ is the complete elliptic integral of the first kind. Figure 1 shows that the solutions are very close when $k = 0.2$. In Figure 2 to Figure 4, with increasing k, the Difference between the perturbation solution and the exact solution increases. However, one can notice the small rate of increase of the difference between the approximate expansion and the exact solution.

The relative percentage errors indicated in (15), (16) are graphically represented at the values of the modulus $k = 0.2, 0.4, 0.6, 0.8$ in Figure 5 to Figure 8. It is obvious from Figure 5 to Figure 8, that the relative percentage errors are undefined at $x = 0$ as all solutions are equal to zero at this point. More importantly, the maximum difference between the errors E_{pert} and E_{approx} increases with k. Moreover,one can note that in Figure 5 to Figure 8, this maximum difference occurs at the largest value $x = K(k)$. Actually, such a behavior of the perturbation solution is expected as this solution was not enforced to satisfy the end condition.

The behavior of the error in Figure 5 to Figure 8 can be attributed to the different terms of small and large magnitudes in both solutions y_p and sn_{approx}. The reason can also go back to the different way each solution is mathematically derived, even though, the perturbation

x	y_{pert}	sn_{approx}	sn_{exact}	E_{pert}	E_{approx}
0	0	0	0	NaN	NaN
0.083519	0.082624	0.083418	0.083418	0.95251	-4.6268e-05
0.16704	0.16465	0.16623	0.16623	0.95253	-0.00018229
0.25056	0.24548	0.24785	0.24784	0.95258	-0.00039997
0.33408	0.32455	0.32767	0.32767	0.95268	-0.00068665
0.4176	0.40128	0.40514	0.40514	0.95286	-0.0010262
0.50112	0.47513	0.47971	0.47971	0.95316	-0.0014004
0.58464	0.54561	0.55087	0.55086	0.95361	-0.00179
0.66815	0.61221	0.61812	0.61811	0.95426	-0.0021762
0.75167	0.67451	0.68103	0.68101	0.95517	-0.0025407
0.83519	0.73209	0.73918	0.73916	0.9564	-0.0028665
0.91871	0.78458	0.79219	0.79217	0.95799	-0.0031363
1.0022	0.83165	0.83974	0.83971	0.96003	-0.003332
1.0858	0.87301	0.88153	0.8815	0.96257	-0.0034321
1.1693	0.90841	0.9173	0.91727	0.96568	-0.0034096
1.2528	0.93765	0.94686	0.94683	0.96945	-0.0032287
1.3363	0.96055	0.97002	0.96999	0.97393	-0.0028421
1.4198	0.97697	0.98665	0.98663	0.97922	-0.002188
1.5033	0.98683	0.99667	0.99665	0.98539	-0.0011877
1.5869	0.99007	1	1	0.99254	0.00025687

Table 1: y_{pert}, sn_{approx}, sn, E_{pert}, and E_{approx} at $k = 0.2$

x	y_{pert}	sn_{approx}	sn_{exact}	E_{pert}	E_{approx}
0	0	0	0	NaN	NaN
0.086316	0.083399	0.086192	0.086192	3.2404	-0.00079054
0.17263	0.16608	0.17165	0.17164	3.2416	-0.0031128
0.25895	0.24733	0.25564	0.25562	3.2438	-0.0068231
0.34526	0.32648	0.33748	0.33744	3.2474	-0.011697
0.43158	0.40289	0.4165	0.41643	3.2525	-0.017449
0.51789	0.47597	0.49212	0.492	3.2598	-0.023755
0.60421	0.54518	0.56378	0.56361	3.2698	-0.030275
0.69053	0.61008	0.63102	0.63079	3.2831	-0.036669
0.77684	0.67023	0.6934	0.69311	3.3003	-0.042609
0.86316	0.72532	0.7506	0.75024	3.3222	-0.047775
0.94947	0.77504	0.80232	0.8019	3.3496	-0.051849
1.0358	0.81918	0.84833	0.84787	3.3834	-0.054485
1.1221	0.85755	0.88845	0.88796	3.4246	-0.055282
1.2084	0.89002	0.92254	0.92205	3.4742	-0.053743
1.2947	0.91647	0.95051	0.95004	3.5333	-0.049231
1.3811	0.93685	0.97227	0.97187	3.6031	-0.040921
1.4674	0.9511	0.98776	0.98749	3.6851	-0.027764
1.5537	0.95918	0.99696	0.99687	3.7807	-0.0084412
1.64	0.96109	0.99981	1	3.8914	0.018666

Table 2: y_{pert}, sn_{approx}, sn, E_{pert}, and E_{approx} at $k = 0.4$

x	y_{pert}	sn_{approx}	sn_{exact}	E_{pert}	E_{approx}
0	0	0	0	NaN	NaN
0.092145	0.087225	0.091972	0.091968	5.1574	-0.0045588
0.18429	0.17342	0.18291	0.18288	5.1722	-0.017926
0.27643	0.2576	0.27183	0.27173	5.1976	-0.039204
0.36858	0.33885	0.35781	0.35757	5.2343	-0.066992
0.46072	0.41634	0.44001	0.43957	5.2834	-0.099508
0.55287	0.48938	0.51772	0.51702	5.3462	-0.13473
0.64501	0.5574	0.59037	0.58937	5.4243	-0.17051
0.73716	0.61995	0.65751	0.65617	5.5193	-0.20469
0.8293	0.67673	0.71881	0.71713	5.6334	-0.23514
0.92145	0.72753	0.77407	0.77207	5.7689	-0.25971
1.0136	0.77224	0.82318	0.82091	5.9288	-0.27624
1.1057	0.81084	0.86611	0.86367	6.1169	-0.28239
1.1979	0.84333	0.90287	0.90039	6.3375	-0.27554
1.29	0.86976	0.93353	0.93118	6.5958	-0.25258
1.3822	0.89019	0.95816	0.95615	6.8981	-0.20977
1.4743	0.90469	0.97682	0.97543	7.2513	-0.1425
1.5665	0.91331	0.98955	0.98911	7.6633	-0.045073
1.6586	0.91607	0.99639	0.99728	8.143	0.08948
1.7508	0.913	0.9973	1	8.7005	0.26966

Table 3: y_{pert}, sn_{approx}, sn, E_{pert}, and E_{approx} at $k = 0.6$

x	y_{pert}	sn_{approx}	sn_{exact}	E_{pert}	E_{approx}
0	0	0	0	NaN	NaN
0.10502	0.10064	0.10472	0.1047	3.8767	-0.018692
0.21003	0.19926	0.20769	0.20753	3.9862	-0.073237
0.31505	0.29398	0.30725	0.30676	4.1665	-0.1592
0.42006	0.38318	0.40195	0.40087	4.4146	-0.26971
0.52508	0.46558	0.49061	0.48867	4.7261	-0.39603
0.6301	0.5403	0.57232	0.56931	5.0957	-0.52824
0.73511	0.60684	0.64649	0.64228	5.5183	-0.6557
0.84013	0.66503	0.71283	0.7074	5.9903	-0.76743
0.94514	0.71495	0.77126	0.76474	6.5109	-0.8523
1.0502	0.75687	0.82189	0.81457	7.084	-0.89918
1.1552	0.79113	0.86499	0.8573	7.7188	-0.89703
1.2602	0.8181	0.90089	0.89343	8.431	-0.83504
1.3652	0.83812	0.92995	0.92346	9.2418	-0.70262
1.4702	0.85143	0.95254	0.9479	10.178	-0.48913
1.5752	0.85821	0.96899	0.96722	11.27	-0.18316
1.6803	0.85856	0.97956	0.98181	12.553	0.22877
1.7853	0.85249	0.98442	0.992	14.063	0.7641
1.8903	0.8399	0.98357	0.99801	15.842	1.4465
1.9953	0.82064	0.97692	1	17.936	2.308

Table 4: y_{pert}, sn_{approx}, sn, E_{pert}, and E_{approx} at $k = 0.8$

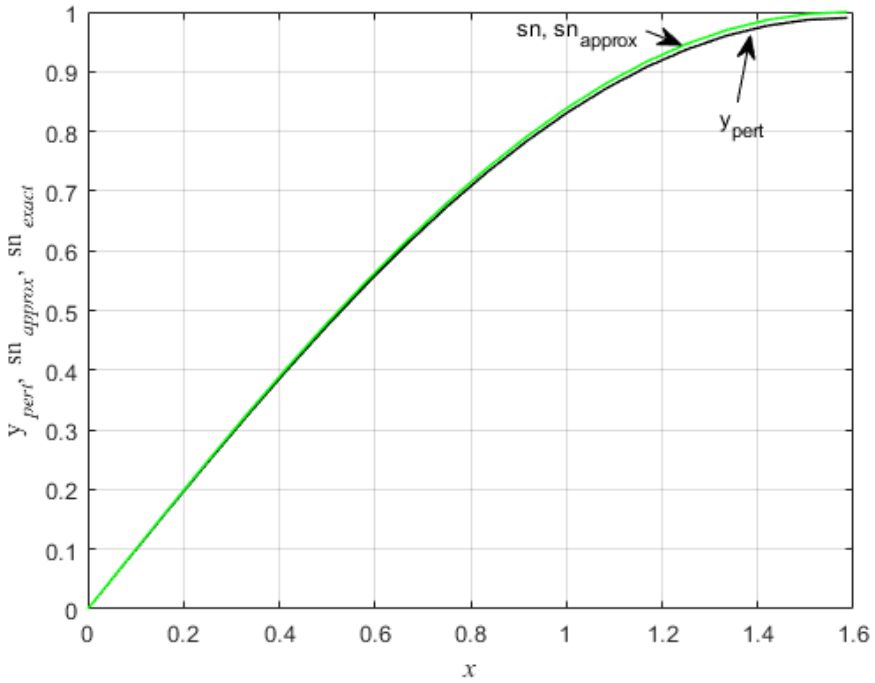

Figure 1: y_p, sn_{approx}, and exact sn at $k = 0.2$.

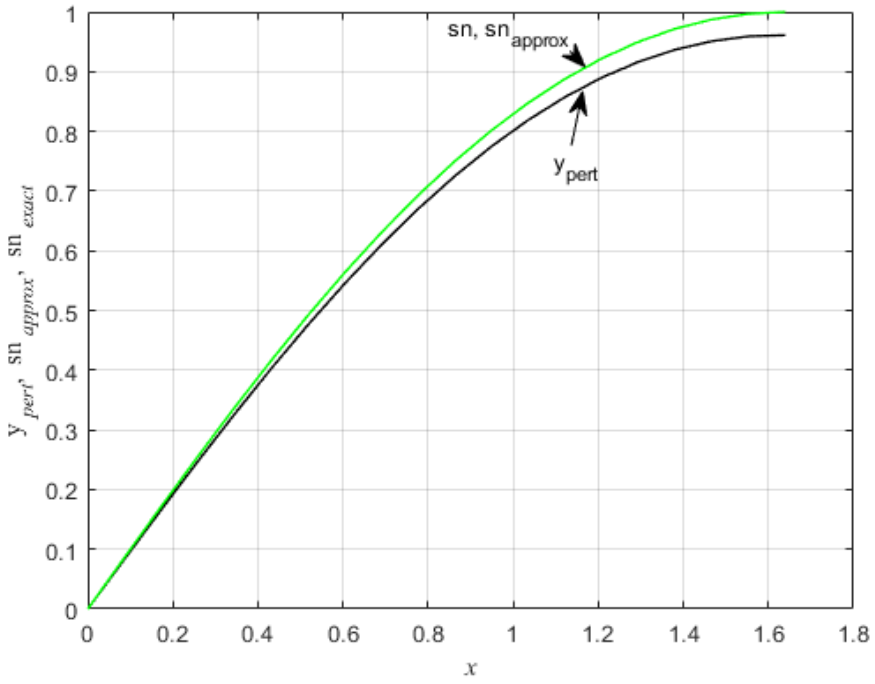

Figure 2: y_p, sn_{approx}, and exact sn at $k = 0.4$.

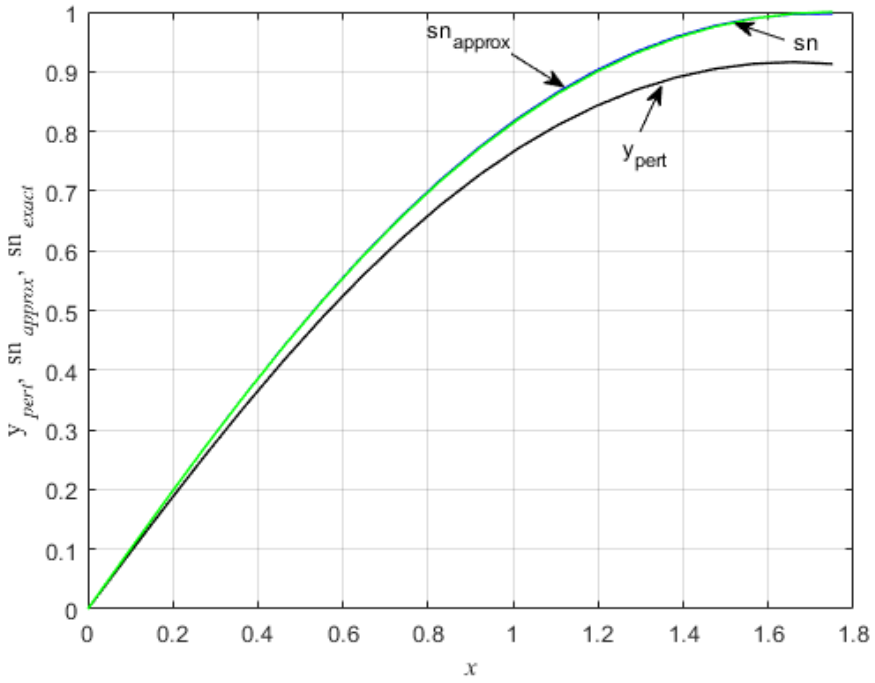

Figure 3: y_p, sn_{approx}, and exact sn at $k = 0.6$.

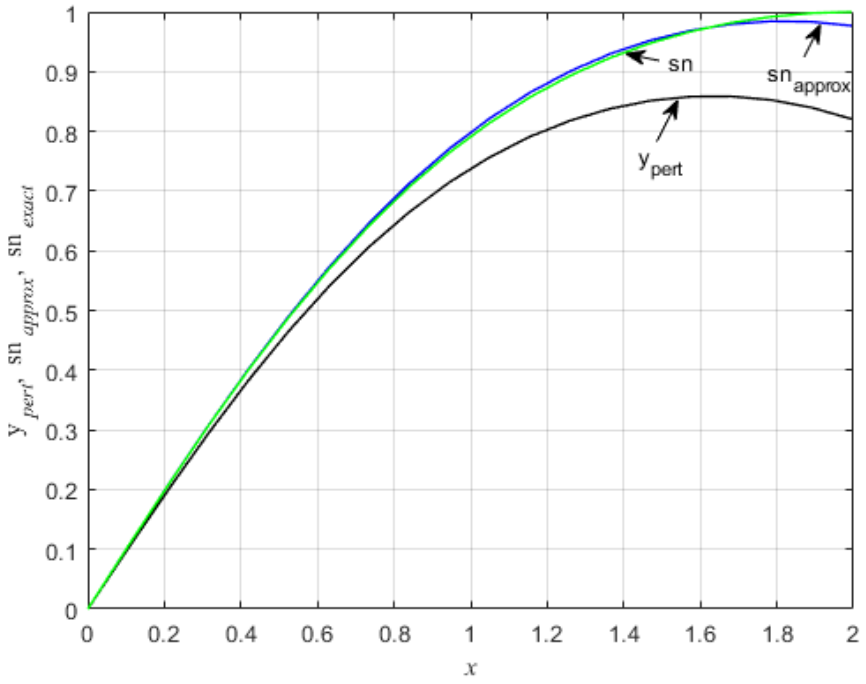

Figure 4: y_p, sn_{approx}, and exact sn at $k = 0.8$.

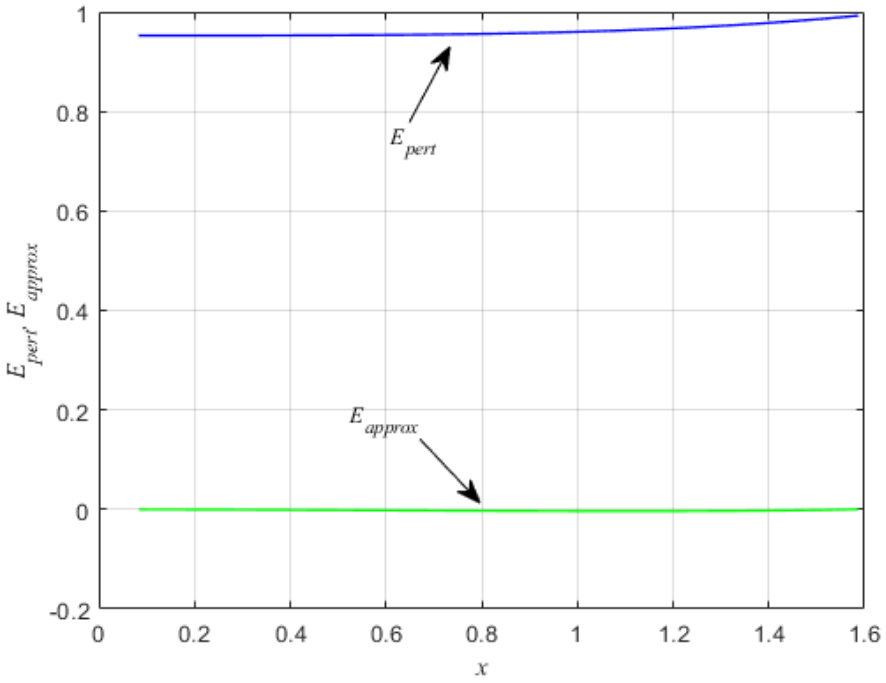

Figure 5: *Relative percentage error at* $k = 0.2$.

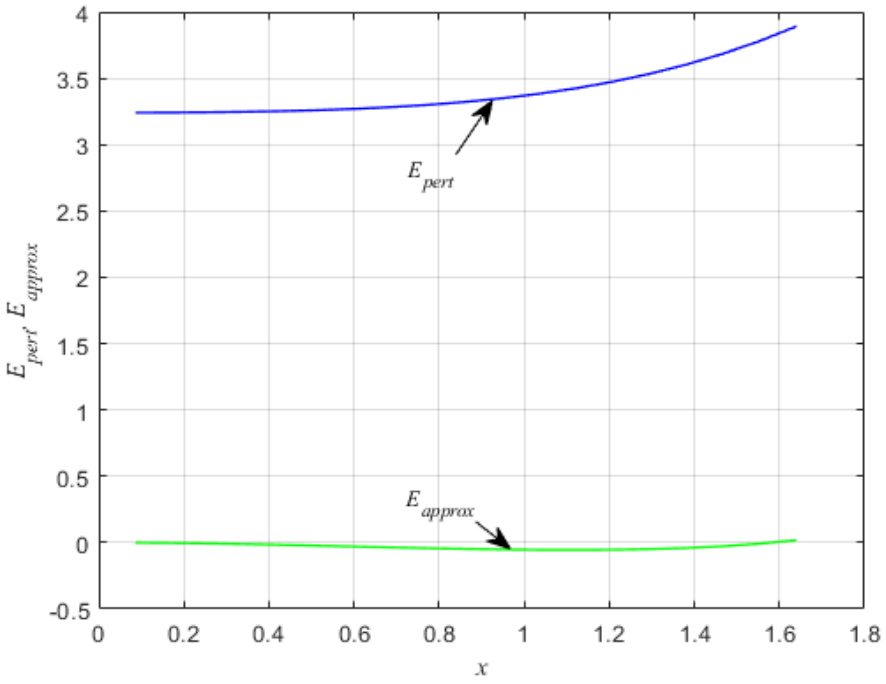

Figure 6: *Relative percentage error at* $k = 0.4$.

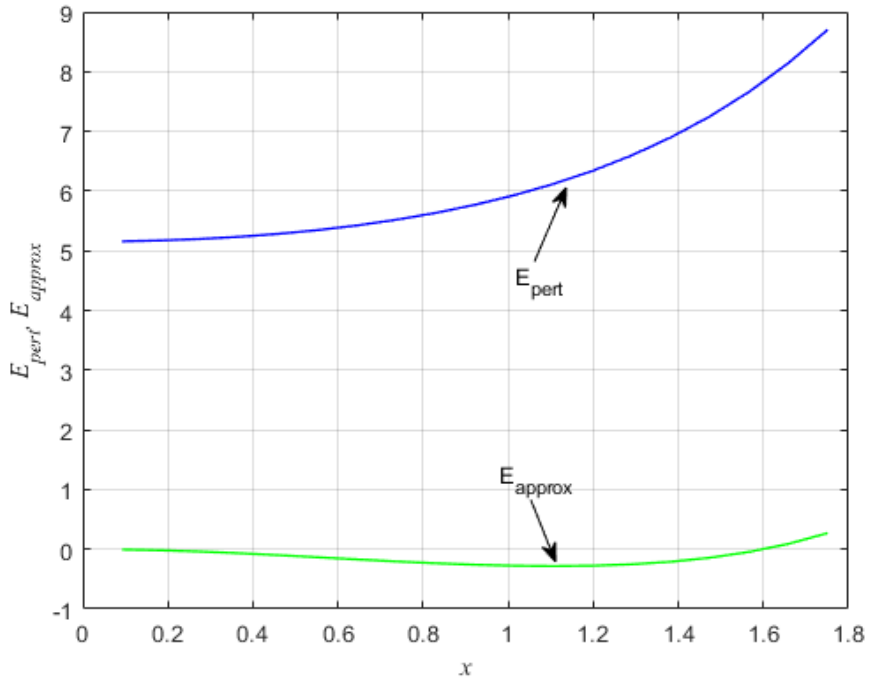

Figure 7: *Relative percentage error at* $k = 0.6$.

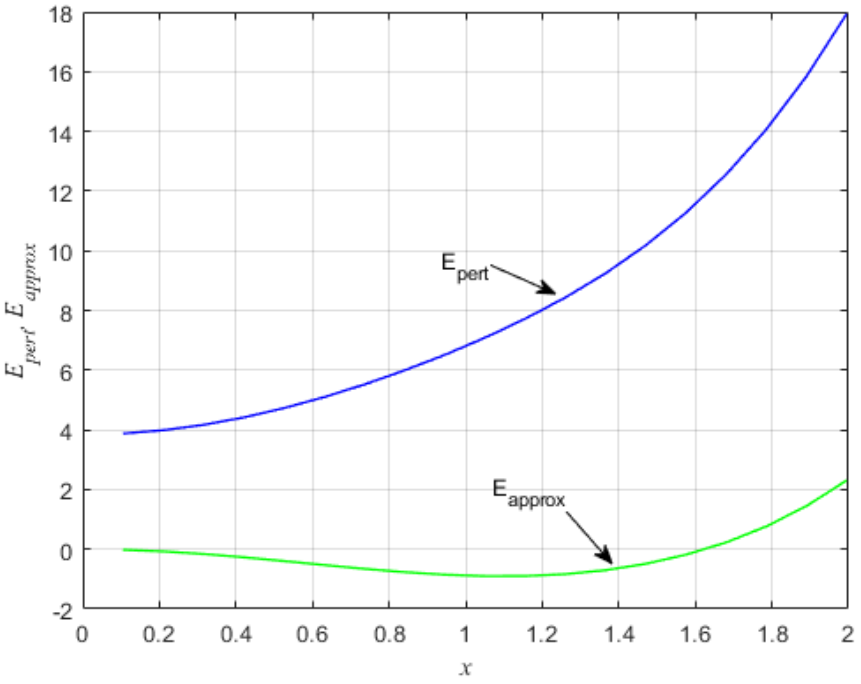

Figure 8: *Relative percentage error at* $k = 0.8$.

solution and the approximate expansion are built on the assumption of a small value of the modulus. The approximate series expansion in (13) includes the function $g(x;k) = (x - \sin x \cos x)$ that does not exist in the perturbation solution in (12). At $k = 0.2, 0.4, 0.6$ the maximum absolute value of E_{pert} is $0.99254\%, 3.8914\%, 8.7005\%$ respectively, while at $k = 0.8$ this value jumps to 17.936%. Thus, for small values of k the relative percentage error is reasonably small and the perturbation solution based on this assumption can be used. Fortunately, in this specific problem we have an exact solution and an approximation of this solution, to compare with the perturbation solution. However, the shown results are indicative of how perturbation solution performs in cases when exact solution doesn't exist.

5. Conclusion

An analytical approximate perturbation solution to the nonlinear ordinary differential equation of the Jacobi elliptic sn function is obtained assuming a small value of the modulus. The relative percentage error between the perturbation solution and the numerical exact one is reasonably small. But, at larger values of the modulus, this error becomes very big. An approximate series expansion of the sn function gives smaller maximum errors than the perturbation solution. However, the magnitude and sign of the error of the series expansion change at different values of the independent variable. Results also give insights into the effect of the mathematical basis of perturbation and approximate series solutions on their accuracy even though they both depend on the small parameter assumption. In future, such results can be considered when applying a Lindstedt-Poincaré perturbation solution to nonlinear problems.

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