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Research Article

Finite-dimensional Leibniz algebra representations of $\mathfrak{sl}_2$

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Abstract

All finite-dimensional Leibniz algebra bimodules of a Lie algebra $\mathfrak{sl}_2$ over a field of characteristic zero are described.

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1. Introduction

Finite-dimensional representations of a finite-dimensional semisimple Lie algebra is a well-studied beautiful classical theory. There is a Weyl's theorem on complete reducibility that claims that any finite-dimensional module over a semisimple Lie algebra is a direct sum of simple modules. A textbook approach starts with the finite-dimensional representations of the simple Lie algebra $\mathfrak{sl}_2$. In this work we find all finite-dimensional representations of $\mathfrak{sl}_2$ in a larger category – Leibniz algebra representations of $\mathfrak{sl}_2$.

The notion of a Leibniz algebra first appeared under the name of a D-algebra, introduced by A. Bloh in [2] as one of the generalizations of Lie algebras, in which multiplication by an element is a derivation. Later, they were discovered independently by J.-L. Loday [8] and gain popularity under the name of Leibniz algebras. Given a Leibniz algebra L there is a two-sided ideal $Leib(L) = Span\{[x, x] \mid x \in L\}$, associated to it, also known as the Leibniz kernel by some authors. The canonical Lie algebra $L/Leib(L)$ is called the liezation of L. Due to Leibniz kernel, there are no simple non-Lie Leibniz algebras. However, by abuse of standard terminology a simple Leibniz algebra is introduced in [4] as an algebra with simple liezation and simple Leibniz kernel. All such algebras are described via irreducible representations of simple Lie algebras.

While originally defined differently (cf. [3, 8]), the representation of a Leibniz algebra is given in [9] as a $\mathbb{K}$-module M with two actions - left and right, satisfying compatibility conditions coming from a so called square-zero construction. It is known that the category of Leibniz representations of a given Leibniz algebra is not semisimple and any non-Lie Leibniz algebra admits a representation, which is neither simple, nor completely reducible

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[7, Proposition 1.2]. In [10] the indecomposable objects of the category of Leibniz representations of a Lie algebra are studied and for $\mathfrak{sl}_2$ the indecomposable objects in that category are described as extensions (see Theorem 2.5 below). Our goal in the current work is to find explicitly indecomposable Leibniz representations. Remarkably, the authors of [10] prove that for $\mathfrak{sl}_n$ $(n \geq 3)$ the category of Leibniz representations is of wild type.

This work is a direct continuation of an investigation started in [7]. If M is an irreducible Leibniz representation of $\mathfrak{sl}_2$, by Weyl's result the left action on M as a Lie algebra representation decomposes into a direct sum of irreducible Lie representations of $\mathfrak{sl}_2$. Hence, the problem of description reduces to the study of the right action. In case the number of such irreducible Lie representations is two, up to a Leibniz algebra representation isomorphism there are exactly two types of irreducible Leibniz representations, whose actions are described in [7, Theorem 3.1]. In the current work, we establish the description in full generality providing an explicit description of actions up to isomorphism in Theorem 3.5.

All representations and algebras in this work are finite-dimensional over a field of characteristic zero.

2. Preliminaries

Definition 2.1. An algebra $(L, [-,-])$ over a field $\mathbb{K}$ is called a (right) *Leibniz algebra* if for all $x, y, z \in L$ the following identity holds:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

In case the bracket is skew-symmetric, the identity above, called Leibniz identity transforms into Jacobi identity. The category of Lie algebras is a full subcategory of the category of Leibniz algebras.

Next we use definition from [9] to define a representation of a Leibniz algebra.

Definition 2.2. A $\mathbb{K}$-vector space M with two bilinear maps $[-,-] : L \times M \to M$ and $[-,-] : M \times L \to M$ is called a *representation* of a Leibniz algebra L if the following holds:

$$[m,[x,y]] = [[m,x],y] - [[m,y],x], \tag{2.1}$$

$$[x,[m,y]] = [[x,m],y] - [[x,y],m], \tag{2.2}$$

$$[x,[y,m]] = [[x,y],m] - [[x,m],y]. \tag{2.3}$$

Note that, these are exactly the conditions for a direct sum $L \oplus M$ of $\mathbb{K}$-vector spaces to be the Leibniz algebra, where L and M are contained as subalgebra and abelian ideal, correspondingly. Such construction is called square-zero construction. Adding identities (2.2) and (2.3) we obtain

$$[x,[m,y] + [y,m]] = 0 \tag{2.4}$$

which is often used instead of identity (2.3).

Given a representation M of a Leibniz algebra L, one defines linear maps $\lambda_x, \rho_x : M \to M$ by $\lambda_x(m) = [x,m]$ and $\rho_x(m) = [m,x]$ for every $x \in L,\ m \in M$. Defining relations of Leibniz representation yield for all $x, y \in L$ the following:

$$\rho_{[x,y]} = \rho_y \circ \rho_x - \rho_x \circ \rho_y, \tag{2.5}$$

$$\lambda_{[x,y]} = \rho_y \circ \lambda_x - \lambda_x \circ \rho_y, \tag{2.6}$$

$$\lambda_x \circ (\rho_y + \lambda_y) = 0. \tag{2.7}$$

A representation of a Leibniz algebra L is called *symmetric* (*anti-symmetric*) if $\rho_x = -\lambda_x$ (respectively, $\lambda_x = 0$) for all $x \in L$. Considering a Lie algebra $\mathfrak{g}$ as a Leibniz algebra, equation (2.5) shows that the map $\rho : \mathfrak{g} \to \mathrm{End}(M)$ defined by $\rho(x) = \rho_x$ coincides with Lie algebra representation of the Lie algebra $\mathfrak{g}$. Moreover, it is known from [9] that the category of symmetric, as well as, the category of anti-symmetric representations of a given Leibniz algebra L is equivalent to the category of Lie algebra representations of the liezation of L.

For the sake of convenience, throughout this work, for a Leibniz algebra L we call representation M a *bimodule* M, and a Lie algebra module N an L-module N or simply a *module* N. Given a module M over a Lie algebra $\mathfrak{g}$, one can introduce symmetric and antisymmetric Leibniz bimodules M^s and M^a, by taking the left action to be negative of the right action for the first, and identically zero for the second bimodule, correspondingly.

A bimodule is called *simple* or *irreducible*, if it does not admit non-trivial subbimodules. It is well-known that the simple objects in the category of Leibniz representations of a given Leibniz algebra are exactly symmetric and anti-symmetric representations [1, Lemma 1.9].

A bimodule is called *indecomposable*, if it is not a direct sum of its subbimodules. Obviously, a simple bimodule is indecomposable, while the converse is not necessarily true (see [7, Proposition 1.2] and a paragraph that follows). To study bimodules it suffices to study indecomposable ones. In case Leibniz algebra is a Lie algebra, we utilize the following Weyl's result on complete reducibility of the right action of the bimodule.

Theorem 2.3. ([6]) *If $\mathfrak{g}$ is a finite-dimensional semi-simple Lie algebra over a field of characteristic zero, then every finite-dimensional module over $\mathfrak{g}$ is completely reducible.*

In order to describe all finite-dimensional indecomposable Leibniz bimodules of a Lie algebra $\mathfrak{sl}_2$ over a field of characteristic zero with basis $\{e, f, h\}$ and the products

$$[e, f] = h, \quad [e, h] = 2e, \quad [f, h] = -2f,$$

we use the following well-known description of simple $\mathfrak{sl}_2$-modules.

Theorem 2.4. ([6]) *For every non-negative integer m there exists up to an $\mathfrak{sl}_2$-module isomorphism one and only one irreducible $\mathfrak{sl}_2$-module $V(m)$ of dimension $m+1$. The module $V(m)$ admits a basis $\{v_0, v_1, \dots, v_m\}$ in which the following holds for all $k = 0, \dots, m$:*

$$\begin{aligned} [h, v_k] &= (m-2k)v_k, \\ [f, v_k] &= v_{k+1}, \\ [e, v_k] &= -k(m+1-k)v_{k-1}. \end{aligned}$$

J.-L. Loday and T. Pirashvili described the Gabriel quiver of Leibniz representations of $\mathfrak{sl}_2$ using Clebsch-Gordon formula in [10], and citing results of [5] and [11] they found all indecomposable objects in the category of Leibniz representations of $\mathfrak{sl}_2$ as extensions of simple objects. For the sake of convenience, we express their result as the following

Theorem 2.5. ([10]) *For every non-negative integers n and $k \leq \lfloor n/2 \rfloor + 1$ there are exactly two indecomposable $\mathfrak{sl}_2$-bimodules M_1 and M_2 determined uniquely by the following extensions:*

$$0 \longrightarrow \bigoplus_{0 \leq i < \frac{k}{2}} V(n-4i-2)^a \longrightarrow M_1 \longrightarrow \bigoplus_{0 \leq i \leq \frac{k-1}{2}} V(n-4i)^s \longrightarrow 0,$$

$$0 \longrightarrow \bigoplus_{0 \leq i \leq \frac{k-1}{2}} V(n-4i)^a \longrightarrow M_2 \longrightarrow \bigoplus_{0 \leq i < \frac{k}{2}} V(n-4i-2)^s \longrightarrow 0,$$

where $V(d)^s$ and $V(d)^a$ are irreducible symmetric and antisymmetric Leibniz representations of $\mathfrak{sl}_2$, correspondingly and $V(0)^a = V(0)^s$ is a trivial one-dimensional representation.

Our goal is to build these extensions explicitly. Let M be a finite-dimensional Leibniz bimodule of $\mathfrak{sl}_2$. As a right module, by Theorem 2.3 it is completely reduces into a direct sum of simple $\mathfrak{sl}_2$-modules $V_1 \oplus \cdots \oplus V_k$, the right action on each simple submodule being described by Theorem 2.4. Hence, the study is reduced to the left action only. In the case $k = 1$ it is $V(d)^s$ and $V(d)^a$, i.e. simple objects in the category of Leibniz representation of $\mathfrak{sl}_2$. In [7, Theorem 3.1] the case $k = 2$ is exploited:

Theorem 2.6. *An $\mathfrak{sl}_2$-module $M = V(n) \oplus V(m)$ is indecomposable as a Leibniz $\mathfrak{sl}_2$-bimodule if and only if $m = n-2$. For any integer $n \geq 2$, up to $\mathfrak{sl}_2$-bimodule isomorphism there are exactly two indecomposable bimodules. The non-zero brackets of the left action are either*

$$\begin{array}{lcl}
[h, v_i] = -(n-2i)v_i - 2iw_{i-1} & & [h, w_j] = 2(m-j+1)v_{j+1} - (m-2j)w_j \\
[f, v_i] = -v_{i+1} + w_i & \text{or} & [f, w_j] = v_{j+2} - w_{j+1} \\
[e, v_i] = i(n-i+1)v_{i-1} + i(i-1)w_{i-2} & & [e, w_j] = (m-j+1)((m-j+2)v_j + iw_{j-1})
\end{array}$$

corresponding to two bimodules, where $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_{n-2}\}$ are bases of $V(n)$ and $V(n-2)$ of the Theorem 2.4.

Note that in the first case of Theorem 2.6, the bimodule $M/V(n-2)$ is symmetric and $V(n-2)$ is anti-symmetric, while in the second one $M/V(n)$ is symmetric and $V(n)$ is anti-symmetric, that is in accordance with Theorem 2.5.

In the current work, we use results on the left action established in [7] using only equality (2.2). Till the rest of the section let M be an $\mathfrak{sl}_2$-bimodule that decomposes as a right module into the direct sum $M = V(n) \oplus V(m)$. The next statements shed light on the general form of the left action depending on n and m, that satisfies only identity (2.2), where $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_m\}$ are bases of $V(n)$ and $V(m)$ of the Theorem 2.4.

Proposition 2.7. [7, Proposition 2.6] *Let $n = m$. Then identity (2.2) implies the following:*

$$\begin{array}{ll}
[h, v_i] = (n-2i)(\psi_1 v_i + \psi_2 w_i), & 0 \leq i \leq n, \\
[f, v_i] = \psi_1 v_{i+1} + \psi_2 w_{i+1}, & 0 \leq i \leq n-1, \\
[e, v_i] = -i(n-i+1)(\psi_1 v_{i-1} + \psi_2 w_{i-1}), & 1 \leq i \leq n, \\
[h, w_i] = (n-2i)(\psi_3 v_i + \psi_4 w_i), & 0 \leq i \leq n, \\
[f, w_i] = \psi_3 v_{i+1} + \psi_4 w_{i+1}, & 0 \leq i \leq n-1, \\
[e, w_i] = -i(n-i+1)(\psi_3 v_{i-1} + \psi_4 w_{i-1}), & 1 \leq i \leq n.
\end{array}$$

Proposition 2.8. [7, Proposition 2.5] *Let $n = m-2$. Then identity (2.2) implies the following:*

$$\begin{array}{l}
[h, v_i] = (n-2i)\phi^{11} v_i - 2i\phi^{12} w_{i-1}, \ 0 \leq i \leq n, \\
[f, v_i] = \phi^{11} v_{i+1} + \phi^{12} w_i, \ 0 \leq i \leq n-1, \\
[e, v_i] = -i(n-i+1)\phi^{11} v_{i-1} + i(i-1)\phi^{12} w_{i-2}, \ 1 \leq i \leq n, \\
[h, w_i] = 2(m-i+1)\phi^{21} v_{i+1} + (m-2i)\phi^{22} w_i, \ 0 \leq i \leq m, \\
[f, w_i] = \phi^{21} v_{i+2} + \phi^{22} w_{i+1}, \ 0 \leq i \leq m, \\
[e, w_i] = (m-i+1)((m-i+2)\phi^{21} v_i - i\phi^{22} w_{i-1}), \ 0 \leq i \leq m.
\end{array}$$

Proposition 2.9. *Let $n - m \geq 4$. Then identity (2.2) implies the following:*

$$\begin{array}{ll}
[f, v_i] = \phi^{11} v_{i+1} & 0 \leq i \leq n-1 \\
[f, w_j] = \phi^{22} w_{j+1} & 0 \leq j \leq m-1 \\
[e, v_i] = -i(n-i+1)\phi^{11} v_{i-1} & 0 \leq i \leq n \\
[e, w_j] = -j(m-j+1)\phi^{22} w_{j-1} & 0 \leq j \leq m \\
[h, v_i] = (n-2i)\phi^{11} v_i & 0 \leq i \leq n \\
[h, w_i] = (m-2i)\phi^{22} w_i & 0 \leq i \leq m
\end{array}.$$

Proof. From [7, Proposition 2.2, 2.3, 2.4] we have the following table of brackets:

$$\begin{array}{ll}
[f, v_i] = \phi^{11} v_{i+1} & 0 \leq i \leq n-1 \\
[f, w_j] = \phi^{22} w_{j+1} & 0 \leq j \leq m-1 \\
[e, v_i] = \dfrac{i(n-i+1)}{n} \epsilon^{11} v_{i-1} & 0 \leq i \leq n \\
[e, w_j] = \dfrac{j(m-j+1)}{m} \epsilon^{22} w_{j-1} & 0 \leq j \leq m \\
[h, v_i] = (\eta^{11} - 2i\phi^{11}) v_i & 0 \leq i \leq n \\
[h, w_i] = (\eta^{22} - 2i\phi^{22}) w_i & 0 \leq i \leq m
\end{array}.$$

Considering identity (2.2) for triples (f, v_0, e) and (f, w_0, e) one obtains $\eta^{11} = n\phi^{11}$ and $\eta^{22} = m\phi^{22}$, correspondingly. Analogously, identity (2.2) for (h, v_i, e) and (h, w_i, e) implies $\epsilon^{11} = -n\phi^{11}$ and $\epsilon^{22} = -m\phi^{22}$, correspondingly. This completes the proof. □

3. Main results

Throughout this section M is an $\mathfrak{sl}_2$-bimodule that decomposes into the direct sum of simple $\mathfrak{sl}_2$-modules $M = V(n_1) \oplus V(n_2) \oplus \cdots \oplus V(n_k)$. Without loss of generality one can assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. By Theorem 2.4 each simple module V_p $(1 \leq p \leq k)$ admits basis $\{v_0^p, v_1^p, \dots, v_{n_p}^p\}$ such that for $0 \leq i \leq n_p$ the following holds:

$$\begin{aligned}
&[v_i^p, h] = (n_p - 2i) v_i^p, \\
&[v_i^p, f] = v_{i+1}^p, \\
&[v_i^p, e] = -i(n_p + 1 - i) v_{i-1}^p.
\end{aligned}$$

In general $[\mathfrak{sl}_2, V_p] \subseteq M$ and let us set the following for all $1 \leq p \leq k$:

$$[h, v_i^p] = \sum_{q=1}^{k} \sum_{j=0}^{n_q} \eta_{ij}^{pq} v_j^q, \ \ [f, v_i^p] = \sum_{q=1}^{k} \sum_{j=0}^{n_q} \phi_{ij}^{pq} v_j^q, \ \ [e, v_i^p] = \sum_{q=1}^{k} \sum_{j=0}^{n_q} \epsilon_{ij}^{pq} v_j^q.$$

The description of Leibniz bimodules over $\mathfrak{sl}_2$ is reduced to simplify the left action. As the following proposition shows, most of the coefficients above are annihilated.

Proposition 3.1. *Set $l_{pq} = \frac{1}{2}(n_p - n_q)$. Then*

$$[h, v_i^p] = \sum_{q=1}^{k} \eta_i^{pq} v_{i-l_{pq}}^q, \ \ [f, v_i^p] = \sum_{q=1}^{k} \phi_i^{pq} v_{i+1-l_{pq}}^q, \ \ [e, v_i^p] = \sum_{q=1}^{k} \epsilon_i^{pq} v_{i-1-l_{pq}}^q,$$

where $\eta_i^{pq} = \phi_i^{pq} = \epsilon_i^{pq} = 0$ if $l_{pq} \notin \mathbb{Z}$.

Proof. From $[h, [m, h]] = [[h, m], h]$ we get

$$\begin{aligned}
(n_p - 2i) \sum_{q=1}^{k} \sum_{j=0}^{n_q} \eta_{ij}^{pq} v_j^q &= (n_p - 2i)[h, v_i^p] = [h, [v_i^p, h]] \\
&= [[h, v_i^p], h] = [\sum_{q=1}^{k} \sum_{j=0}^{n_q} \eta_{ij}^{pq} v_j^q, h] = \sum_{q=1}^{k} \sum_{j=0}^{n_q} (n_q - 2j) \eta_{ij}^{pq} v_j^q.
\end{aligned}$$

Thus $\eta_{ij}^{pq} = 0$ unless $j = \frac{1}{2}(n_q - n_p) + i$. Denote by $\eta_i^{pq} := \eta_{i,i-l_{pq}}^{pq}$.

From $[f, [m, h]] = [[f, m], h] - 2[f, m]$, as above we obtain

$$(n_p - 2i) \sum_{q=1}^{k} \sum_{j=0}^{n_q} \phi_{ij}^{pq} v_j^q = (n_p - 2i)[f, v_i^p] = [f, [v_i^p, h]] = \sum_{q=1}^{k} \sum_{j=0}^{n_q} (n_q - 2j - 2) \phi_{ij}^{pq} v_j^q.$$

Therefore $\phi_{ij}^{pq} = 0$ unless $j = \frac{1}{2}(n_q - n_p) + i + 1$. Denote by $\phi_i^{pq} := \phi_{i,i+1-l_{pq}}^{pq}$

From $[e,[m,h]] = [[e,m],h] - 2[e,m]$ we get

$$(n_p - 2i)\sum_{q=1}^{k}\sum_{j=0}^{n_q}\epsilon_{ij}^{pq}v_j^q = \sum_{q=1}^{k}\sum_{j=0}^{n_q}(n_q - 2j - 2)\epsilon_{ij}^{pq}v_j^q.$$

Hence, $\epsilon_{ij}^{pq} = 0$ unless $j = \frac{1}{2}(n_q - n_p) + i - 1$ and denote by $\epsilon_i^{pq} := \epsilon_{i,i-1-l_{pq}}^{pq}$. $\square$

The next proposition is the main tool in partially reducing the general case to the case $k = 2$.

Proposition 3.2. *For any $x \in \mathfrak{sl}_2$ and $1 \le i \le j \le k$, the restriction of the left action λ_x on $V(n_i) \oplus V(n_j)$ coincides with the left action described in Propositions 2.7–2.9.*

Proof. Let $x \in \mathfrak{sl}_2$ and for any i, j from $\{1, \dots, k\}$ let us denote by $\pi_{i,j}$ the linear projection from M to $V(n_i) \oplus V(n_j)$. Consider $m = v_m^1 + \dots + v_m^k \in \oplus_{i=1}^k V(n_i)$. Using the fact that $\rho_x(V(n_i)) \subseteq V(n_i)$ for all $1 \le i \le k$ we have

$$\pi_{ij}(\rho_x(m)) = \rho_x(v_m^i) + \rho_x(v_m^j) = \rho_x(v_m^i + v_m^j) = \rho_x(\pi_{ij}(m)).$$

This implies that π_{ij} and ρ_x commute. Moreover, using equality (2.6) we have

$$\pi_{ij}(\lambda_x \circ \rho_y) = \pi_{ij}(\rho_y \circ \lambda_x - \lambda_{[x,y]}) = \rho_y \circ (\pi_{ij} \circ \lambda_x) - \pi_{ij} \circ \lambda_{[x,y]}.$$

Denote by $\lambda_x^{ij} := \pi_{ij} \circ \lambda_x$. Then $\lambda_x^{ij}\rho_y = \rho_y\lambda_x^{ij} - \lambda_{[x,y]}^{ij}$ that shows that λ_x^{ij} satisfies equation (2.2). However, for fixed i and j linear maps satisfying such condition are studied in [7, Section 2] and are described in Propositions 2.7-2.9. $\square$

Although it is known from Theorem 2.5 that for a bimodule M to be indecomposable the sequence $n_1 \ge n_2 \ge \dots \ge n_k$ must decrease by 2, there is a direct proof why M is decomposable if $n_1 = n_2 = \dots = n_k$.

Proposition 3.3. *Let $M = \oplus_{i=1}^k V_i$, where $\dim V_i = n + 1$. Then bimodule M is decomposable.*

Proof. By Proposition 3.1 for all $1 \le i \le n+1,\ 1 \le p \le k$ we have

$$[h, v_i^p] = \sum_{q=1}^{k}\eta_i^{pq}v_i^q,\ [f, v_i^p] = \sum_{q=1}^{k}\phi_i^{pq}v_{i+1}^q,\ [e, v_i^p] = \sum_{q=1}^{k}\epsilon_i^{pq}v_{i-1}^q.$$

Furthermore, by Proposition 3.2 for $(1 \le s, j \le k)$ and Proposition 2.7 we get the following for all $1 \le i \le n+1,\ 1 \le p \le k$:

$$[h, v_i^p] = (n-2i)\sum_{q=1}^{k}\phi^{pq}v_i^q,\ \ [f, v_i^p] = \sum_{q=1}^{k}\phi^{pq}v_{i+1}^q,\ \ [e, v_i^p] = -i(n-i+1)\sum_{q=1}^{k}\phi^{pq}v_{i+1}^q. \quad (3.1)$$

In the matrix form, we can write the first equality of (3.1) as follows:

$$\begin{bmatrix} [h, v_i^1] \\ [h, v_i^2] \\ \vdots \\ [h, v_i^k] \end{bmatrix}^T = (n-2i)\left(\begin{bmatrix} \phi^{11} & \phi^{12} & \dots & \phi^{1k} \\ \phi^{21} & \phi^{22} & \dots & \phi^{2k} \\ \vdots & \vdots & \dots & \vdots \\ \phi^{k1} & \phi^{k2} & \dots & \phi^{kk} \end{bmatrix} \cdot \begin{bmatrix} v_i^1 \\ v_i^2 \\ \vdots \\ v_i^k \end{bmatrix}\right)^T = (n-2i)\cdot[v_i^1\ v_i^2\ \dots\ v_i^k]\Phi^T,$$

where $\Phi = (\phi^{ij})_{1 \le i,j \le k}$ is a matrix. Verifying identity (2.4) for h and v_i^p, $(1 \le p \le k)$ we

$$\begin{bmatrix} 1+\phi^{11} & \phi^{12} & \dots & \phi^{1k} \\ \phi^{21} & 1+\phi^{22} & \dots & \phi^{2k} \\ \vdots & \vdots & \dots & \vdots \\ \phi^{k1} & \phi^{k2} & \dots & 1+\phi^{kk} \end{bmatrix} \cdot \Phi \cdot \begin{bmatrix} v_i^1 \\ v_i^2 \\ \vdots \\ v_i^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence, $(I+\Phi)\Phi = O$ and therefore, Φ is diagonalizable. Let $\vec{x} = \sum_{q=1}^{k} x_q v_i^q \in M$ $(0 \le i \le n)$ be an eigenvector of Φ^T with an eigenvalue λ. Then

$$[h, \vec{x}] = [h, x_1 v_i^1 + x_2 v_i^2 + \ldots + x_k v_i^k] = [[h, v_i^1]\ [h, v_i^2]\ \ldots\ [h, v_i^k]] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} =$$

$$= (n-2i)[v_i^1\ v_i^2\ \ldots\ v_i^k] \cdot \Phi^T \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = (n-2i)[v_i^1\ v_i^2\ \ldots\ v_i^k]\lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = (n-2i)\lambda\vec{x}.$$

Consequently, this implies that $[\mathfrak{sl}_2, V_i] \subseteq V_i$, which means the module M is decomposable. □

The following statement describes all subbimodules of M when all n_i's are different.

Proposition 3.4. *Let N be a subbimodule of M and $n_i \neq n_j$ for all $1 \le i \neq j \le k$. Then N is expressed as $N = V_{n_{i_1}} \oplus V_{n_{i_2}} \oplus \cdots \oplus V_{n_{i_t}}$ for some $1 \le i_1 < i_2 < \cdots < i_t \le k$.*

Proof. Let N be a subbimodule of M and $u = (\alpha_1 v_{p_1}^{i_1} + \ldots) + (\alpha_2 v_{p_2}^{i_2} + \ldots) + \cdots + (\alpha_t v_{p_t}^{i_t} + \ldots) \in N$ with $\alpha_1\alpha_2\ldots\alpha_t \neq 0$. Acting with f from the right $(n_{i_1} - p_1)$-times on u we obtain

$$\alpha_1 v_{n_{i_1}}^{i_1} + (\alpha_2 v_{q_2}^{i_2} + \ldots) + \cdots + (\alpha_t v_{q_t}^{i_t} + \ldots) \in N \tag{3.2}$$

If all the brackets vanish, then $v_{n_{i_1}}^{i_1} \in N$ and acting from the right with e consecutively, one has $V_{n_{i_1}} \subseteq N$. Therefore, $u \mod V_{n_{i_1}} \in N$ and recursively, the process continues.

If some of the brackets are non-zero, apply h from the right to expression (3.2) and add it to expression (3.2) multiplied by n_{i_1}, to deduce

$$(\alpha_2(n_{i_1} + n_{i_2} - 2q_2)v_{q_2}^{i_2} + \ldots) + \cdots + (\alpha_t(n_1 + n_{i_t} - 2q_t)v_{q_t}^{i_t} + \ldots) \in N.$$

Note that due to $n_1 > n_2 > \cdots > n_k$, none of the first coefficients is equal to zero in the brackets that did not vanish in expression (3.2). Hence, we reduce the number of components to one less and recursively we obtain $v_t^{i_t} \in N$. Applying e from the right continuously one has $V_{i_t} \subseteq N$. Therefore, $u \mod V_{i_p} \in N$ and applying the arguments recursively from the start we are done. □

Note that if $n_i = n_j$ for some i and j, the result of Proposition 3.4 is not true (cf. there are two subbimodules constructed in Case 1 of the proof of [7, Proposition 3.1]).

Theorem 3.5. *Let M be an $\mathfrak{sl}_2$-bimodule and as a right $\mathfrak{sl}_2$-module let it decompose as $M = V(n_1) \oplus V(n_2) \oplus \cdots \oplus V(n_k)$, where $V(n_i)$ are simple $\mathfrak{sl}_2$-modules of Theorem 2.4 with base $\{v_0^i, \ldots, v_{n_i}^i\}$, $1 \le i \le k$ and $n_1 \ge n_2 \ge \cdots \ge n_k$. Then M is an indecomposable Leibniz $\mathfrak{sl}_2$-bimodule only if $n_i - n_{i+1} = 2$ for all $1 \le i \le k-1$. Moreover, up to $\mathfrak{sl}_2$-bimodule isomorphism there are exactly two indecomposable $\mathfrak{sl}_2$-bimodules. The non-zero brackets of the left action is either*

$$\begin{aligned}
&[h, v_i^{2p}] = 2(n-2p-i+3)v_{i+1}^{2p-1} - (n-2p-2i+2)v_{i+1}^{2p} - 2iv_{i-1}^{2p+1},\\
&[f, v_i^{2p}] = v_{i+2}^{2p-1} - v_{i+1}^{2p} + v_i^{2p+1},\\
&[e, v_i^{2p}] = (n-2p-i+3)((n-2p-i+4)v_i^{2p-1} + iv_{i-1}^{2p}) + i(i-1)v_{i-2}^{2p+1},
\end{aligned}$$

for all $0 \le p \le k/2$ *or*

$$
\begin{aligned}
&[h, v_i^1] = -(n-2i)v_i^1 - 2iv_{i-1}^2,\\
&[f, v_i^1] = -v_{i+1}^1 + v_i^2,\\
&[e, v_i^1] = i(n-i+1)v_{i-1}^1 + i(i-1)v_{i-2}^2,\\
&[h, v_i^{2p+1}] = (n-4p-i+1)v_{i+1}^{2p} - (n-4p-2i)v_{i+1}^{2p+1} - 2iv_{i-1}^{2p+2},\\
&[f, v_i^{2p+1}] = v_{i+2}^{2p} - v_{i+1}^{2p+1} + v_i^{2p+2},\\
&[e, v_i^{2p+1}] = (n-4p-i+1)((n-4p-i+2)v_i^{2p} + iv_{i-1}^{2p+1} + i(i-1)v_{i-2}^{2p+2},
\end{aligned}
$$

for all $1 \le p \le (k-1)/2$, *where* $n = n_1$.

Proof. By Theorem 2.5 it is clear that the sequence $\{n_i \mid 1 \le i \le k\}$ must decrease by two. Let us denote by $n = n_1$ and for the sake of convenience, denote by $V_i = V(n-2i+2) = \{v_0^i, v_1^i, \dots, v_{n-2i+2}^i\},\ 1 \le i \le k$. First we use Proposition 3.2 for pair $(j, j+1)$ for all $1 \le j \le k-1$ and Proposition 2.8, then we use Proposition 3.2 for pairs $(j, s),\ 1 \le j \le k-2,\ j+2 \le s \le k$ and Proposition 2.9 to obtain the following:

$$
\begin{aligned}
&[h, v_i^1] = (n-2i)\phi_{1,1}v_i^1 - 2i\phi_{1,2}v_{i-1}^2, \qquad 0 \le i \le n,\\
&[f, v_i^1] = \phi_{1,1}v_{i+1}^1 + \phi_{1,2}v_i^2, \qquad 0 \le i \le n-1,\\
&[e, v_i^1] = -i(n-i+1)\phi_{1,1}v_{i-1}^1 + i(i-1)\phi_{1,2}v_{i-2}^2, \qquad 1 \le i \le n,
\end{aligned}
$$

$$
\begin{aligned}
&2 \le j \le k-1,\ 0 \le i \le n-2j+2:\\
&[h, v_i^j] = 2(n-2j-i+3)\phi_{j,j-1}v_{i+1}^{j-1} + (n-2j-2i+2)\phi_{j,j}v_i^j - 2i\phi_{j,j+1}v_{i-1}^{j+1},\\
&[f, v_i^j] = \phi_{j,j-1}v_{i+2}^{j-1} + \phi_{j,j}v_{i+1}^j + \phi_{j,j+1}v_i^{j+1},\\
&[e, v_i^j] = (n-2j+3-i)((n-2j+4-i)\phi_{j,j-1}v_i^{j-1} - i\phi_{j,j}v_{i-1}^j) + i(i-1)\phi_{j,j+1}v_{i-2}^{j+1},
\end{aligned}
$$

$$
\begin{aligned}
&0 \le i \le n-2k+2:\\
&[h, v_i^k] = 2(n-2k+3-i)\phi_{k,k-1}v_{i+1}^{k-1} + (n-2k+2-2i)\phi_{k,k}v_i^k,\\
&[f, v_i^k] = \phi_{k,k-1}v_{i+2}^{k-1} + \phi_{k,k}v_{i+1}^k,\\
&[e, v_i^k] = (n-2k-i+3)((n-2k+4-i)\phi_{k,k-1}v_i^{k-1} - i\phi_{k,k}v_{i-1}^k.
\end{aligned}
$$

Consider identity (2.4) for corresponding triples:

- For (f, v_0^1, f) we have

$$(1+\phi_{1,1}+\phi_{2,2})\phi_{1,2} = 0, \tag{3.3}$$
$$(1+\phi_{1,1})\phi_{1,1} + \phi_{1,2}\phi_{2,1} = 0, \tag{3.4}$$
$$\phi_{1,2}\phi_{2,3} = 0. \tag{3.5}$$

- For (f, v_0^1, h) we get

$$(1+\phi_{1,1})\phi_{1,2} = 0, \tag{3.6}$$
$$(1+\phi_{1,1})\phi_{1,1} = 0. \tag{3.7}$$

- For $(f, v_0^j, f),\ 2 \le j \le k-1$ we obtain

$$(1+\phi_{j-1,j-1}+\phi_{j,j})\phi_{j,j-1} = 0, \tag{3.8}$$
$$(1+\phi_{j,j})\phi_{j,j} + \phi_{j,j-1}\phi_{j-1,j} + \phi_{j,j+1}\phi_{j+1,j} = 0, \tag{3.9}$$
$$(1+\phi_{j,j}+\phi_{j+1,j+1})\phi_{j,j+1} = 0. \tag{3.10}$$

- For $(f, v_0^j, h),\ 2 \le j \le k-1$ we have

$$(1+\phi_{j,j})\phi_{j,j+1} = 0. \tag{3.11}$$

• For $(f, v_0^j, e),\ 2 \le j \le k-1$ we get

$$\phi_{j,j-1}\phi_{j-1,j-2} = \phi_{j,j-1}\phi_{j-1,j-1} = \phi_{j,j-1}\phi_{j-1,j} = 0. \tag{3.12}$$

Suppose k is odd and consider the following cases (the case when k is an even is carried out analogously).

Case 1. Let $\phi_{1,1} = 0$. Then by (3.6) we have $\phi_{1,2} = 0$, hence $[\mathfrak{sl}_2, V_1] = 0$.

If $\phi_{2,1} = \phi_{2,3} = 0$, then bimodule M is decomposable. Thus $\phi_{2,1} \neq$ and $\phi_{2,3} \neq 0$. Hence, from (3.8) and (3.12) we get $\phi_{2,2} = -1$ and $\phi_{3,2} = 0$, respectively. Since $\phi_{2,3} \neq 0$, then from equality (3.10) we have $\phi_{3,3} = 0$, hence from (3.11) we obtain $\phi_{3,4} = 0$. Thus $[\mathfrak{sl}_2, V_3] = 0$.

Let $\phi_{4,3} \neq 0,\ \phi_{4,5} \neq 0$, otherwise M is decomposable. Then equalities (3.8) and (3.9) imply $\phi_{4,4} = -1$ and $\phi_{5,5} = 0$. Hence by (3.11) and (3.12) we obtain $\phi_{5,6} = 0$ and $\phi_{5,4} = 0$, correspondingly. This means that $[\mathfrak{sl}_2, V_5] = 0$. Continuing this process we will get the following:

$$\begin{aligned}
&[f, v_i^1] = 0,\\
&[f, v_i^2] = \phi_{2,1}v_{i+2}^1 - v_{i+1}^2 + \phi_{2,3}v_i^3,\\
&[f, v_i^3] = 0,\\
&[f, v_i^4] = \phi_{4,3}v_{i+2}^3 - v_{i+1}^4 + \phi_{4,5}v_i^5,\\
&[f, v_i^5] = 0,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&[f, v_i^{2p}] = \phi_{2p,2p-1}v_{i+2}^{2p-1} - v_{i+1}^{2p} + \phi_{2p,2p+1}v_i^{2p+1},\\
&[f, v_i^{2p+1}] = 0,
\end{aligned}$$

where $6 \le p \le \frac{k-1}{2}$. Make a basis change

$$(v_i^1)' = v_i^1,\ \ (v_i^2)' = \frac{1}{\phi_{2,1}}v_i^2,\ \ (v_i^3)' = \frac{\phi_{2,3}}{\phi_{2,1}}v_i^3,\ \ (v_i^4)' = \frac{\phi_{2,3}}{\phi_{2,1}\phi_{4,3}}v_i^4,\ \ (v_i^5)' = \frac{\phi_{2,3}\phi_{4,5}}{\phi_{2,1}\phi_{4,3}}v_i^5,$$
$$(v_i^6)' = \frac{\phi_{2,3}\phi_{4,5}}{\phi_{2,1}\phi_{4,3}\phi_{6,5}}v_i^6, \dots, (v_i^{2p})' = \frac{\phi_{2,3}\phi_{4,5}\cdots\phi_{2p-2,2p-1}}{\phi_{2,1}\phi_{4,3}\cdots\phi_{2p-2,2p-3}\phi_{2p,2p-1}}v_i^{2p},$$
$$(v_i^{2p+1})' = \frac{\phi_{2,3}\phi_{4,5}\cdots\phi_{2p,2p+1}}{\phi_{2,1}\phi_{4,3}\cdots\phi_{2p,2p-1}}v_i^{2p+1}$$

to obtain the following

$$\begin{aligned}
&[f, v_i^1] = 0,\\
&[f, v_i^2] = v_{i+2}^1 - v_{i+1}^2 + v_i^3,\\
&[f, v_i^3] = 0,\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&[f, v_i^{2p}] = v_{i+2}^{2p-1} - v_{i+1}^{2p} + v_i^{2p+1},\\
&[f, v_i^{2p+1}] = 0.
\end{aligned}$$

Thus for all $1 \le p \le \dfrac{k-1}{2}$ we obtain

$$\begin{aligned}
&[h, v_i^{2p}] = 2(n-2p-i+3)v_{i+1}^{2p-1} - (n-2p-2i+2)v_{i+1}^{2p} - 2iv_{i-1}^{2p+1}, \quad 0 \le i \le n-4p+2,\\
&[f, v_i^{2p}] = v_{i+2}^{2p-1} - v_{i+1}^{2p} + v_i^{2p+1}, \quad 0 \le i \le n-4p+2,\\
&[e, v_i^{2p}] = (n-2p-i+3)((n-2p-i+4)v_i^{2p-1} + iv_{i-1}^{2p}) + i(i-1)v_{i-2}^{2p+1}, \quad 0 \le i \le n-4p+2,
\end{aligned}$$

$$\begin{aligned}
&[h, v_i^{2p+1}] = 0, \quad 0 \le i \le n-4p,\\
&[f, v_i^{2p+1}] = 0, \quad 0 \le i \le n-4p,\\
&[e, v_i^{2p+1}] = 0, \quad 0 \le i \le n-4p.
\end{aligned}$$

Using Proposition 3.4 it is easy to see that M is indecomposable.

Case 2. Let $\phi_{1,1} \neq 0$. Then by (3.7) we have $\phi_{1,1} = -1$. Hence in (3.12) we get $\phi_{2,1} = 0$. If $\phi_{1,2} = 0$, then bimodule M is decomposable. So we may assume that $\phi_{1,2} \neq 0$. Then by equations (3.3) and (3.5) one has $\phi_{2,2} = 0$ and $\phi_{2,3} = 0$. Hence $[\mathfrak{sl}_2, V_2] = 0$. Continuing a similar reasoning as in the Case 1, we obtain for all $1 \leq p \leq \dfrac{k-1}{2}$ the following:

$$[h, v_i^1] = -(n-2i)v_i^1 - 2iv_{i-1}^2, \quad 0 \leq i \leq n,$$
$$[f, v_i^1] = -v_{i+1}^1 + v_i^2, \quad 0 \leq i \leq n,$$
$$[e, v_i^1] = i(n-i+1)v_{i-1}^1 + i(i-1)v_{i-2}^2, \quad 0 \leq i \leq n,$$

$$[h, v_i^{2p}] = 0, \quad 0 \leq i \leq n-4p+2,$$
$$[f, v_i^{2p}] = 0, \quad 0 \leq i \leq n-4p+2,$$
$$[e, v_i^{2p}] = 0, \quad 0 \leq i \leq n-4p+2,$$

$$[h, v_i^{2p+1}] = (n-4p-i+1)v_{i+1}^{2p} - (n-4p-2i)v_{i+1}^{2p+1} - 2iv_{i-1}^{2p+2}, \quad 0 \leq i \leq n-4p,$$
$$[f, v_i^{2p+1}] = v_{i+2}^{2p} - v_{i+1}^{2p+1} + v_i^{2p+2}, \quad 0 \leq i \leq n-4p,$$
$$[e, v_i^{2p+1}] = (n-4p-i+1)((n-4p-i+2)v_i^{2p} + iv_{i-1}^{2p+1} + i(i-1)v_{i-2}^{2p+2}, \quad 0 \leq i \leq n-4p,$$

Once again, indecomposability is proved by using Proposition 3.4. $\square$

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