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Disjunctive Total Domination of Some Shadow Distance Graphs

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Abstract

Let G be a graph having vertex set $V(G)$. For $S \subseteq V(G)$, if each vertex is adjacent to a vertex in S or has at least two vertices in S at distance two from it, then the set S is a disjunctive total dominating set of G. The disjunctive total domination number is the minimum cardinality of such a set. In this work, we discuss the disjunctive total domination of shadow distance graphs of some graphs such as cycle, path, star, complete bipartite and wheel graphs.

1. Introduction

Domination in graphs [1] has received considerable attention in graph theory due to the various applications for real world problems such as the chess problem, communication network problems, location of radar stations, routing and coding theory [2]-[4]. There are several variations of domination; one of which is total domination [5]. Since implementations of dominating and total dominating sets in modern networks are expensive, some restrictions are added to them. Then Henning and Naicker [6] defined the disjunctive total domination as a relaxation of total domination. For a set $S \subseteq V(G)$, if each vertex is adjacent to a vertex in S or has at least two vertices in S at distance two from it, then the set S is a *disjunctive total dominating set*, briefly DTD-set, of G. When a vertex u satisfies one of these two conditions, it is known that u is *disjunctively totally dominated*, briefly DT-dominated, by vertices of S. Furthermore, when u satisfies the first condition (the second condition, respectively), it is known that u is *totally dominated* (*disjunctively dominated*, respectively) by vertices of S. The disjunctive total domination number, $\gamma_t^d(G)$, is the minimum cardinality of a DTD-set in G. A DTD-set which gives the value $\gamma_t^d(G)$ is called $\gamma_t^d(G)$-set. This parameter is studied on grids, trees, permutation graphs, claw-free graphs and it is applied on some graph modifications such as bondage and subdivision [6]-[12]. This paper is about disjunctive total domination number of shadow distance graph of some special graphs.

Let G be a graph having vertex set $V(G)$ and edge set $E(G)$. For two vertices u and v if there is an edge joining them, then they are adjacent (or neighbors). The *distance* $d_G(u,v)$ between u and v is the length of the shortest path joining them in G. The greatest distance between any pair of vertices of G is the *diameter* of G and denoted by $diam(G)$. We follow [1] for graph theory terminology and notation which are not defined here for simplicity.

The distance graph [13] $D(G,D_s)$ of G has vertex set $V(G)$ and two vertices u and v are neighbors in $D(G,D_s)$ if $d(u,v) \in D_s$, in which D is the set of all distances between distinct pairs of vertices in G and D_s is a subset of D. The shadow graph $D_2(G)$ [14] of a connected graph G is obtained by taking two copies of G and joining each vertex u in the first copy to the neighbors of the corresponding vertex v in the second copy.

The shadow distance graph $D_{sd}(G,D_s)$ of a connected graph G is defined by Kumar and Muralli [15] and is obtained from G

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with the following properties:

(i) The graph $D_{sd}(G,D_s)$ consists of two copies of G say G itself and G'.
(ii) For $v \in V(G)$, the corresponding vertex is denoted by $v' \in V(G')$.
(iii) $V(D_{sd}(G,D_s)) = V(G) \cup V(G')$.
(iv) $E(D_{sd}(G,D_s)) = E(G) \cup E(G') \cup E_{ds}$, in which E_{ds} is the set of all edges between two distinct vertices $v \in V(G)$ and $w' \in V(G')$ satisfying $d(v,w) \in D_s$ in G.

If $D_s = \{1\}$, then this gives the definition of shadow graph $D_2(G)$. The shadow graph $D_2(P_6)$ and shadow distance graphs $D_{sd}(P_6,\{2\})$, $D_{sd}(P_6,\{3\})$ are shown in Figure 1.1.

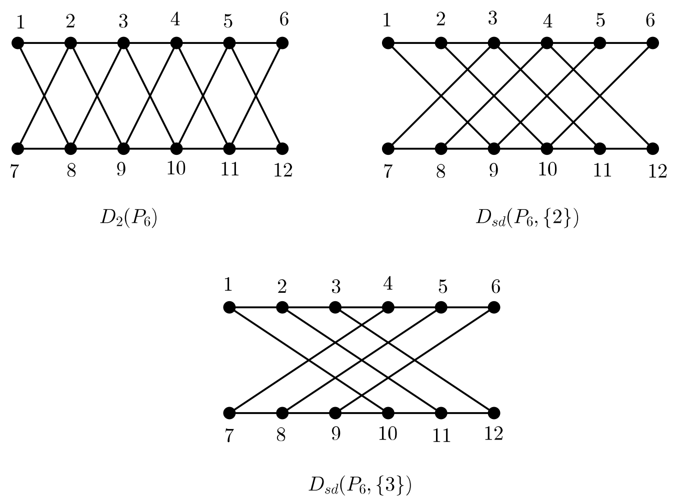

Figure 1.1: The shadow and shadow distance graphs of a path P_6

Now, we make use of the following known theorems in our results.

Theorem 1.1. *[6] Let G be a cycle with $n \geq 3$. Then $\gamma_t^d(G) = 2n/5$ when $n \equiv 0$ (mod 5) and $\gamma_t^d(G) = \lceil 2(n+1)/5 \rceil$ otherwise.*

Observation 1.2. *[11] If $diam(G) \in \{1,2\}$ for a connected graph G having at least two vertices, then $\gamma_t^d(G) = 2$.*

2. Disjunctive total domination of shadow distance graphs

We, in this section, determine the disjunctive total domination number of shadow distance graph of some special graphs such as cycle, path, star, complete bipartite and wheel graphs. Throughout the paper, we will label vertices of $D_2(G)$ and $D_{sd}(G,D_s)$ for $G \ncong W_{1,n}, K_{r,s}$ as the vertices in the first copy of G by $1,2,...,n$ and the vertices in the second copy of G by $n+1, n+2, ..., 2n$ starting from the left.

Theorem 2.1. *If $D_2(C_n)$ is a shadow graph of a cycle with $n \geq 3$, then*

$$\gamma_t^d(D_2(C_n)) = \begin{cases} \lceil \frac{2n}{5} \rceil + 1, & \text{if } n \equiv 2 \text{ (mod 5)} \\ \lceil \frac{2n}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proof. We first establish the upper bound for $\gamma_t^d(D_2(C_n))$. Let

$$S = \left\{ 5i+1 \mid 0 \leq i \leq \left\lceil \frac{n}{5} \right\rceil - 1 \right\} \cup \left\{ n+5i+2 \mid 0 \leq i \leq \left\lceil \frac{n-1}{5} \right\rceil - 1 \right\}.$$

In all cases of n based on mod 5, the set S is a DTD-set of $D_2(C_n)$. Thus, if $n \equiv 2$ (mod 5), then $|S| = \lceil \frac{2n}{5} \rceil + 1$ and for other cases $|S| = \lceil \frac{2n}{5} \rceil$. Therefore,

$$\gamma_t^d(D_2(C_n)) \leq |S| = \begin{cases} \lceil \frac{2n}{5} \rceil + 1, & \text{if } n \equiv 2 \text{ (mod 5)} \\ \lceil \frac{2n}{5} \rceil, & \text{otherwise.} \end{cases}$$

Now, we will prove the reverse inequality. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ is a γ_t^d-set of $D_2(C_n)$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$

and $n+1 \le v_j \le 2n$ for $j \in \{m+1, m+2, \ldots, t\}$. Let $f_x = v_{x+1} - v_x$ for $x \in \{1,2,\ldots,t-1\}$ with $x \ne m$. We must prove $f_x \le 5$ for each $x \in \{1,2,\ldots,t-1\}$ provided that $x \ne m$.

Let us suppose that $f_x \ge 6$ for every x. We claim that $f_x = 6$ for some $x \in \{1,2,\ldots,t-1\}$ with $x \ne m$. In accordance with this claim, we construct the set

$$\left\{6i+1 \mid 0 \le i \le \left\lceil \frac{n}{6} \right\rceil - 1\right\} \cup \left\{n+6i+2 \mid 0 \le i \le \left\lceil \frac{n-1}{6} \right\rceil - 1\right\}.$$

However, some vertices, i.e. vertices 4 and 5 are not DT-dominated by this set. Thus, it is needed to add some new vertices. This makes $f_x < 6$ for some x, which contradicts our claim. Therefore, $f_x \le 5$ for all $x \in \{1,2,\ldots,t-1\}$ with $x \ne m$. Thus, it is clear that $\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 5(t-2)$. This yields

$$5\left(\left\lceil \frac{n}{5} \right\rceil - 1\right) + 5\left(\left\lceil \frac{n-1}{5} \right\rceil - 1\right) = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 5(t-2).$$

Therefore, we have $|T| = t \ge \lceil \frac{2n}{5} \rceil + 1$ for $n \equiv 2 \pmod 5$ and $|T| = t \ge \lceil \frac{2n}{5} \rceil$ for the other cases of n. The proof is completed by combining the lower and upper bounds for $\gamma_t^d(D_2(C_n))$. □

Theorem 2.2. *If $D_2(P_n)$ is a shadow graph of a path with $n \ge 3$, then*

$$\gamma_t^d(D_2(P_n)) = \begin{cases} \lceil \frac{2n+2}{5} \rceil + 1, & \text{if } n \equiv 1 \text{ (mod 5)} \\ \lceil \frac{2n+2}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound on $\gamma_t^d(D_2(P_n))$, let

$$S = \left\{5i+3 \mid 0 \le i \le \left\lceil \frac{n-2}{5} \right\rceil - 1\right\} \cup \left\{n+5i+2 \mid 0 \le i \le \left\lceil \frac{n-1}{5} \right\rceil - 1\right\}.$$

If $n \equiv 0,2 \pmod 5$, then let $S' = S \cup \{n-1\}$; if $n \equiv 1 \pmod 5$, then let $S' = S \cup \{n, 2n-1\}$ and if $n \equiv 3,4 \pmod 5$, then let $S' = S$. The set S' is a DTD-set of $D_2(P_n)$ in all cases. Thus, if $n \equiv 1 \pmod 5$, then $\gamma_t^d(D_2(P_n)) \le |S'| = \lceil \frac{2n+2}{5} \rceil + 1$ and for other cases $\gamma_t^d(D_2(P_n)) \le |S'| = \lceil \frac{2n+2}{5} \rceil$.

We now prove the lower bound on $\gamma_t^d(D_2(P_n))$. Assume that $T = \{v_1, v_2, \ldots, v_i, \ldots, v_j, \ldots, v_t\}$ is a γ_t^d-set of $D_2(P_n)$ with $v_1 < v_2 < \ldots < v_i < \ldots < v_m < v_{m+1} < \ldots < v_j < \ldots < v_t$, where v_i and v_j are any positive integers such that $1 \le v_i \le n$ for $i \in \{1,2,\ldots,m\}$ and $n+1 \le v_j \le 2n$ for $j \in \{m+1, m+2, \ldots, t\}$. Let $f_x = v_{x+1} - v_x$ for $x \in \{1,2,\ldots,t-1\}$ with $x \ne m$. As similar as the proof of Theorem 2.1 we conclude $f_x \le 5$ for each $x \in \{1,2,\ldots,t-1\}$ with $x \ne m$. This yields

$$\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 5(t-2).$$

Since $v_1 = 3$ and $v_{m+1} = n+2$ in all cases of n, it follows $\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x = v_m + v_t - (n+5)$.

If $n \equiv 0 \pmod 5$, then $v_t = 2n-3$ and $v_m = n-1$. Thus,

$$2n-9 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 5(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+1}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \ge \lceil \frac{2n+2}{5} \rceil$.

If $n \equiv 1,3 \pmod 5$, then $v_t = 2n-1$ and $v_m = n$. Thus,

$$2n-6 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 5(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+4}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \ge \lceil \frac{2n+2}{5} \rceil + 1$ for $n \equiv 1 \pmod 5$ and $\gamma_t^d(D_2(P_n)) \ge \lceil \frac{2n+2}{5} \rceil$ for $n \equiv 3 \pmod 5$.

If $n \equiv i \pmod 5$ for $i \in \{2,4\}$, then $v_t = 2n-i+2$ and $v_m = n-1$. Thus,

$$2n-i-4 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \leq 5(t-2),$$

and hence $|T| = t \geq \lceil \frac{2n-i+6}{5} \rceil$. This implies that $\gamma_t^d(D_2(P_n)) \geq \lceil \frac{2n+2}{5} \rceil$.

The proof is completed by combining the lower and upper bounds for $\gamma_t^d(D_2(P_n))$. □

Theorem 2.3. *Let $K_{1,s}, W_{1,n}, K_{r,s}$ denote a star, a wheel and a complete bipartite graph, respectively, and if $G \cong H$, where $H \in \{K_{1,s}, W_n, K_{r,s}\}$, then $\gamma_t^d(D_2(G)) = 2$.*

Proof. Since $diam(D_2(G)) = 2$ for $G \cong H$, where $H \in \{K_{1,s}, W_n, K_{r,s}\}$, the result follows from Observation 1.2. □

Theorem 2.4. *For $n \geq 6$,*

$$\gamma_t^d(D_{sd}(P_n,\{2\})) = \begin{cases} \lceil \frac{3n}{8} \rceil + 1, & \text{if } n \equiv 5 \ (\text{mod } 8) \\ \lceil \frac{3n}{8} \rceil, & \text{otherwise.} \end{cases}$$

Proof. We first establish the upper bound for $\gamma_t^d(D_{sd}(P_n,\{2\}))$. Let

$$S = \left\{ \{8i+3, 8i+5\} \;\middle|\; 0 \leq i \leq \left\lceil \frac{n-4}{8} \right\rceil - 1 \right\} \cup \left\{ n+8i+6 \;\middle|\; 0 \leq i \leq \left\lceil \frac{n-5}{8} \right\rceil - 1 \right\}.$$

If $n \equiv 0,6,7 \pmod 8$, then let $S' = S$; if $n \equiv 1,5 \pmod 8$, then let $S' = S \cup \{2n-1\}$; if $n \equiv 2 \pmod 8$, then let $S' = S \cup \{2n-2\}$; if $n \equiv 3 \pmod 8$, then let $S' = S \cup \{n-2, 2n-1\}$ and if $n \equiv 4 \pmod 8$, then let $S' = S \cup \{n-3, 2n-2\}$. The set S' is a DTD-set of $D_{sd}(P_n,\{2\})$ in all cases. Thus, if $n \equiv 5 \pmod 8$, then $\gamma_t^d(D_{sd}(P_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil + 1$ and for other cases $\gamma_t^d(D_{sd}(P_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil$.

Let T be a γ_t^d-set of $D_{sd}(P_n,\{2\})$ to prove the lower bound. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. Let $f_x = v_{x+2} - v_x$ for $x \in \{1,2,...,m-2\}$ and $f_y = v_{y+1} - v_y$ for $y \in \{m+1, m+2, ..., t-1\}$. We must prove $f_x \leq 8$ for $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for $y \in \{m+1, m+2, ..., t-1\}$. Suppose that at least one inequality is not true. Without loss of generality, let $f_y > 8$ for at least one y. We claim that $f_{m+1} = 9$ for $y = m+1$. In accordance with this claim, one of the set can be constructed is

$$\{n+6\} \cup \left\{ \{8i+3, 8i+5\} \;\middle|\; 0 \leq i \leq \left\lceil \frac{n-4}{8} \right\rceil - 1 \right\} \cup \left\{ n+8i+7 \;\middle|\; 0 \leq i \leq \left\lceil \frac{n}{8} \right\rceil - 2 \right\}.$$

However, all vertices of this set are not DT-dominated. Therefore, $f_x \leq 8$ for each $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for each $y \in \{m+1, m+2, ..., t-1\}$. This yields $\sum\limits_{x=1}^{m-2} f_x + \sum\limits_{y=m+1}^{t-1} f_y \leq 8(t-3)$.

Since $v_1 = 3$, $v_2 = 5$ and $v_{m+1} = n+6$ in all cases of n, it follows $\sum\limits_{x=1}^{m-2} f_x + \sum\limits_{y=m+1}^{t-1} f_y = v_{m-1} + v_m + v_t - (n+14)$.

If $n \equiv i \pmod 8$ for $i \in \{1,2\}$, then $v_{m-1} = n-i-5$, $v_m = n-i-3$ and $v_t = 2n-i$. Thus,

$$3n-3i-22 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3)$$

and hence $|T| = t \geq \lceil \frac{3n-3i+2}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv i \pmod 8$ for $i \in \{3,4\}$, then $v_{m-1} = n-i-3$, $v_m = n-i+1$ and $v_t = 2n-i+2$. Thus,

$$3n-3i-14 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+10}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv i \pmod 8$ for $i \in \{5,6\}$, then $v_{m-1} = n-i+3$, $v_m = n-i+5$ and $v_t = 2n+i-6$. Thus,

$$3n-i-12 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-i+12}{8} \rceil$. This implies that $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil + 1$ for $n \equiv 5 \pmod 8$ and $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$ for $n \equiv 6 \pmod 8$.

Let $n \equiv i \pmod 8$ for $i \in \{0,7\}$. We take $i = 8$ for $n \equiv 0 \pmod 8$. Then $v_{m-1} = n-i+3$, $v_m = n-i+5$ and $v_t = 2n-i+6$. Thus,

$$3n-3i = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+24}{8} \rceil$. This implies $\gamma_t^d(D_{sd}(P_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

Consequently, the proof follows from the lower and upper bounds. □

Theorem 2.5. *For $n \geq 3$,*

$$\gamma_t^d(D_{sd}(C_n,\{2\})) = \begin{cases} \lceil \frac{3n}{8} \rceil + 1, & \text{if } n \equiv 3,4,5 \pmod 8 \\ \lceil \frac{3n}{8} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound on $\gamma_t^d(D_{sd}(C_n,\{2\}))$, let

$$S = \left\{\{8i+1, 8i+3\} \mid 0 \leq i \leq \left\lceil \frac{n-2}{8} \right\rceil - 1\right\} \cup \left\{n+8i+4 \mid 0 \leq i \leq \left\lceil \frac{n-3}{8} \right\rceil - 1\right\}.$$

If $n \equiv 1 \pmod 8$, then let $S' = S \cup \{n\}$; if $n \equiv 2 \pmod 8$, then let $S' = S \cup \{n-1\}$; if $n \equiv 3 \pmod 8$, then let $S' = S \cup \{2n\}$ and otherwise let $S' = S$. The set S' is a DTD-set of $D_{sd}(C_n,\{2\})$ in all cases. Thus, if $n \equiv 3,4,5 \pmod 8$, then $\gamma_t^d(D_{sd}(C_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil + 1$ and for other cases $\gamma_t^d(D_{sd}(C_n,\{2\})) \leq |S'| = \lceil \frac{3n}{8} \rceil$.

Now, we need to prove the lower bound to complete the proof. Let T be a γ_t^d-set of $D_{sd}(C_n,\{2\})$. Assume that $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1, m+2, ..., t\}$. As similar as the proof of Theorem 2.4, we define functions $f_x = v_{x+2} - v_x$ for $x \in \{1,2,...,m-2\}$ and $f_y = v_{y+1} - v_y$ for $y \in \{m+1, m+2, ..., t-1\}$. It is easily seen that $f_x \leq 8$ for each $x \in \{1,2,...,m-2\}$ and $f_y \leq 8$ for each $y \in \{m+1, m+2, ..., t-1\}$ as in the proof of Theorem 2.4. This means that $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3)$.

Since $v_1 = 1$, $v_2 = 3$ and $v_{m+1} = n+4$ in all cases of n, it follows $\sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y = v_{m-1} + v_m + v_t - (n+8)$.

If $n \equiv i \pmod 8$ for $i \in \{1,2\}$, then $v_{m-1} = n-i-5$, $v_m = n-i+1$ and $v_t = 2n-i-4$. Thus,

$$3n-3i-16 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+8}{8} \rceil$. This means that $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$.

If $n \equiv 3 \pmod 8$, then $v_{m-1} = n-2$, $v_m = n$ and $v_t = 2n$. Thus,

$$3n-10 = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n}{8} \rceil + 1$.

Let $n \equiv i \pmod 8$ for $i \in \{0,4,5,6,7\}$. We take $i = 8$ for $n \equiv 0 \pmod 8$. Then $v_{m-1} = n-i+1$, $v_m = n-i+3$ and $v_t = 2n-i+4$. Thus,

$$3n-3i = \sum_{x=1}^{m-2} f_x + \sum_{y=m+1}^{t-1} f_y \leq 8(t-3),$$

and hence $|T| = t \geq \lceil \frac{3n-3i+24}{8} \rceil$. This implies that $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil + 1$ for $n \equiv 4,5 \pmod 8$ and $\gamma_t^d(D_{sd}(C_n,\{2\})) \geq \lceil \frac{3n}{8} \rceil$ for otherwise. □

Theorem 2.6. *For $r \geq 1$ and $s \geq 2$, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) = 3$.*

Proof. Let $V(D_{sd}(K_{r,s},\{2\})) = V(K_{r,s}) \cup V(K'_{r,s})$ be vertex set of $D_{sd}(K_{r,s},\{2\})$, in which $V(K_{r,s}) = \{u_1,u_2,...,u_r,v_1,v_2,...,v_s\}$ and $V(K'_{r,s}) = \{u'_1,u'_2,...,u'_r,v'_1,v'_2,...,v'_s\}$. We first establish the upper bound for $\gamma_t^d(D_{sd}(K_{r,s},\{2\}))$. If $S = \{u_1,u_2,v_1\}$, then the set S is a DTD-set of $D_{sd}(K_{r,s},\{2\})$. Thus, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) \leq 3$.

For the lower bound, let T be a $\gamma_t^d(D_{sd}(K_{r,s},\{2\}))$-set. Suppose that $|T| = 2$, this means that the vertices of T are adjacent. Then we have the following cases.

Case 1. Let $T = \{u_i,v_j\}$ for any $i \in \{1,2,...,r\}$ and $j \in \{1,2,...,s\}$ (The case $T = \{u'_i,v'_j\}$ for any $i \in \{1,2,...,r\}$ and $j \in \{1,2,...,s\}$ is similar). All vertices except u'_i and u'_j are totally dominated by the vertices of T. However, since $d(u'_i,v_j) = 2$ and $d(u'_i,u_i) = 3$, the vertex u'_i is not DT-dominated by the vertices of T.

Case 2. Let $T = \{u_i,u'_j\}$ for any $i,j \in \{1,2,...,r\}$ and $i \neq j$. (The case $T = \{v_i,v'_j\}$ for any $i,j \in \{1,2,...,s\}$ and $i \neq j$ is similar.) Since $d(u_j,u_i) = 2$ and $d(u_j,u'_j) = 3$, the vertex u_j is not DT-dominated by the vertices of T.

Therefore, $\gamma_t^d(D_{sd}(K_{r,s},\{2\})) = |T| \geq 3$, and this concludes the proof. □

Theorem 2.7. *For $n \geq 3$, $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) = 3$.*

Proof. Let $V(D_{sd}(W_{1,n},\{2\})) = V(W_{1,n}) \cup V(W'_{1,n})$ be vertex set of $D_{sd}(W_{1,n},\{2\})$ in which $V(W_{1,n}) = \{c,u_1,u_2,...,u_n\}$ and $V(W'_{1,n}) = \{c',u'_1,u'_2,...,u'_n\}$, where c is the center vertex of $W_{1,n}$. We first establish the upper bound for $\gamma_t^d(D_{sd}(W_{1,n},\{2\}))$. If $S = \{c,u_1,u'_2\}$, then the set S is a DTD-set of $D_{sd}(W_{1,n},\{2\})$. Thus, $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) \leq 3$.

Now, we need to prove the lower bound. Let T be a $\gamma_t^d(D_{sd}(W_{1,n},\{2\}))$-set. Suppose that $|T| = 2$. We have following cases.

Case 1. Let $T = \{c,u_i\}$ for $i \geq 1$. (The case $T = \{c'_1,u'_i\}$ for $i \geq 1$ is similar.) Since $d(u'_i,u_i) = 3$ and $d(c'_1,c_1) = 3$, then vertices c' and u'_i are not DT-dominated.

Case 2. Let $T = \{u_i,u_{i+1}\}$ for $i \in \{1,2,...,n-1\}$. (The case $T = \{u'_i,u'_{i+1}\}$ for $i \in \{1,2,...,n-1\}$ is similar.) Since $d(u'_i,u_i) = 3$ and $d(u'_{i+1},u_{i+1}) = 3$, vertices u'_i and u'_{i+1} are not DT-dominated.

Case 3. Let $T = \{u_i,u'_j\}$ for $j \notin \{i,(i-1)(\text{mod } n),(i+1)(\text{mod } n)\}$. Note that we take $j = n$ when $j = 0$. In this case, since $d(u'_i,u_i) = 3$ and $d(u_j,u'_j) = 3$, vertices u'_i and u_j are not DT-dominated.

In all cases, the assumption is false and $\gamma_t^d(D_{sd}(W_{1,n},\{2\})) = |T| \geq 3$, which completes the proof. □

Theorem 2.8. *For $n \geq 14$,*

$$\gamma_t^d(D_{sd}(P_n,\{3\})) = \begin{cases} \frac{n}{3}+2, & \text{if } n \equiv 0 \ (\text{mod } 6) \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound for $\gamma_t^d(D_{sd}(P_n,\{3\}))$, let $D = \{3\} \cup \{\{6i+5, n+6i+5\} \mid 0 \leq i \leq \lceil \frac{n-4}{6} \rceil - 2\}$. If $n \equiv 0$ (mod 6), then let $S = D \cup \{n-1, 2n-3, 2n-1\}$; if $n \equiv 0,2$ (mod 6), then let $S = D \cup \{n-3, 2n-3, 2n-1\}$; if $n \equiv 1$ (mod 6), then let $S = D \cup \{n-2, 2n, 2n-2\}$; if $n \equiv 3$ (mod 6), then let $S = D \cup \{n-4, 2n-4, 2n-2\}$ and if $n \equiv 4$ (mod 6), then let $S = D \cup \{n-5, n-2, 2n-5, 2n-2\}$ and if $n \equiv 5$ (mod 6), then let $S = D \cup \{n-3, 2n-3\}$. Then the set S is a DTD-set of $D_{sd}(P_n,\{3\})$ in all cases of n. Thus, if $n \equiv 0$ (mod 6), then $\gamma_t^d(D_{sd}(P_n,\{3\})) = |S| \leq \frac{n}{3}+2$ and otherwise $\gamma_t^d(D_{sd}(P_n,\{3\})) = |S| \leq \lceil \frac{n}{3} \rceil + 1$.

Now, we prove the lower bound for $\gamma_t^d(D_{sd}(P_n,\{3\}))$. Let $T = \{v_1,v_2,...,v_i,...,v_j,...,v_t\}$ be a γ_t^d-set of $D_{sd}(P_n,\{3\})$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \leq v_i \leq n$ for $i \in \{1,2,...,m\}$ and $n+1 \leq v_j \leq 2n$ for $j \in \{m+1,m+2,...,t\}$. Assume that $f_1 = v_2 - v_1$ and $f_y = v_{y+1} - v_y$ for $y \in \{2,...,t-1\}$ with $y \neq m$. We will show that $f_1 \leq 2$ and $f_y \leq 6$ for each y. Suppose first that $f_1 \geq 3$. In order to DT-dominate v_1, the condition $f_y \leq 6$ must be hold for at least one y. Thus, the set

$$T' = \{2,5,n+4\} \cup \{\{6i+8, n+6i+8\} \mid 0 \leq i \leq \lceil \frac{n-7}{6} \rceil - 2\}$$

is constructed. However, this contradicts with our upper bound. For example, if $n \equiv 1$ (mod 6), then $T = T' \cup \{n-5, n-2, 2n-5, 2n-2\}$ and $|T| = \frac{n+7}{3}$, a contradiction.

Suppose now that $f_1 \le 2$ and $f_y \ge 7$ for at least one y. Then the set

$$T' = \{3,5,n+5,n+7,12,n+12\} \cup \{\{6i+15,n+6i+15\} \mid 0 \le i \le \lceil \frac{n-14}{6} \rceil - 1\}$$

is constructed. However, this contradicts with our upper bound. For example, if $n \equiv 1 \pmod 6$, then $T = T' \cup \{2n-2\}$ and $|T| = \frac{n+8}{3}$, a contradiction.

Therefore, $f_1 \le 2$ and $f_y \le 6$ for each $y \in \{2,...,t-1\}$. This yields $f_1 + \sum_{y=2}^{t-1} f_y \le 2+6(t-3)$. Since $v_2 = 5$ and $v_{m+1} = n+5$, it follows $2 + \sum_{y=2}^{t-1} f_y = 2 + v_m - v_1 + v_t - v_{m+1} = v_m + v_t - (n+8)$.

If $n \equiv 0 \pmod 6$, then $v_m = n-1$ and $v_t = 2n-1$. This yields

$$2n-10 = 2 + \sum_{y=2}^{t-1} f_y \le 2+6(t-3),$$

and hence $|T| = t \ge \lceil \frac{2n+6}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{1,2,3\}$, then $v_m = n-i-1$ and $v_t = 2n-i+1$. This yields

$$2n-2i-8 = 2 + \sum_{y=2}^{t-1} f_y \le 2+6(t-3),$$

and hence $|T| = t \ge \lceil \frac{2n-2i+8}{6} \rceil$.

If $n \equiv 4 \pmod 6$, then $v_m = n-2$ and $v_t = 2n-2$. This yields

$$12(\lceil \frac{n-4}{6} \rceil - 1) + 8 = 2 + \sum_{y=2}^{t-1} f_y \le 8+6(t-5),$$

and hence $|T| = t \ge \frac{n+5}{3}$.

If $n \equiv 5 \pmod 6$, then $v_m = n-3$ and $v_t = 2n-3$. This yields

$$2n-14 = 2 + \sum_{y=2}^{t-1} f_y \le 2+6(t-3),$$

and hence $|T| = t \ge \lceil \frac{2n+4}{6} \rceil$.

Consequently, if $n \equiv 0 \pmod 6$, then $\gamma_t^d(D_{sd}(P_n,\{3\})) \ge \frac{n}{3}+2$ and otherwise $\gamma_t^d(D_{sd}(P_n,\{3\})) \ge \lceil \frac{n}{3} \rceil + 1$. This completes the proof. □

Since $D_{sd}(P_n,\{3\}) \cong C_8$ for $n=4$, by Theorem 1.1 we have $\gamma_t^d(D_{sd}(P_4,\{3\})) = 4$. Therefore, we give the result of $\gamma_t^d(D_{sd}(P_n,\{3\}))$ for $5 \le n \le 13$ in Table 1.

n	5	6	7	8	9	10	11	12	13
$\gamma_t^d(D_{sd}(P_n,\{3\}))$	4	4	4	4	4	4	4	5	5

Table 1: The values of $\gamma_t^d(D_{sd}(P_n,\{3\}))$ for $5 \le n \le 13$

Theorem 2.9. *For $n \ge 15$,*

$$\gamma_t^d(D_{sd}(C_n,\{3\})) = \begin{cases} \frac{n+5}{3}, & \text{if } n \equiv 1 \ (mod\ 6) \\ \lceil \frac{n+2}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. For the upper bound for $\gamma_t^d(D_{sd}(C_n,\{3\}))$, let $D = \{\{6i+3,n+6i+3\} \mid 0 \le i \le \lceil \frac{n-2}{6} \rceil - 2\}$. If $n \equiv 0 \pmod 6$, then let $S = D \cup \{n-3,2n-3,2n-1\}$; if $n \equiv 1 \pmod 6$, then let $S = D \cup \{n-4,2n-4,n,2n\}$; if $n \equiv 2 \pmod 6$, then let $S = D \cup \{n-5,2n-5,n-1,2n-1\}$; if $n \equiv 3 \pmod 6$, then let $S = D \cup \{n,2n\}$; if $n \equiv 4 \pmod 6$, then let $S = D \cup \{n-1,2n-1\}$ and if $n \equiv 5 \pmod 6$, then let $S = D \cup \{n,n-3,2n-6\}$. The set S is a DTD-set of $D_{sd}(C_n,\{3\})$ in all cases of n. Thus, if $n \equiv 1 \pmod 6$, then $\gamma_t^d(D_{sd}(C_n,\{3\})) = |S| \le \frac{n+5}{3}$ and otherwise $\gamma_t^d(D_{sd}(C_n,\{3\})) = |S| \le \lceil \frac{n+2}{3} \rceil$.

We need to prove the opposite inequality to complete the proof. Let $T = \{v_1, v_2, ..., v_i, ..., v_j, ..., v_t\}$ be a γ_t^d-set of $D_{sd}(C_n, \{3\})$ with $v_1 < v_2 < ... < v_i < ... < v_m < v_{m+1} < ... < v_j < ... < v_t$, where v_i and v_j are any positive integers such that $1 \le v_i \le n$ for $i \in \{1, 2, ..., m\}$ and $n+1 \le v_j \le 2n$ for $j \in \{m+1, m+2, ..., t\}$. Assume that $f_x = v_{x+1} - v_x$ for $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. As similar as the proof of Theorem 2.1 we can show that $f_x \le 6$ for each $x \in \{1, 2, ..., t-1\}$ with $x \neq m$. This yields

$$\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2).$$

Since $v_1 = 3$ and $v_{m+1} = n+3$ in all cases of n, we have $\sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x = v_m - v_1 + v_t - v_{m+1} = v_m + v_t - (n+6)$.

If $n \equiv 0 \pmod 6$, then $v_m = n-3$ and $v_t = 2n-1$. This yields

$$2n - 10 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+2}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{1, 3\}$, then $v_m = n$ and $v_t = 2n$. This yields

$$2n - 6 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+6}{6} \rceil$.

If $n \equiv i \pmod 6$ for $i \in \{2, 4\}$, then $v_m = n-1$ and $v_t = 2n-1$. This yields

$$2n - 8 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 6(t-2),$$

and hence $|T| = t \ge \lceil \frac{2n+4}{6} \rceil$.

If $n \equiv 5 \pmod 6$, then $v_m = n$ and $v_t = 2n-6$. This yields

$$12\left(\left\lceil \frac{n-2}{6} \right\rceil - 2\right) + 10 = \sum_{x=1}^{m-1} f_x + \sum_{x=m+1}^{t-1} f_x \le 10 + 6(t-5),$$

and hence $|T| = t \ge \frac{n+4}{3}$.

As a consequence, if $n \equiv 1 \pmod 6$, then $\gamma_t^d(D_{sd}(C_n, \{3\})) \ge \frac{n+5}{3}$ and otherwise $\gamma_t^d(D_{sd}(C_n, \{3\})) \ge \lceil \frac{n+2}{3} \rceil$, and this completes the proof. □

For $n = 4$, since diameter of $D_{sd}(C_n, \{3\})$ is two, it is clear that $\gamma_t^d(D_{sd}(C_n, \{3\})) = 2$. For $5 \le n \le 14$, the result is given in Table 2.

n	5	6	7	8	9	10	11	12	13	14
$\gamma_t^d(D_{sd}(C_n, \{3\}))$	3	3	4	4	4	4	4	5	5	5

Table 2: The values of $\gamma_t^d(D_{sd}(C_n, \{3\}))$ for $5 \le n \le 14$

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