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ON THE PERIODICITY OF SOLUTIONS OF A SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we have investigated the periodicity of the well-defined solutions of the system of difference equations

$$u_{n+1}=\frac{u_{n-1}+v_n}{\alpha v_n u_{n-1}-1},\ v_{n+1}=\frac{v_{n-1}+u_n}{\alpha u_n v_{n-1}-1},\ w_{n+1}=\frac{u_n}{v_n}$$

where $u_0,u_{-1},v_0,v_{-1},w_0,w_{-1}\in\mathbb{R}\setminus\{0\}$ and $\alpha>0$.

Keywords: Difference equation; system; solutions; periodicity.

1. Introduction

In recent years, there has been a lot of interest in studying difference equations an their systems [1-24]. One of the reasons for this is a necessity for some techniques which can be used in investigating difference equations and their systems arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics etc. There are many papers with related to the systems of difference equations for example,

In [3] Cinar studied the solutions of the systems of the difference equations

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [2] Camouzis and Papaschinnopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations

$$x_{n+1}=1+\frac{x_n}{y_{n-m}},\ y_{n+1}=1+\frac{y_n}{x_{n-m}}.$$

In [11] Kulenović and Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+y_n},\ y_{n+1}=\frac{c+y_n}{d+z_n},\ z_{n+1}=\frac{e+z_n}{f+x_n}.$$

In [22] Yalcinkaya and Cinar studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}}.$$

In [12] Kurbanli et al. studied the periodicity of solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1}.$$

In [13] Kurbanli et al. studied the behavaior of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In this paper, we investigated the periodicity of the well-defined solutions of the difference equation system

$$u_{n+1}=\frac{u_{n-1}+v_n}{\alpha v_n u_{n-1}-1},\ v_{n+1}=\frac{v_{n-1}+u_n}{\alpha u_n v_{n-1}-1},\ w_{n+1}=\frac{u_n}{v_n} \tag{1}$$

where $u_0,u_{-1},v_0,v_{-1},w_0,w_{-1}\in\mathbb{R}\setminus\{0\}$ and $\alpha>0$. Note that system (1) can be written as

$$x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1},\ z_{n+1}=\frac{x_n}{y_n} \tag{2}$$

by the change of variables $u_n=\frac{x_n}{\sqrt{\alpha}},\ v_n=\frac{y_n}{\sqrt{\alpha}},\ w_n=z_n.$ That's why, we will consider system (2) instead of system (1) for the remaining part of the paper.

2. Main Result

Our main result in this paper is the following:

Theorem 1. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f$ be nonzero arbitrary real numbers and $\{x_n,y_n,z_n\}$ be a solution of system (2). Also, assume that $ad\neq 1$, $bc\neq 1$, $(b+c)\neq 0$ and $(d+a)\neq 0$. Then, all solutions of system (2) are as following:

$$x_n=\begin{cases}\frac{d+a}{ad-1}, & n=6k+1\\ b, & n=6k+2\\ a, & n=6k+3\\ \frac{b+c}{cb-1}, & n=6k+4\\ d, & n=6k+5\\ c, & n=6k+6\end{cases}\quad \text{for } k\in\mathbb{N}_0,$$

$$y_n=\begin{cases}\frac{b+c}{cb-1}, & n=6k+1\\ d, & n=6k+2\\ c, & n=6k+3\\ \frac{d+a}{ad-1}, & n=6k+4\\ b, & n=6k+5\\ a, & n=6k+6\end{cases}\quad \text{for } k\in\mathbb{N}_0,$$

$$z_n = \begin{cases} \dfrac{c}{a}, & n = 6k+1 \\ \dfrac{(d+a)(cb-1)}{(ad-1)(b+c)}, & n = 6k+2 \\ \dfrac{b}{d}, & n = 6k+3 \\ \dfrac{a}{c}, & n = 6k+4 \\ \dfrac{(b+c)(ad-1)}{(cb-1)(d+a)}, & n = 6k+5 \\ \dfrac{d}{b}, & n = 6k+6 \end{cases} \quad \text{for } k \in \mathbb{N}_0.$$

Proof. We prove the theorem by induction for k. If $k = 0$, from system (2) we have

$$x_1 = \frac{x_{-1} + y_0}{y_0 x_{-1} - 1} = \frac{d+a}{ad-1},$$

$$y_1 = \frac{y_{-1} + x_0}{x_0 y_{-1} - 1} = \frac{b+c}{cb-1},$$

$$z_1 = \frac{x_0}{y_0} = \frac{c}{a},$$

$$x_2 = \frac{x_0 + y_1}{y_1 x_0 - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{bc^2 - c + b + c}{bc + c^2 - bc + 1} = b,$$

$$y_2 = \frac{y_0 + x_1}{x_1 y_0 - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2 d - a + d + a}{da + a^2 - ad + 1} = d,$$

$$z_2 = \frac{x_1}{y_1} = \frac{\frac{d+a}{ad-1}}{\frac{b+c}{cb-1}} = \frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_3 = \frac{x_1 + y_2}{y_2 x_1 - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{d + a + ad^2 - d}{d^2 + da - ad + 1} = a,$$

$$y_3 = \frac{y_1 + x_2}{x_2 y_1 - 1} = \frac{\frac{b+c}{cb-1} + b}{b\frac{b+c}{cb-1} - 1} = \frac{b + c + b^2 c - b}{b^2 + bc - cb + 1} = c,$$

$$z_3 = \frac{x_2}{y_2} = \frac{b}{d},$$

$$x_4 = \frac{x_2 + y_3}{y_3 x_2 - 1} = \frac{b+c}{cb-1},$$

$$y_4 = \frac{y_2 + x_3}{x_3 y_2 - 1} = \frac{d+a}{ad-1},$$

$$z_4 = \frac{x_3}{y_3} = \frac{a}{c},$$

$$x_5 = \frac{x_3 + y_4}{y_4 x_3 - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2 d - a + d + a}{da + a^2 - ad + 1} = d,$$

$$y_5 = \frac{y_3 + x_4}{x_4 y_3 - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{bc^2 - c + b + c}{bc + c^2 - cb + 1} = b,$$

$$z_5 = \frac{x_4}{y_4} = \frac{\frac{b+c}{cb-1}}{\frac{d+a}{ad-1}} = \frac{(b+c)(ad-1)}{(cb-1)(d+a)},$$

$$x_6 = \frac{x_4 + y_5}{y_5 x_4 - 1} = \frac{\frac{b+c}{cb-1} + b}{b\frac{b+c}{cb-1} - 1} = \frac{b + c + b^2 c - b}{b^2 + bc - cb + 1} = c,$$

$$y_6 = \frac{y_4 + x_5}{x_5 y_4 - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{d + a + ad^2 - d}{d^2 + da - ad + 1} = a,$$

$$z_6 = \frac{x_5}{y_5} = \frac{d}{b}.$$

Now, suppose that $k > 0$ and that our assumption holds for $k = n-1$. That is;

$$x_{6(n-1)+1} = x_{6n-5} = \frac{d+a}{ad-1}, \quad y_{6(n-1)+1} = y_{6n-5} = \frac{b+c}{cb-1}, \quad z_{6(n-1)+1} = z_{6n-5} == \frac{c}{a},$$

$$x_{6(n-1)+2} = x_{6n-4} = b, \quad y_{6(n-1)+2} = y_{6n-4} = d, \quad z_{6(n-1)+2} = z_{6n-4} = \frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_{6(n-1)+3} = x_{6n-3} = a, \quad y_{6(n-1)+3} = y_{6n-3} = c, \quad z_{6(n-1)+3} = z_{6n-3} = \frac{b}{d},$$

$$x_{6(n-1)+4} = x_{6n-2} = \frac{b+c}{cb-1}, \quad y_{6(n-1)+4} = y_{6n-2} = \frac{d+a}{ad-1}, \quad z_{6(n-1)+4} = z_{6n-2} = \frac{a}{c},$$

$$x_{6(n-1)+5} = x_{6n-1} = d, \quad y_{6(n-1)+5} = y_{6n-1} = b, \quad z_{6(n-1)+5} = z_{6n-1} = \frac{(b+c)(ad-1)}{(cb-1)(d+a)},$$

$$x_{6(n-1)+6} = x_{6n} = c, \quad y_{6(n-1)+6} = y_{6n} = a, \quad z_{6(n-1)+6} = z_{6n} = \frac{d}{b}.$$

From system (2), we have the following for $k = n$:

$$x_{6n+1} = \frac{x_{6n-1} + y_{6n}}{y_{6n}x_{6n-1} - 1} = \frac{d+a}{ad-1},$$

$$y_{6n+1} = \frac{y_{6n-1} + x_{6n}}{x_{6n}y_{6n-1} - 1} = \frac{b+c}{cb-1},$$

$$z_{6n+1} = \frac{x_{6n}}{y_{6n}} = \frac{c}{a},$$

$$x_{6n+2} = \frac{x_{6n} + y_{6n+1}}{y_{6n+1}x_{6n} - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{bc^2 - c + b + c}{bc + c^2 - cb + 1} = b,$$

$$y_{6n+2} = \frac{y_{6n} + x_{6n+1}}{x_{6n+1}y_{6n} - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2d - a + d + a}{ad + a^2 - ad + 1} = d,$$

$$z_{6n+2} = \frac{x_{6n+1}}{y_{6n+1}} = \frac{\frac{d+a}{ad-1}}{\frac{b+c}{cb-1}} = \frac{(d+a)(cb-1)}{(ad-1)(b+c)},$$

$$x_{6n+3} = \frac{x_{6n+1} + y_{6n+2}}{y_{6n+2}x_{6n+1} - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{d + a + ad^2 - d}{d^2 + da - ad + 1} = a,$$

$$y_{6n+3} = \frac{y_{6n+1} + x_{6n+2}}{x_{6n+2}y_{6n+1} - 1} = \frac{\frac{b+c}{cb-1} + b}{b\frac{b+c}{cb-1} - 1} = \frac{b + c + b^2c - b}{b^2 + bc - cb + 1} = c,$$

$$z_{6n+3} = \frac{x_{6n+2}}{y_{6n+2}} = \frac{b}{d},$$

$$x_{6n+4} = \frac{x_{6n+2} + y_{6n+3}}{y_{6n+3}x_{6n+2} - 1} = \frac{b+c}{cb-1},$$

$$y_{6n+4} = \frac{y_{6n+2} + x_{6n+3}}{x_{6n+3}y_{6n+2} - 1} = \frac{d+a}{ad-1},$$

$$z_{6n+4} = \frac{x_{6n+3}}{y_{6n+3}} = \frac{a}{c},$$

$$x_{6n+5} = \frac{x_{6n+3} + y_{6n+4}}{y_{6n+4}x_{6n+3} - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2d - a + d + a}{da + a^2 - ad + 1} = d,$$

$$y_{6n+5}=\frac{y_{6n+3}+x_{6n+4}}{x_{6n+4}y_{6n+3}-1}=\frac{c+\frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c-1}=\frac{c^2b-c+b+c}{bc+c^2-cb+1}=b,$$

$$z_{6n+5}=\frac{x_{6n+4}}{y_{6n+4}}=\frac{\frac{b+c}{cb-1}}{\frac{d+a}{ad-1}}=\frac{(b+c)(ad-1)}{(cb-1)(d+a)}$$

and

$$x_{6n+6}=\frac{x_{6n+4}+y_{6n+5}}{y_{6n+5}x_{6n+4}-1}=\frac{\frac{b+c}{cb-1}+b}{b\frac{b+c}{cb-1}-1}=\frac{b+c+b^2c-b}{b^2+bc-cb+1}=c,$$

$$y_{6n+6}=\frac{y_{6n+4}+x_{6n+5}}{x_{6n+5}y_{6n+4}-1}=\frac{\frac{d+a}{ad-1}+d}{d\frac{d+a}{ad-1}-1}=\frac{d+a+ad^2-d}{d^2+da-ad+1}=a,$$

$$z_{6n+6}=\frac{x_{6n+5}}{y_{6n+5}}=\frac{d}{b}.$$

Therefore, the proof is completed by induction.

The followig Corollary is a natural result of Theorem 1:

Corollary 1. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f$ be nonzero arbitrary real numbers and $\{x_n,y_n,z_n\}$ be a solutions of the system (2). Also, assume that $ad\neq 1$, $bc\neq 1$, $(b+c)\neq 0$ and $(d+a)\neq 0$. Then the sequences (x_n), (y_n) and (z_n) are six periodic.

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