Lightlike Hypersurfaces of Semi-Euclidean Spaces Satisfying Curvature Conditions of Semisymmetry Type

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Abstract

In this paper, we investigate lightlike hypersurfaces which are semi-symmetric, Ricci semi-symmetric, parallel or semi-parallel in a semi-Euclidean space. We obtain that every screen conformal lightlike hypersurface of the Minkowski spacetime is semi-symmetric. For higher dimensions, we show that the semi-symmetry condition of a screen conformal lightlike hypersurface reduces to the semi-symmetry condition of a leaf of its screen distribution. We also obtain that semi-symmetric and Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under certain conditions. Moreover, we show that there exist no non-totally geodesic parallel hypersurfaces in a Lorentzian space.

Key Words: Degenerate metric, Screen conformal lightlike hypersurface, Parallel lightlike hypersurface, Semi-symmetric lightlike hypersurface.

1. Introduction

The class of semi-Riemannian manifolds, satisfying the condition

\[ \nabla R = 0, \]

is a natural generalization of the class of manifolds of constant curvature, where \( \nabla \) is the Levi-Civita connection on semi-Riemannian manifold and \( R \) is the corresponding curvature tensor.
curvature tensor. For precise definitions of the symbols used, we refer to Section 2.1.

A semi-Riemannian manifold is called semi-symmetric if

\[ R \cdot R = 0, \tag{1.2} \]

where \( R \) is the curvature operator corresponding to \( R \) and the \( \cdot \) operation is defined in Section 2.1. Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [15] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [17].

A semi-Riemannian manifold is said to be Ricci semi-symmetric [7], if the following condition is satisfied:

\[ R \cdot Ric = 0. \tag{1.3} \]

It is clear that every semi-symmetric manifold is Ricci semi-symmetric; the converse is not true in general and a brief discussion of this issue is given in Section 2.1.

If a manifold \( M \) is immersed into a manifold \( \bar{M} \), the immersion is said to be parallel if the second fundamental form is covariantly constant, i.e., \( \nabla h = 0 \), where \( \nabla \) is an affine connection \( \bar{M} \) and \( h \) is the second fundamental form of the immersion. The general classification of parallel submanifolds of Euclidean space was obtained in [13] by D. Ferus. He showed that such an immersion is an isometric immersion into an \( n \)-dimensional symmetric \( R \)-space imbedded in \( R^{n+p} \) in the standard way. The general theory of lightlike submanifolds was introduced and presented in a book by Duggal-Bejancu [10]. The theory of lightlike submanifolds is a new area of differential geometry and it is very different from Riemannian geometry as well as semi-Riemannian geometry.

In third section of this paper, we consider a lightlike hypersurface of the semi-Euclidean space and study semi-symmetry conditions on this hypersurface. Our main result, in this section, states that every screen conformal lightlike hypersurface (Definition 3) of the Minkowski spacetime \( R_{4}^{1} \) is semi-symmetric. For \( R_{q}^{n+2}, n \geq 3 \) we show that semi-symmetry of a lightlike hypersurface depends on the geometry of a leaf of screen distribution.

In section four, we study Ricci semi-symmetric lightlike hypersurfaces and obtain that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic under a certain condition. In this section, we also obtain that semi-symmetric lightlike hypersurfaces are totally geodesic under a condition in terms of the Ricci tensor.

In section five, we investigate parallel hypersurface of a Lorentzian manifold. In fact, we show that every parallel lightlike hypersurface must be totally geodesic. Then we
study semi-parallel lightlike hypersurfaces in a semi-Euclidean space. We note that the semi-parallel hypersurfaces were defined in [8] as a generalization of parallel hypersurfaces for Riemannian case.

2. Preliminaries

In this section, we will give a brief review of curvature conditions of semi-symmetry type and lightlike submanifolds of semi-Riemannian manifolds. A full discussion of the contents of this section can be found in [7] and [10], respectively. In this paper, we will assume that every object in hand is smooth.

2.1. Curvature Conditions of Symmetry Type

Let \((M, g)\) be a semi-Riemannian manifold. We denote its curvature operator by \(R(X, Y)\)

\[ R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \]

for \(X, Y \in \Gamma(TM)\), where \(\nabla\) denotes the Levi-Civita connection on \(M\). Then the Riemannian Christoffel curvature tensor \(R\) and the Ricci tensor \(Ric\) are defined by

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W), \] (2.4)

\[ Ric(X, Y) = \text{trace}\{Z \rightarrow R(X, Y)Z\}, \] (2.5)

respectively.

For a \((0, k)\)-tensor field \(T\) on \(M\), \(k \geq 1\), the \((0, k+2)\) tensor field \(R \cdot T\) is defined by

\[ (R \cdot T)(X_1, \ldots, X_k, X, Y) = -T(R(X, Y)X_1, X_2, \ldots, X_k) \]

\[ -\ldots - T(X_1, \ldots, X_{k-1}, R(X, Y)X_k) \] (2.6)

for \(X, Y, X_1, \ldots, X_k \in \Gamma(TM)\). Curvature conditions, involving the form \(R \cdot T = 0\), are called curvature conditions of semi-symmetric type [7].

A semi-Riemannian manifold \(M\) is said to be semi-symmetric if it satisfies the condition \(R \cdot R = 0\). Thus, from (2.6) and properties of curvature tensor, we have


\[ - R(R(X, Y)U, V)W - R(U, R(X, Y)V)W = 0 \] (2.7)

for any \(X, Y, U, V, W \in \Gamma(TM)\).
A semi-Riemannian manifold $M$ is said to be Ricci semi-symmetric if it satisfies the condition $R \cdot Ric = 0$, i.e.,

$$
(R(X,Y) \cdot Ric)(X_1, X_2) = -Ric(R(X,Y)X_1, X_2)
- Ric(X_1, R(X,Y)X_2) = 0,
$$

(2.8)

for $X, Y, X_1, X_2 \in \Gamma(TM)$.

In [8], Deprez defined and studied semi-paralel hypersurfaces in Euclidean $n$ space. We recall that a hypersurface $M$ of a semi-Riemannian manifold $\tilde{M}$ is said to be semi-parallel if the following condition is satisfied for every point $p \in M$ and every vector fields $X, Y, Z, W \in \Gamma(TM)$:

$$
(R(X,Y)h)(Z,W) = -h(R(X,Y)Z,W) - h(Z,R(X,Y)W) = 0,
$$

(2.9)

where $h$ is the second fundamental form and $R$ is the curvature tensor field of $M$.

Although conditions (1.2) and (1.3) are not equivalent for manifolds in general, P.J. Ryan [16] raised the following question for hypersurfaces of Euclidean spaces in 1972: “Are the conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ equivalent for hypersurfaces of Euclidean spaces?” Although there are many results which contributed to the solution of the above question in the affirmative under some conditions (see [5], [6], [14], [19]), Abdalla and Dillen [1] gave an explicit example of a hypersurface in Euclidean space $E^{n+1}$ ($n \geq 4$) that is Ricci semi-symmetric but not semi-symmetric (See also [7] for another example.). This result shows that the conditions $R \cdot R = 0$ and $R \cdot Ric = 0$ are not equivalent for hypersurfaces of Euclidean space in general. A recent survey on Ricci semi-symmetric spaces and contributions to the solution of above problem can be found in [7]. We note that, in [20], I. Van de Woestijne and L. Verstraelen used the standard forms of a symmetric operator in a Lorentzian vector space to give an algebraic proof that the shape operator of a semisymmetric hypersurface at a point with type number greater than 2 is diagonalizable with exactly two eigenvalues, one of which is zero.

**2.2. Lightlike Hypersurfaces**

Let $(\tilde{M}, \tilde{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with the indefinite metric $\tilde{g}$ of index $q \in \{1, \ldots, m+1\}$ and $M$ be a hypersurface of $\tilde{M}$. We denote the tangent space at $x \in M$ by $T_x M$. Then

$$
T_x M^\perp = \{ V_x \in T_x \tilde{M} | \tilde{g}_x(V_x, W_x) = 0, \forall W_x \in T_x M \}.
$$

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and

$$RadT_xM = T_xM \cap T_xM^\perp.$$  

Then, $M$ is called a lightlike hypersurface of $\bar{M}$ if $RadT_xM \neq \{0\}$ for any $x \in M$. Thus $TM^\perp = \bigcap_{x \in M} T_xM^\perp$ becomes a one-dimensional distribution on $M$. We denote $F(M)$ the algebra of differential functions on $M$ and by $\Gamma(E)$ the $F(M)$-module of differentiable sections of a vector bundle $E$ over $M$.

**Definition 1.** ([10], p:78): Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. A complementary vector subbundle $S(TM)$ to $TM^\perp$ in $TM$ is called a screen distribution of $M$.

It is known from ([10], Proposition 2.1, p:5) that $S(TM)$ is non-degenerate. Thus, we have the orthogonal direct sum

$$TM = TM^\perp \oplus S(TM),$$  

(2.10)

where $\oplus \perp$ denotes the orthogonal direct sum. From (2.10), we observe that $TM^\perp$ lies in the tangent bundle of the lightlike hypersurface $M$. Thus a vital problem of this theory is to replace the intersecting part by a vector bundle of $T\bar{M}|_M$ whose sections are nowhere tangent to $M$. Next theorem shows that there exists a such complementary (non-orthogonal) vector bundle to $M$ in $T\bar{M}$.

**Theorem 2.1.** ([10], p: 79): Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over $M$, such that for any non-zero section $\xi$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique section $N$ of $tr(TM)$ on $U$ such that

$$\bar{g}(\xi, N) = 1, \bar{g}(N, N) = \bar{g}(N, X) = 0 \quad \forall X \in \Gamma(S(TM|_U)).$$  

(2.11)

It follows from (2.11) that $tr(TM)$ is a lightlike vector bundle such that $tr(TM)|_x \cap T_xM = \{0\}$ for any $x \in M$. Thus from (2.10) and (2.11) we have

$$TM|_M = S(TM) \oplus \perp (TM^\perp \oplus tr(TM)) = TM \oplus tr(TM).$$  

(2.12)

**Definition 2.** ([10], p:79): Let $M$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the complementary (non-orthogonal) vector bundle $tr(TM)$ to the tangent bundle $TM$ in $T\bar{M}|_M$ is called the lightlike transversal bundle of $M$ with respect to
screen distribution $S(TM)$.

Suppose $M$ is a lightlike hypersurface of $\bar{M}$ and $\nabla$ is the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.12) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

(2.13)

and

$$\bar{\nabla}_X \xi = -A^\xi X + \nabla^\xi_X \xi$$

(2.14)

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(tr(TM))$, where $\nabla_X Y$ and $A^\xi X$ belong to $\Gamma(TM)$, $h(X, Y)$ and $\nabla^\xi_X \xi$ belong to $\Gamma(tr(TM))$. We note that it is easy to see that $\nabla$ is a torsion free connection, $h$ is a $tr(TM)$ valued, symmetric $F(M)$– bilinear form on $TM$, $A^\xi$ is a $F(M)$– linear operator on $\Gamma(TM)$ and $\nabla^\xi$ is a linear connection on $tr(TM)$. $h$ and $A^\xi$ are called the second fundamental form and shape operator of the lightlike hypersurface $M$, respectively.

Locally suppose $\{\xi, N\}$ is a pair of vector fields on $U$ in Theorem 2.1. Then we define a symmetric bilinear form $B$ and 1– form $\tau$ on $U$ by

$$B(X, Y) = \bar{g}(h(X, Y), \xi) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla^\xi_X N, \xi)$$

for $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$. Thus (2.13) and (2.14) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N$$

(2.15)

and

$$\bar{\nabla}_X \xi = -A^\xi X + \tau(X)N$$

(2.16)

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$.

Let $P$ denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.10). We obtain

$$\nabla_X P Y = \nabla^*_X P Y + C(X, P Y)\xi$$

(2.17)

and

$$\nabla_X \xi = -A^\xi_X + v(X)\xi$$

(2.18)
for any \( X, Y \in \Gamma(TM) \), where \( \nabla_X^*PY, A^*_X \in \Gamma(S(TM)) \) and \( C \) is a 1– form on \( U \) defined by

\[
C(X, PY) = \bar{g}(\nabla_X PY, N) \tag{2.19}
\]

for \( X, Y \in \Gamma(TM) \). \( C \) and \( A^* \) are called the second fundamental form and shape operator of the screen distribution \( S(TM) \), respectively. From (2.11), (2.15), (2.16) and (2.18) we obtain \( v(X) = -\tau(X) \), thus (2.18) becomes

\[
\nabla_X \xi = -A^*_X X - \tau(X)\xi. \tag{2.20}
\]

By direct calculations, using (2.15), (2.16), (2.17) and (2.20) we obtain the following lemma.

**Lemma 2.1.** ([10], p:85) Let \( M \) be a lightlike hypersurface of a semi-Riemannian manifold \( \bar{M} \). Then we have

\[
\begin{align*}
g(A_N Y, PW) &= C(Y, PW), \quad g(A_N Y, N) = 0 \tag{2.21} \\
g(A^*_X X, PY) &= B(X, PY) \tag{2.22}
\end{align*}
\]

for \( X, Y, W \in \Gamma(TM), \xi \in \Gamma(TM^\perp) \) and \( N \in \Gamma(tr(TM)) \).

We note that the second equation of (2.21) implies that \( A_N X \in \Gamma(S(TM)) \) for \( X \in \Gamma(TM) \), i.e., \( A_N \) is \( \Gamma(S(TM)) \)– valued. On the other hand, from \( \bar{g}(\nabla_X \xi, \xi) = 0 \) we have

\[
B(X, \xi) = 0. \tag{2.23}
\]

We now recall the definition of screen conformal lightlike hypersurfaces of a semi-Riemannian manifold \( \bar{M} \).

**Definition 3.** ([2]). A lightlike hypersurface \((M, g, S(TM))\) of a semi-Riemannian manifold is screen conformal if the shape operators \( A_N \) and \( A^*_X \) of \( M \) and its screen distribution \( S(TM) \) are related by

\[
A_N = \varphi A^*_X, \tag{2.24}
\]

where \( \varphi \) is a non-vanishing smooth function on a neighborhood \( U \) in \( M \). In case \( U = M \) the screen conformality is said to be global.
We note that there are many examples of screen conformal lightlike hypersurfaces of semi-Riemannian manifolds. Next, we give two examples of screen conformal lightlike hypersurfaces of semi-Euclidean spaces; for more examples, see [2].

**Examples.**

**1.** The Lightlike Cone $\Lambda_{0}^{3}$ of $R^{4}_{1}$: Let $R^{4}_{1}$ be the space $R^{4}$ endowed with the semi-Euclidean metric

$$g(x, y) = -x_{1}y_{1} + x_{2}y_{2} + x_{3}y_{3} + x_{4}y_{4}, \quad x = \sum_{i=1}^{4} x_{i} \frac{\partial}{\partial x_{i}}.$$ 

The lightlike cone is given by the equation $-(x_{1})^{2} + (x_{2})^{2} + (x_{3})^{2} + (x_{4})^{2} = 0$, $x \neq 0$. It is known that the lightlike cone is a screen conformal lightlike hypersurface [2].

**2.** Lightlike Monge Hypersurfaces of $R^{4}_{1}$: Let $D$ be an open set of $R^{4}_{1}$ and $F : D \rightarrow R$ be a smooth function on $D$. Then the set

$$M = \{(x^{1}, x^{2}, x^{3}, x^{4}) \in R^{4} : x^{1} = F(x^{2}, x^{3}, x^{4})\}$$

is called a Monge hypersurface. A Monge hypersurface of $R^{4}_{1}$ is lightlike if and only if $F$ is a solution of the partial differential equation

$$1 + \left(\frac{\partial F}{\partial x_{1}}\right)^{2} = \left(\frac{\partial F}{\partial x_{2}}\right)^{2} + \left(\frac{\partial F}{\partial x_{3}}\right)^{2} + \left(\frac{\partial F}{\partial x_{4}}\right)^{2}.$$ 

It is known that a lightlike Monge hypersurface is screen conformal [2].

3. **Semi-symmetric Lightlike Hypersurfaces in Semi-Euclidean Spaces**

In this section, we consider semi-symmetric lightlike hypersurfaces in a semi-Euclidean space. First, we give the Gauss equation for a lightlike hypersurface of a semi-Euclidean space $R^{(n+2)}_{q}$. Then we show that every screen conformal lightlike hypersurface of the Minkowski spacetime is semi-symmetric. For higher dimensions, we show that the semi-symmetry condition of a screen conformal lightlike hypersurface $M$ has close relation with the semi-symmetry condition of a leaf of its screen distribution. From now on, we denote a lightlike hypersurface by $M$ and use $A$ for $A_{N}$. 

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Proposition 3.1. Let $M$ be a lightlike hypersurface of a semi-Euclidean space $R_q^{(n+2)}$. Then the Gauss equation of $M$ is given by

$$R(X,Y)Z = B(Y,Z)AX - B(X,Z)AY$$

(3.1)

for any $X, Y, Z \in \Gamma(TM)$ and $N \in \Gamma(\text{tr}(TM))$.

Proof. For a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$, from ([10], p:93) we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z),$$

(3.2)

where $\bar{R}$ and $R$ are curvature tensor fields of $\bar{M}$ and $M$, respectively. We note that $(\nabla_X h)(Y,Z)$ is defined by

$$(\nabla_X h)(Y,Z) = \nabla^i_X h(Y,Z) - h(\nabla_Y h, X, Z).$$

(3.3)

By assumption, $\bar{M} = R_q^{(n+2)}$ is a semi-Euclidean space, hence $\bar{R} = 0$. Then (3.2) becomes

$$R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0.$$

On the other hand, (2.13) and (2.15) imply that $h(X,Y) = B(X,Y)N$ for $X, Y \in \Gamma(TM)$ and $N \in \Gamma(\text{tr}(TM))$. Thus, we get


Then comparing the tangential and transversal parts of the above equation, we obtain (3.1).

We note that $g(R(X,Y)Z, W) \neq -g(R(X,Y)W, Z)$, $\forall X, Y, Z, W \in \Gamma(TM)$, for a lightlike hypersurface in general.

Definition 4. Let $M$ be a lightlike hypersurface of a semi-Euclidean space. We say that $M$ is a semi-symmetric if the following condition is satisfied

$$(R(X,Y) \cdot R)(X_1, X_2, X_3, X_4) = 0$$

(3.4)

for $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$. 

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Notice that it is easy to see that

\[(R(X, Y) \cdot R)(X_1, X_2, X_3, \xi) = 0\]

for \(\xi \in \Gamma(TM^\perp)\). Thus the condition (3.4) is equivalent to the following condition

\[(R(X, Y) \cdot R)(X_1, X_2, X_3, PX_4) = 0\] (3.5)

for \(X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)\). We also note that (3.4) and (3.5) do not imply the equation (2.7) due to \(g(R(X, Y)Z, W) \neq -g(R(X, Y)W, Z)\) in general, for \(X, Y, Z, W \in \Gamma(TM)\).

Now, from (3.5) and (3.1), we obtain

\[(R(X, Y) \cdot R)(X_1, X_2, X_3, PX_4) = B(Y, X_1)[B(AX, X_3)g(AX_2, PX_4)
- B(X_2, X_4)g(A^2X, PX_4)] + B(X, X_1)[B(X_2, X_3)g(A^2Y, PX_4)
- B(AY, X_3)g(AX_2, PX_4)] + g(AX_1, PX_4)[B(Y, X_2)B(AX, X_3)
+ B(X, X_2)B(AY, X_3)] + B(X_1, X_4)[B(Y, X_2)g(A^2X, PX_4)
- B(X, X_2)g(A^2Y, PX_4)] + g(AX_1, PX_4)[B(Y, X_2)B(AX, X_3)
+ B(X, X_2)B(AY, X_3)] + g(AX_2, PX_4)[B(X, X_1)B(Y, AX)
- B(X, X_1)B(X_2, AX)] + g(AX_2, PX_4)[B(X_2, X_3)B(AY, X_1)AX)
- B(X_2, X_3)B(X_1, AX)] + B(X_1, X_4)[B(Y, PX_4)g(AX_2, AX)
- B(X, PX_4)g(AX_1, AX)] + B(X_1, X_3)[B(Y, PX_4)g(AX_2, AX)
- B(X, PX_4)g(AX_1, AX)]\] (3.6)

for any \(X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)\).

**Proposition 3.2.** Every screen conformal lightlike hypersurface of the Minkowski spacetime is a semi-symmetric lightlike hypersurface.
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\textbf{Proof.} First, from (3.6), we have
\begin{align*}
(R(X, Y) \cdot R)(\xi, X_2, X_3, PX_4) &= B(Y, \xi)[B(AX, X_3)g(AX_2, PX_4) \\
&- B(X_2, X_3)g(A^2X, PX_4)] \\
&+ B(X, \xi)[B(X_2, X_3)g(A^2Y, PX_4) - B(AY, X_3)g(AX_2, PX_4)] \\
&+ g(A\xi, PX_4)[B(Y, X_2)B(AX, X_3) + B(X, X_2)B(AY, X_3)] \\
&+ B(\xi, X_3)[B(Y, X_2)g(A^2X, PX_4) - B(X, X_2)g(A^2Y, PX_4)] \\
&+ g(A\xi, PX_4)[-B(X_3, Y)B(AX_2, AX) + B(X, X_3)B(X_2, AX)] \\
&+ B(X_2, X_3)[-B(Y, PX_4)B(A\xi, AX) + B(X, PX_4)g(A\xi, AX)].
\end{align*}
for any $X, Y, X_2, X_3, X_4 \in \Gamma(TM)$ and $\xi \in \Gamma(\text{RadTM}).$ Then, from (2.23), we get
\begin{align*}
(R(X, Y) \cdot R)(\xi, X_2, X_3, PX_4) &= g(A\xi, PX_4)[-B(Y, X_2)B(AX, X_3) \\
&+ B(X, X_2)B(AX, X_3)] \\
&+ g(A\xi, PX_4)[-B(X_3, Y)B(AX_2, AX) + B(X, X_3)B(X_2, AX)] \\
&+ B(X_2, X_3)[-B(Y, PX_4)B(A\xi, AX) + B(X, PX_4)g(A\xi, AX)].
\end{align*}
Then, (2.24) implies that
\begin{align*}
(R(X, Y) \cdot R)(\xi, X_2, X_3, PX_4) &= \varphi g(A^2\xi, PX_4)[-B(Y, X_2)B(AX, X_3) \\
&+ B(X, X_2)B(AX, X_3)] \\
&+ \varphi g(A^2\xi, PX_4)[-B(X_3, Y)B(AX_2, AX) + B(X, X_3)B(X_2, AX)] \\
&+ \varphi B(X_2, X_3)[-B(Y, PX_4)B(A^2\xi, AX) + B(X, PX_4)g(A^2\xi, AX)].
\end{align*}
From (2.22) and (2.23), we have $A^2\xi = 0.$ Thus, we derive
\begin{align*}
(R(X, Y) \cdot R)(\xi, X_2, X_3, PX_4) &= 0.
\end{align*}
In a similar way, we obtain
\begin{align*}
(R(X, Y) \cdot R)(X_1, X_2, \xi, PX_4) &= 0, (R(\xi, Y) \cdot R)(X_1, X_2, X_3, PX_4) = 0
\end{align*}
and
\begin{align*}
(R(X, Y) \cdot R)(X_1, \xi, X_3, PX_4) &= 0, (R(X, \xi) \cdot R)(X_1, X_2, X_3, PX_4) = 0.
\end{align*}
for \(X_1, X_2, X_3, X_4 \in \Gamma(TM)\) and \(\xi \in \Gamma(TM^\perp)\). Let \(\{X_1, X_2, \xi, N\}\) be a quasi-ortonormal basis of \(R^4\) such that \(S(TM) = \text{span}\{X_1, X_2\}\) and \(\text{tr}(TM) = \text{span}\{N\}\). From (3.6), we have

\[
(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = B(X_2, X_1)[B(AX_1, X_1)g(AX_2, X_2)
- B(X_2, X_1)g(A^2 X_1, PX_2)]
+ B(X_1, X_1)[B(X_2, X_1)g(A^2 X_2, X_2) - B(AX_2, X_3)g(AX_2, PX_3)]
+ g(AX_1, X_2)[-B(X_1, X_2)B(AX_1, X_1) + B(X_1, X_2)B(AX_2, X_1)]
+ g(X_1, X_2)[B(X_2, X_1)g(A^2 X_1, X_2) - B(X_1, X_2)g(A^2 X_1, X_2)]
+ g(AX_1, X_2)[-B(X_1, X_2)B(X_2, AX_1) + B(X_1, X_2)B(X_2, AX_1)]
+ g(AX_2, X_2)[B(X_1, X_2)B(AX_1, AX_1) - B(X_1, X_1)B(X_1, AX_2)]
+ B(X_2, X_1)[-B(X_1, X_2)B(AX_1, AX_1) + B(X_1, X_2)g(AX_1, AX_1)]
+ B(X_1, X_1)[B(X_2, X_2)g(AX_2, AX_1) - B(X_1, X_2)g(AX_2, AX_2)].
\]

Since \(AX \in \Gamma(S(TM))\) for any \(X \in \Gamma(TM)\) and \(N \in \Gamma(\text{tr}(TM))\) and \(A = A_N\) is self-adjoint on \(S(TM)\), we get

\[
(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = B(X_2, X_1)[B(AX_1, X_1)g(AX_2, X_2)
- B(X_2, X_1)g(AX_1, AX_2)]
+ B(X_1, X_1)[B(X_2, X_1)g(AX_2, AX_2) - B(AX_2, X_1)g(AX_2, AX_2)]
+ g(AX_1, X_2)[-B(X_1, X_2)B(AX_1, X_1) + B(X_1, X_2)B(AX_2, X_1)]
+ g(X_1, X_2)[B(X_2, X_1)g(A^2 X_1, X_2) - B(X_1, X_2)g(A^2 X_1, X_2)]
+ g(AX_1, X_2)[-B(X_1, X_2)B(X_2, AX_1) + B(X_1, X_2)B(X_2, AX_1)]
+ g(AX_2, X_2)[B(X_1, X_2)B(AX_1, AX_1) - B(X_1, X_1)B(AX_1, AX_2)]
+ B(X_2, X_1)[-B(X_1, X_2)B(AX_1, AX_1) + B(X_1, X_2)g(AX_1, AX_1)]
+ B(X_1, X_1)[B(X_2, X_2)g(AX_2, AX_1) - B(X_1, X_2)g(AX_2, AX_2)].
\]
Then, using (2.24), we arrive at
\[
(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = \varphi B(X_2, X_1)[B(AX_1, X_1)g(A_x^*_X X_2, X_2)
- B(X_2, X_1)g(A_x^*_X X_1, AX_2)]
+ \varphi B(X_1, X_1)[B(X_2, X_1)g(A_x^*_X X_2, AX_2) - B(AX_2, X_1)g(A_x^*_X X_2, X_2)]
+ \varphi g(A_x^*_X X_2, X_2)\{ - B(X_2, X_2)B(AX_1, X_1) + B(X_1, X_2)B(AX_2, X_1) \}
+ \varphi B(X_1, X_1)[B(X_2, X_2)g(A_x^*_X X_1, AX_2) - B(X_1, X_1)g(A_x^*_X X_2, AX_2)]
+ \varphi g(A_x^*_X X_2, X_2)\{ - B(X_1, X_2)B(AX_2, AX_1) + B(X_1, X_1)B(AX_2, AX_2) \}
+ \varphi B(X_1, X_1)[B(X_2, X_2)g(A_x^*_X X_2, AX_1) - B(X_1, X_2)g(A_x^*_X X_2, AX_2)].
\]

Thus, using (2.22), we obtain
\[
(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = \varphi B(X_2, X_1)[B(AX_1, X_1)B(X_2, X_2)
- B(X_2, X_1)B(AX_1, AX_2)]
+ \varphi B(X_1, X_1)[B(X_2, X_1)B(X_2, AX_2) - B(AX_2, X_1)B(X_2, X_2)]
+ \varphi B(X_1, X_1)[B(X_2, X_2)B(X_1, AX_2) - B(X_1, X_2)B(AX_1, X_2)]
+ \varphi B(X_1, X_1)[ - B(X_1, X_2)B(X_2, AX_1) + B(X_1, X_1)B(AX_2, AX_2)]
+ \varphi B(X_2, X_1)[B(X_2, X_2)B(X_1, AX_1) - B(X_1, X_1)B(AX_2, AX_2)]
+ \varphi B(X_1, X_1)[ - B(X_2, X_2)B(X_1, AX_1) + B(X_1, X_2)B(AX_1, AX_2)]
+ \varphi B(X_1, X_1)[B(X_2, X_2)B(X_2, AX_1) - B(X_1, X_2)B(AX_2, AX_2)].
\]

Since $B$ is symmetric, by direct computations, we get
\[
(R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = \varphi \{(B(X_2, X_1))^2 B(X_1, AX_2)
- (B(X_1, X_2))^2 B(X_2, AX_1)
- B(X_2, X_2)B(X_1, X_1)B(AX_1, AX_2)
+ B(X_1, X_1)B(X_2, X_2)B(AX_2, AX_1)\}.
\]

(3.7)

On the other hand, from (2.22) and (2.24), we have
\[
B(AX_2, X_1) = g(A_x^*_X X_1, AX_2) = g(\varphi A_x^*_X X_1, A_x^*_X X_2) = g(AX_1, A_x^*_X X_2).
\]
Thus, using again (2.22), we get
\[ B(AX_2, X_1) = B(X_2, AX_1). \]  \hspace{1cm} (3.8)

Then, from (3.7) and (3.8), we obtain
\[ (R(X_1, X_2) \cdot R)(X_1, X_2, X_1, X_2) = 0. \]

In a similar way, we have
\[ (R(X_1, X_2) \cdot R)(X_2, X_1, X_1, X_2) = 0, \]
\[ (R(X_1, X_2) \cdot R)(X_2, X_1, X_1, X_1) = 0. \]

and
\[ (R(X_1, X_2) \cdot R)(X_1, X_2, X_2, X_1) = 0. \]

Thus proof is complete. \hspace{1cm} \(\square\)

**Remark 1.** From Proposition 3.2, it follows that lightlike cone of \(R^1_3\), lightlike Monge hypersurface of \(R^4_1\), and lightlike surfaces of \(R^3_1\) are examples of semi-symmetric lightlike hypersurfaces. We also note that Proposition 3.1 is valid for a semi-Euclidean space \(R^q_1\), \(1 \leq q < 4\).

Let \(M\) be a screen conformal lightlike hypersurface of an \((n + 2)\) dimensional semi-Euclidean space. Then, it is known that the screen distribution of \(M\) is integrable [2]. We denote a leaf of the screen distribution by \(M'\). Then, we have the following theorem.

**Theorem 3.1.** Let \(M\) be a screen conformal lightlike hypersurface of an \((n + 2)\) dimensional semi-Euclidean space, \(n \geq 3\). Then \(M\) is semi-symmetric if and only if any leaf \(M'\) of \(S(TM)\) is semi-symmetric in semi-Euclidean space, that is, the curvature tensor of \(M'\) satisfies the condition (2.7) in semi-Euclidean space.

**Proof.** Using (3.1) and (2.24) we obtain
\[ g(R(X, Y)PZ, PW) = \varphi \{ B(Y, Z)B(X, PW) - B(X, Z)B(Y, PW) \} \]  \hspace{1cm} (3.9)
for any $X,Y,Z,W \in \Gamma(TM)$. Then, by straightforward computations, using (2.17), (2.20), (2.21), (2.23) and (2.24), we get
\[
g(R(X,Y)PZ,PW) = g(R^*(X,Y)PZ,PW) - \varphi \{ B(Y,PZ)B(X,PW) \\
+ B(X,PZ)B(Y,PW) \}
\]
for any $X,Y,Z,W \in \Gamma(TM)$. Thus, from (3.9) and (3.10), we derive
\[
g(R(X,Y)PZ,PW) = \frac{1}{1+\varphi} g(R^*(X,Y)PZ,PW)
\]
(3.11)
On the other hand, from (2.21) and (3.1), we get
\[
g(R(X,Y)Z,N) = 0, \forall X,Y,Z,N \in \Gamma(TM), N \in \Gamma(tr(TM)).
\]
(3.12)
Thus, from (3.11) and (3.12), we conclude that
\[
R(X,Y)PZ = \frac{1}{1+\varphi} R^*(X,Y)PZ
\]
(3.13)
Hence, using algebraic properties of the curvature tensor field, we have
\[
(R(X,Y) \cdot R)(U,V,W,Z) = \frac{1}{(1+\varphi)^2} (R^*(X,Y) \cdot R^*)(U,V,W,Z)
\]
(3.14)
for any $X,Y,U,V,W \in \Gamma(S(TM))$. Thus the proof is complete.

\[\Box\]

**Remark 2.** The above theorem shows us that the semi-symmetry of a screen conformal lightlike hypersurface of an $(n+2)$ semi-Euclidean space is related with the semi-symmetry of a leaf $M'$ of its integrable screen distribution. In Lorentzian case, since screen distribution is Riemannian, studying semi-symmetry of a screen conformal lightlike hypersurface is exactly same with a Riemannian manifold. In fact, we can see from proof of Theorem 3.1, the curvature conditions of a screen conformal lightlike hypersurface reduces to the curvature conditions of a leaf of its screen distribution.

4. Ricci Semi-symmetric Lightlike Hypersurfaces in Semi-Euclidean Spaces

In this section, we study Ricci semi-symmetric lightlike hypersurfaces of semi-Euclidean spaces and obtain that Ricci semi-symmetric lightlike hypersurfaces are totally geodesic.
under a condition. We also give a theorem on semi-symmetric lightlike hypersurfaces of semi-Euclidean spaces in terms of the Ricci tensor. First, we need the expression of the Ricci tensor of a lightlike hypersurface.

**Lemma 4.1.** Let $M$ be a lightlike hypersurface of semi-Euclidean $(n + 2)$ space. Then the Ricci tensor $\text{Ric}$ of $M$ is given by

$$\text{Ric}(X, Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X, Y)C(w_i, w_i)\} - \bar{g}(A_\xi^Y, AX)$$

for any $X, Y \in \Gamma(TM)$, where $\epsilon_i = \pm 1$ and $\{w_i\}_{i=1}^{n}$ is an orthonormal basis of $S(TM)$.

**Proof.** The Ricci tensor of a lightlike hypersurface is given by

$$\text{Ric}(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(X, w_i)Y, w_i) - \bar{g}(R(X, \xi)Y, N)$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$, where $\{w_i\}_{i=1}^{n}$ is a basis of $S(TM)$. Then, from (2.21) and (3.1), we have

$$\text{Ric}(X, Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X, Y)C(w_i, w_i)\} - \bar{g}(\sum_{i=1}^{n} \epsilon_i g(A_\xi^Y, w_i)w_i, AX).$$

Using (2.21) and (2.22), we get

$$\text{Ric}(X, Y) = -\sum_{i=1}^{n} \epsilon_i \{B(X, Y)C(w_i, w_i)\} - g(\sum_{i=1}^{n} \epsilon_i g(A_\xi^Y, w_i)w_i, AX).$$

Hence, we have (4.1). \[\square\]

**Definition 5.** Let $M$ be a lightlike hypersurface of a semi-Euclidean space. Then we say that $M$ is Ricci semi-symmetric if the following condition is satisfied

$$(R(X, Y) \cdot \text{Ric})(X_1, X_2) = 0$$

(4.2)

for $X, Y, X_1, X_2 \in \Gamma(TM)$. 

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Next we give a theorem which shows the effect of Ricci semi-symmetric condition on the geometry of lightlike hypersurfaces of a semi-Euclidean space.

**Theorem 4.1.** Let $M$ be a Ricci semi-symmetric lightlike hypersurface of an $(n+2)$-dimensional semi-Euclidean space. Then either $M$ is totally geodesic or $\text{Ric}(\xi, A\xi) = 0$ for $\xi \in \Gamma(TM^\perp)$, where $\text{Ric}$ is the Ricci tensor of $M$ and $A$ denotes the shape operator defined in (2.16).

**Proof.** From (3.1), (2.8) and (4.2), we obtain

$$\langle R(X, Y) \cdot \text{Ric}(X_1, X_2) \rangle = \alpha \{ -B(X, X_1)B(AY, X_2) + B(Y, X_1)B(AX, X_2)$$

$$- B(X, X_2)B(X_1, AY) + B(Y, X_2)B(X_1, AX) \}$$

$$- B(X, X_1)B(X_2, A^2 Y) + B(Y, X_1)B(X_2, A^2 X)$$

$$- B(X, X_2)B(AY, AX_1) + B(Y, X_2)B(AX, AX_1)$$

for $X, Y, X_1, X_2 \in \Gamma(TM)$, where $\alpha = \sum_{i=1}^{n} \epsilon_i C(w_i, w_i)$. Now, suppose that $M$ is Ricci semi-symmetric lightlike hypersurface. Taking $X_1 = \xi$ in the above equation and using (2.23), we obtain

$$-B(X, X_2)B(AY, A\xi) + B(Y, X_2)B(AX, A\xi) = 0.$$ 

Hence for $Y = \xi$ we derive

$$B(X, X_2)B(A\xi, A\xi) = 0.$$ 

So, if $B(X, X_2) = 0$ for any $X, X_2 \in \Gamma(TM)$, then $M$ is totally geodesic. If $M$ is not totally geodesic, it follows that $B(A\xi, A\xi) = 0$, then from (4.1) we obtain $\text{Ric}(\xi, A\xi) = 0$. \hfill \Box

**Theorem 4.2.** Let $M$ be a lightlike hypersurface of a semi-Euclidean $(n+2)$ space such that $\text{Ric}(\xi, X) = 0$, $\forall X \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $A\xi$ is a non-null vector field. Then $M$ is semi-symmetric if and only if $M$ is totally geodesic, where $\text{Ric}$ is the Ricci tensor of $M$ and $A$ is the shape operator of $M$.

**Proof.** Suppose that $M$ is a semi-symmetric lightlike hypersurface of a semi-Euclidean
(n + 2) space. Taking $X_1 = \xi$ in (3.6), we obtain

\[
\{-B(Y, X_2)B(AX, X_3) + B(X, X_2)B(AY, X_3)\}g(\xi, PX_4) \\
\{-B(X_3, Y)B(X_2, AX) + B(X, X_3)B(X_2, AY)\}g(\xi, PX_4) \\
\{-B(Y, PX_4)g(\xi, AX) + B(X, PX_4)g(\xi, AY)\}B(X_2, X_3) = 0.
\]

Then, for $Y = \xi$, we have

\[
B(X, X_2)B(AX, X_3)g(AX, PX_4) + B(X, X_3)B(X_2, AX)g(AX, PX_4) \\
+ B(X, PX_4)g(AX, AX)B(X_2, X_3) = 0.
\]

Thus, by assumption, $R(\xi, X) = 0$, we have $B(X, AX) = 0$. Hence, we get

\[
B(X, PX_4)g(AX, AX)B(X_2, X_3) = 0.
\]

Since $AX$ is a non-null vector field by hypothesis, for $X = X_3$ and $X_4 = X_2$ we arrive at

\[
B(X_2, X_3) = 0.
\]

Thus, $M$ is totally geodesic. The converse is clear from (3.6).

For Lorentzian space $R^{(n+2)}_1$, we have the following corollary. \hfill \square

**Corollary 4.1.** Let $M$ be a lightlike hypersurface of a Lorentzian space $R^{(n+2)}_1$ such that $\text{Ric}(\xi, X) = 0$, $\forall X \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$. Then $M$ is totally geodesic if and only if $M$ is semi-symmetric, where $\text{Ric}$ is the Ricci tensor of $M$.

**Proof.** If $M$ is a lightlike hypersurface of $R^{(n+2)}_1$. Then the screen distribution of $M$ is a Riemannian vector bundle. From (2.21), we can see that $AX \in \Gamma(S(TM))$, $\forall X \in \Gamma(TM)$. Then, the proof follows from Theorem 4.2. \hfill \square

5. **Parallel and Semi-Parallel Lightlike Hypersurfaces**

In this section, we give a characterization on parallel lightlike hypersurfaces of a Lorentzian manifold. In fact, it shows that there do not exist non-totally geodesic parallel lightlike hypersurfaces in a Lorentzian manifold. Moreover, we investigate the effect of semi-parallel condition on the geometry of lightlike hypersurfaces in a semi-Euclidean
Theorem 5.1. Let $M$ be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then the second fundamental form of $M$ is parallel if and only if $M$ is totally geodesic.

Proof. Let $M$ be a lightlike hypersurface of a Lorentzian manifold. We suppose that the second fundamental form $h$ is parallel. Then, from (3.3) and (2.15) we have

$$(\nabla_X h)(Y, Z) = X(B(Y, Z)N) - B(\nabla_X Y, Z)N - B(Y, \nabla_X Z)N = 0. \hspace{1cm} (5.1)$$

Thus, from (2.23), for $Y = \xi$, we obtain

$$-B(\nabla_X \xi, Z)N = 0.$$ 

By using (2.18), we have

$$B(A^*_\xi X, Z)N = 0.$$ 

Hence we derive $B(A^*_\xi X, Z) = 0$. Considering (2.23) we can assume that $Z \in \Gamma(S(TM))$. Thus, from (2.22), we obtain $g(A^*_\xi X, A^*_\xi Z) = 0$. Then, for $X = Z$ we get $g(A^*_\xi X, A^*_\xi X) = 0$. On the other hand, any screen distribution $S(TM)$ of a lightlike hypersurface of a Lorentzian manifold is Riemannian. Then, we have $A^*_\xi X = 0$ for any $X \in \Gamma(TM)$. Thus, proof follows from this and (2.23). The converse is clear. □

Theorem 5.2. Let $M$ be a semi-parallel lightlike hypersurface of semi-Euclidean $(n + 2)$ space. Then either $M$ is totally geodesic or $C(\xi, A^*_\xi U) = 0$ for any $U \in \Gamma(S(TM))$ and $\xi \in \Gamma(TM^\perp)$, where $C$ and $A^*_\xi$ are the second fundamental form and shape operator of the screen distribution $S(TM)$ defined in (2.19) and (2.18), respectively.

Proof. Since $M$ is a semi-parallel lightlike hypersurface, we have

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0.$$
By using (3.1), we obtain
\[ B(X, Z)B(AY, W) - B(Y, Z)B(AX, W) + B(X, W)B(Z, AY) - B(Y, W)B(AX, Z) = 0 \] (5.2)
for any \( X, Y, Z, W \in \Gamma(TM) \). Then, from (2.23) and (5.2), for \( X = \xi \), we have
\[ B(Y, Z)B(A\xi, W) + B(Y, W)B(A\xi, Z) = 0. \]
Thus, for \( Z = W \), we obtain
\[ B(Y, Z)B(A\xi, Z) = 0. \]
Now, if \( B(Y, Z) = 0 \), then \( M \) is totally geodesic. If \( B(Y, Z) \neq 0 \), then from (2.21), we have \( C(\xi, A^*_\xi U) = 0 \) for any \( U \in \Gamma(S(TM)) \).

**Example 3.** Consider a hypersurface \( M \) in \( \mathbb{R}^4_2 \) given by
\[ x_1 = x_2 + \sqrt{2}\sqrt{x_3^2 + x_4^2}. \]
Then, it is easy to check that \( M \) is a lightlike hypersurface. Its radical distribution is spanned by
\[ \xi = \sqrt{x_3^2 + x_4^2} \frac{\partial}{\partial x_1} - \sqrt{x_3^2 + x_4^2} \frac{\partial}{\partial x_2} + \sqrt{2} x_3 \frac{\partial}{\partial x_3} + \sqrt{2} x_4 \frac{\partial}{\partial x_4}. \]
Then the lightlike transversal vector bundle is spanned by
\[ tr(TM) = span\{N = \frac{1}{4(x_3^2 + x_4^2)}(-\sqrt{x_3^2 + x_4^2} \frac{\partial}{\partial x_1} + \sqrt{x_3^2 + x_4^2} \frac{\partial}{\partial x_2} + \sqrt{2} x_3 \frac{\partial}{\partial x_3} + \sqrt{2} x_4 \frac{\partial}{\partial x_4}) \} \]
It follows that the corresponding screen distribution \( S(TM) \) is spanned by
\[ \{Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Z_2 = -x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\}. \]
By direct computations, we obtain
\[ \nabla_X Z_1 = \nabla Z_1 X = 0, \quad \nabla_\xi Z_1 = \nabla Z_1 \xi, \quad \nabla_{Z_2} Z_1 = \nabla Z_1 Z_2, \]
and
\[ \nabla_{Z_2} Z_2 = -x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}. \]
for any \( X \in \Gamma(TM) \). Then, by using Gauss formula, we obtain

\[
\nabla_X Z_1 = 0, \nabla_{Z_2} Z_2 = -\frac{1}{2\sqrt{2}} \xi, \nabla_\xi Z_2 = \nabla_{Z_2} \xi = \sqrt{2} Z_2, \nabla_{Z_1} Z = 0
\]

and

\[
B(Z_2, Z_2) = -\sqrt{2}(x_3^2 + x_4^2), B(Z_1, Z_2) = 0, B(Z_1, Z_1) = 0.
\]

On the other hand, we have

\[
\nabla_\xi N = \frac{1}{2\sqrt{2}} \frac{\partial}{\partial x_1} - \frac{1}{2\sqrt{2}} \frac{\partial}{\partial x_2} - \frac{x_3}{2(x^2_3 + x^2_4)} \frac{\partial}{\partial x_3} - \frac{x_4}{2(x^2_3 + x^2_4)} \frac{\partial}{\partial x_4},
\]

\[
\nabla_{Z_1} N = 0,
\]

\[
\nabla_{Z_2} N = \frac{x_4}{2\sqrt{2}(x^2_3 + x^2_4)} \frac{\partial}{\partial x_3} + \frac{x_3}{2\sqrt{2}(x^2_3 + x^2_4)} \frac{\partial}{\partial x_4}.
\]

Thus, from Weingarten formula (2.16), we have

\[
A_N \xi = 0, A_N Z_1 = 0, A_N Z_2 = \frac{1}{2\sqrt{2}(x^2_3 + x^2_4)} Z_2.
\]

Then, from the above equations, one can show that the following equations are satisfied

\[
(R(Z_1, Z_2) h)(Z_1, Z_1) = 0, (R(Z_1, Z_2) h)(Z_1, Z_2) = 0, (R(Z_1, Z_2) h)(Z_2, Z_2) = 0.
\]

Finally, using (2.23) and definition of \((R(X, Y) h)(X, Y) = 0\) for any \( X, Y, U \in \Gamma(TM) \) and \( \xi \in \Gamma(TM^+) \). Thus, \( M \) is a non-totally geodesic semi-parallel hypersurface of \( R^4_2 \).

6. Concluding Remarks

It is known that the second fundamental forms of a lightlike hypersurface \( M \) do not depend on the vector bundles \( S(TM), S(TM^+) \) and \( tr(TM) \). Thus, the results of this paper are stable with respect to any change in the above vector bundles.
In [10], Duggal-Bejancu showed that the geometry of a Monge lightlike hypersurface of $R^4_1$ essentially reduces to the geometry of a leaf of its canonical screen distribution. Thus the following question naturally arises: Are there other classes of lightlike hypersurfaces whose geometry is essentially the same as that of their chosen screen distribution?

The above problem has been studied in [3], [4], [11], [12] and [18]. On the other hand it is known that the shape operator plays a key role in studying geometry of submanifolds. In [2], Atindogbe and Duggal introduced screen conformal lightlike hypersurfaces whose shape operators are conformal to shape operators of their corresponding screen distributions. Moreover, they showed that lightlike hypersurface $M$ of a semi-Riemannian manifold $\bar{M}$ is totally geodesic, totally umbilical or minimal if and only if any leaf $M'$ of its integrable distribution is so immersed in $\bar{M}$ as a codimension 2 non-degenerate submanifold.

In this paper, we have shown that the curvature tensor field of a screen conformal lightlike hypersurface in a semi-Euclidean space has directly related with the curvature tensor field of a leaf of its screen distribution $S(TM)$ (Theorem 3.1). Thus we have made further progress in solving above stated problem.

Finally, we note that the results of this paper are valid for a lightlike hypersurface of a flat semi-Riemannian manifold.

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