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Three-Dimensional CR Submanifolds in $S^6(1)$ with Umbilical Direction Normal to $\mathcal{D}_3$

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

It is well known that the sphere $S^6(1)$ admits an almost complex structure J which is nearly Kähler. A submanifold M of an almost Hermitian manifold is called a CR submanifold if it admits a differentiable almost complex distribution $\mathcal{D}_1$ such that its orthogonal complement is a totally real distribution. In this case the normal bundle of the submanifold also splits into two distributions $\mathcal{D}_3$, which is almost complex, and a totally real complement. In the case of the proper three-dimensional CR submanifold of a six-dimensional manifold the distribution $\mathcal{D}_3$ is non-trivial. Here, we investigate three-dimensional CR submanifolds of the sphere $S^6(1)$ admitting an umbilic direction orthogonal to $\mathcal{D}_3$ and show that such submanifolds do not exist.

Keywords: *CR submanifolds, umbilical direction, nearly Kähler 6-sphere.*

AMS Subject Classification (2020): *Primary: 53B25; Secondary: 53C42; 53C40.*

1. Introduction

Let $\widetilde{\nabla}$ and J be the Levi-Civita connection and an almost complex structure of the almost Hermitian manifold $\widetilde{M}$. In the case that the almost complex structure is parallel, i.e. $\widetilde{\nabla}J = 0$ we say that M is a Kähler manifold. If this condition is relaxed to skew-symmetry of the tensor $G(X,Y) = (\widetilde{\nabla}_X J)Y$ then M is a nearly Kähler manifold. A six-dimensional nearly Kähler manifold is strict if it is not Kähler. The investigation of these manifolds and their properties has been initiated by A. Gray, see [11, 12]. In particular, six-dimensional nearly Kähler manifolds play a special role, since in [16], Nagy showed that any complete strict nearly Kähler manifold is locally a Riemannian product of six-dimensional nearly Kähler manifolds, some homogeneous nearly Kähler spaces and twistor spaces over quaternionic Kähler manifolds with positive scalar curvature. We recall that it has been shown by Butruille [7] that there exists only four homogeneous, six-dimensional, nearly Kähler manifolds: the nearly Kähler sphere $S^6(1)$, product manifold $S^3 \times S^3$, the projective space CP^3 and the flag manifold $SU(3)/U(1)\times U(1)$, where only the sphere $S^6(1)$ is endowed with its standard metric.

In this paper we are interested in the study of special types of submanifolds of the nearly Kähler sphere $S^6(1)$. The two most natural classes to regard are the following: those for which J maps the tangent space of the submanifold into the tangent space, i.e. $JT_pM \subset T_pM$, $p \in M$ (and hence $JT_p^{\perp}M \subset T_p^{\perp}M$), and those for which J maps the tangent space into the normal space, i.e. $JT_pM \subset T_p^{\perp}M$. The first type of the submanifolds are called almost complex submanifolds and the second type of submanifolds are totally real submanifolds. If in the second case the dimension and codimension of the submanifold coincide the submanifold is Lagrangian. It is well known that an almost complex submanifold of the sphere S^6 is two-dimensional, see [13], and minimal. Also in [10] it was shown that a Lagrangian submanifold of the sphere $S^6(1)$ is always minimal and orientable.

The notion of CR submanifolds was introduced by Aurel Bejancu, first in [3] for the Kähler manifolds, and later, in [4] its extension was given for submanifolds of almost Hermitian manifolds. The class of CR submanifolds is set in between of the classes of almost complex and totally real submanifolds, as they represent

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a natural generalization of both of these types of submanifolds. A submanifold M of an almost Hermitian manifold is called a CR submanifold if there exists on M a differentiable almost complex distribution $\mathcal{D}_1$ (i.e. $J\mathcal{D}_1 = \mathcal{D}_1$) such that its orthogonal complement $\mathcal{D}_1^\perp \subset TM$ is a totally real distribution. Obviously, if the distribution $\mathcal{D}_1$ or its complement are trivial, we deal with a totally real or an almost complex submanifold. If none of these two distributions is trivial the submanifold is a proper CR submanifold. Trivially any hypersurface of an almost Hermitian manifold is a proper CR submanifold. If we denote by $\mathcal{D}_2 = \mathcal{D}_1^\perp \oplus J(\mathcal{D}_1^\perp)$ and by $\mathcal{D}_3$ the orthogonal complement of $J(\mathcal{D}_1^\perp)$ in the normal bundle we have that $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are almost complex distributions such that $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3 = TM \oplus TM^\perp$. In the case of the sphere $S^6(1)$, proper non-trivial cases are of dimensions three and four, here we refer the reader for the results in three-dimensional case to [17, 9, 14, 2, 1].

In [4] the geometric properties of several special types of CR submanifolds of almost Hermitian manifolds were investigated, among others the properties of totally umbilical CR submanifolds. If for a normal direction $V\xi$ of the submanifold M it holds that $A_\xi = \lambda I$, where A_ξ is the shape operator of the immersion with respect to the section ξ, λ is a differentiable function, and I the identity map, then M is umbilical with the respect of the normal section ξ. If M umbilical with respect to arbitrary normal section then M is said to be totally umbilical. Totally umbilical CR submanifolds of Kähler manifolds have been investigated and classified, see [5, 8, 6]. Also, in the case of the nearly Kaähler manifolds some of the geometric properties are known, see [15]. However, in the case of the sphere $S^6(1)$ it is known that there does not exist a totally umbilical, three-dimensional proper, CR submanifold. Therefore, here we investigate a weaker condition: if a three-dimensional, proper CR submanifold of $S^6(1)$ admits a particular umbilical direction and prove the following theorem.

Theorem 1.1. *There exists no proper three-dimensional CR submanifold of $S^6(1)$ that admits an umbilical direction orthogonal to the distribution $\mathcal{D}_3$.*

2. Preliminaries

First, we give a brief exposition of how the standard nearly Kähler structure J on $S^6(1)$ arises in a natural manner from the Cayley multiplication. The multiplication on the Cayley numbers $\mathcal{O}$ may be used to define a vector cross product $\times$ on the purely imaginary Cayley numbers $\mathbb{R}^7$ using the formula $u \times v = \frac{1}{2}(uv - vu)$. This cross product has many similarities to the cross product in the space $\mathbb{R}^3$, in particular the triple scalar product $\langle u \times v, w\rangle$ is skew symmetric in $u,\ v,\ w$. Then the vectors of the standard orthonormal basis of the space $\mathbb{R}^7$ satisfy the relations given in the following multiplication table.

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

We note that any orthonormal basis or a frame that satisfies relations given in this table is said to be a G_2 basis or a G_2 frame. Such basis or frame is uniquely determined by mutually orthogonal unit vectors e_1, e_2 and e_4 at a given point, such that e_4 is orthogonal to $e_1 \times e_2$. Then other vectors of the G_2 basis are generated by e_1, e_2 and e_4 subject to the relations:

$$e_i \times (e_j \times e_k) + (e_i \times e_j) \times e_k = 2\delta_{ik}e_j - \delta_{ij}e_k - \delta_{jk}e_i.$$

Now, the standard almost complex structure on $S^6(1)$ is obtained as

$$JX = p \times X, \quad X \in T_pS^6(1), \quad p \in S^6(1), \tag{2.1}$$

and moreover, it is a nearly Kähler structure in the sense that the $(2, 1)$-tensor field G on $S^6(1)$ defined by $G(X, Y) = (\bar{\nabla}_X J)Y$, where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$, is skew-symmetric. A straightforward computation also shows that

$$G(X,Y) = X \times Y + \langle X, p \times Y\rangle p.$$

It is important to note the relation between the cross product and the Levi-Civita connection D in $\mathbb{R}^7$. Namely, the following Lemma holds.

Lemma 2.1. *Arbitrary vector fields X, Y, Z in $\mathbb{R}^7$ satisfy*

$$D_X(Y \times Z) = D_X Y \times Z + Y \times D_X Z.$$

Let M be a Riemannian submanifold of a nearly Kähler sphere $S^6(1)$ and let us denote by ∇ and $\nabla^\perp$ the Riemannian connection of M and the normal connection induced from $\bar{\nabla}$ in the normal bundle $T^\perp M$ of M in $S^6(1)$, respectively. Then the formulas of Gauss and Weingarten are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$
$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.3}$$

where $X, Y \in TM$, $\xi \in T^\perp M$ and h and A_ξ are the second fundamental form and the shape operator with respect to the secion ξ, respectively. The second fundamental form and the shape operator are related by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

Also, if we denote by

$$(\nabla h)(X,Y,Z) = \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(X, \nabla_Y Z),$$

for $X, Y, Z \in TM$, we have that the Gauss, Codazzi and Ricci equations yield

$$R(X,Y,Z,W) = \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \tag{2.4}$$
$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z) \tag{2.5}$$
$$\langle R^\perp(X,Y)\xi, \mu\rangle = \langle [A_\xi, A_\mu]X, Y\rangle, \tag{2.6}$$

where we denote by R the Riemannian curvature tensor of M.

A submanifold M is said to be umbilical with respect to the normal section $\xi \in T^\perp M$ if the second fundamental form $h : TM \times TM \to T^\perp M$ satisfies $\langle h(X,Y), \xi\rangle = \lambda\langle X,Y\rangle$, for all $X, Y \in TM$ for some differentiable function λ. If the submanifold is umbilical with respect to any normal section then it is a totally umbilical submanifold, and clearly such submanifolds represent the simplest generalization of the totally geodesic ones.

3. The construction of the suitable moving frame

Here we are dealing with three-dimensional, proper CR submanifolds of $S^6(1)$. Since the almost complex distribution is even dimensional, and nontrivial, it follows that dim $\mathcal{D}_1 = 2$ and then dim $\mathcal{D}_1^\perp = 1$. We present one of the convenient moving frames to work with and the relations between the connection coefficients in it, for details see [9], [2].

We denote by p the position vector field of the submanifold. Let then E_1 and $E_2 = JE_1$ be the unit vector fields which span the tangent almost complex distribution $\mathcal{D}_1$, and E_3 the unit vector field which spans the totally real distribution. Straightforwardly, we have that the vector fields $E_4 = JE_3$, $E_5 = E_1 \times E_3$ and $E_6 = E_2 \times E_3 = JE_5$ are sections of the normal bundle, unit and mutually orthogonal. Here, we have that E_5 and E_6 span the normal almost complex distribution $\mathcal{D}_3$. Note, that by assuming that E_1, E_2 and E_3 are positively oriented, we have that the choice of E_3 is unique. Nevertheless, the choice of the pair E_1, E_2 is not unique, by rotating initial pair E_1, E_2 in the distribution $\mathcal{D}_3$ we obtain another frame

$$\begin{aligned} \bar{E}_1 &= \cos\theta E_1 + \sin\theta E_2, & \bar{E}_2 &= J\bar{E}_1 = -\sin\theta E_1 + \cos\theta E_2, \\ \bar{E}_3 &= E_3, & \bar{E}_4 &= E_4, \\ \bar{E}_5 &= \cos\theta E_5 + \sin\theta E_6, & \bar{E}_6 &= -\sin\theta E_5 + \cos\theta E_6, \end{aligned} \tag{3.1}$$

so the rotation in one of the distributions $\mathcal{D}_1$ or $\mathcal{D}_3$ induces a rotation in the other. For one such frame, let us denote by

$$g_{ijk} = \langle D_{E_i} E_j, E_k \rangle, \qquad h_{ijk} = \langle D_{E_i} E_j, E_{k+3} \rangle, \qquad \eta_{ijk} = \langle D_{E_i} E_{j+3}, E_{k+3} \rangle, \qquad 1 \leq i, j, k \leq 3. \tag{3.2}$$

The connection D is metric, which gives the skew-symmetry of the coefficients g_{ijk} and ηijk, and the second fundamental form symmetric, so we have that

$$g_{ijk} = -g_{ikj}, \quad h_{ijk} = h_{ikj}, \quad \eta_{ijk} = -\eta_{ikj}. \tag{3.3}$$

Moreover, by using the relations of the Lemma 2.2 for the vector fields $X \in \{E_1, E_2, E_3\}$ and $Y, Z \in \{p, E_1, ..., E_6\}$, it follows

Lemma 3.1. *([9]) For the previously defined coefficients the following relations hold*

$$\begin{array}{llll} h_{111} = -g_{123}, & h_{121} = g_{113}, & h_{122} = h_{113}, & h_{123} = -h_{112}, \\ \eta_{113} = -h_{132}, & g_{223} = -g_{113}, & h_{221} = g_{213}, & h_{222} = -h_{112}, \\ h_{223} = -h_{113}, & \eta_{212} = h_{233}, & \eta_{213} = 1 - h_{232}, & h_{131} = -g_{323}, \\ h_{231} = g_{313}, & h_{232} = h_{133} - 1, & h_{233} = -h_{132}, & \eta_{312} = h_{333}, \\ \eta_{313} = -h_{332}, & \eta_{123} = g_{112} - g_{323}, & \eta_{223} = g_{212} + g_{313}, & \eta_{323} = g_{312} + h_{331}, \\ \eta_{112} = 1 + h_{133}. & & & \end{array} \tag{3.4}$$

4. Proof of the main theorem

Now we assume that M has admits an umbilical direction orthogonal to $\mathcal{D}_3$, i.e. in the direction of the vector field E_4, so it satisfies

$$\langle h(X, Y), E_4 \rangle = \lambda \langle X, Y \rangle, \quad X, Y \in TM, \tag{4.1}$$

for some differentiable function λ.

Lemma 4.1. *The coefficients (3.3) satisfy the following relations*

$$\begin{aligned} g_{113} = g_{313} = g_{323} = 0, \\ h_{331} = g_{213} = -g_{123} = \lambda. \end{aligned}$$

Proof. Using the relations from the Lemma 3.1 and relations (4.1) it straightforwardly follows that

$$\begin{aligned} -g_{123} = h_{111} = \langle h(E_1, E_1), E_4 \rangle = \lambda \langle E_1, E_1 \rangle = \lambda, \\ g_{113} = h_{121} = \langle h(E_1, E_2), E_4 \rangle = \lambda \langle E_1, E_2 \rangle = 0, \\ -g_{323} = h_{131} = \langle h(E_1, E_3), E_4 \rangle = \lambda \langle E_1, E_3 \rangle = 0, \\ g_{213} = h_{221} = \langle h(E_2, E_2), E_4 \rangle = \lambda \langle E_2, E_2 \rangle = \lambda, \\ g_{313} = h_{231} = \langle h(E_2, E_3), E_4 \rangle = \lambda \langle E_2, E_3 \rangle = 0, \\ h_{331} = \langle h(E_3, E_3), E_4 \rangle = \lambda \langle E_3, E_3 \rangle = \lambda. \end{aligned}$$

□

We recall that these relations hold for any such frame. Also, the vector field E_3 is uniquely determined up to a sign, so the vector field $D_{E_3} E_3$ is independent of the choice of the frame, while the pairs of the vector fields (E_1, E_2) and (E_5, E_6) can be simultaneously rotated in the distributions $\mathcal{D}_1$ and $\mathcal{D}_3$, respectively. Therefore, we can choose the pair of the vector fields (E_5, E_6) in the distribution $\mathcal{D}_3$, i.e. chose one particular frame, such that the projection of the vector field $D_{E_3} E_3$ to $\mathcal{D}_3$ is in direction of the corresponding vector field E_5. Therefore, it is orthogonal to the vector filed E_6 and we have

$$h_{333} = \langle D_{E_3} E_3, E_6 \rangle = 0.$$

Now, the Gauss, Codazzi and Ricci equations (2.4), (2.5), (2.6) further yield some new relations among the coefficients. In particular we first obtain the expressions for the derivatives of some of the coefficients in directions of the vector fields E_1, E_2 and E_3.

Lemma 4.2. *The derivatives of the differentiable functions (3.3) satisfy*

$$
\begin{aligned}
E_1(g_{212}) &= -1 - g_{112}^2 - g_{212}^2 + 2h_{112}^2 + 2h_{113}^2 - 2g_{312}h_{331} - 2h_{331}^2 + E_2(g_{112}),\\
E_1(h_{331}) &= h_{112} + 2h_{113}h_{132} - 2h_{112}h_{133},\\
E_2(h_{331}) &= h_{113} - 2h_{112}h_{132} - 2h_{113}h_{133},\\
E_1(h_{113}) &= -3g_{112}h_{112} - 3g_{212}h_{113} - 4h_{132}h_{331} + E_2(h_{112}),\\
E_2(h_{113}) &= -3g_{212}h_{112} + 3g_{112}h_{113} - 2h_{331} + 4h_{133}h_{331} - E_1(h_{112}),\\
E_1(h_{133}) &= g_{212} - 2g_{212}h_{133} - 2(g_{112}h_{132} + h_{331}h_{332}) + E_2(h_{132}),\\
E_2(h_{133}) &= -2g_{212}h_{132} + g_{112}(-1 + 2h_{133}) - E_1(h_{132},\\
E_1(g_{312}) &= -g_{212}g_{312} + h_{112} + 2h_{113}h_{132} - 2h_{112}h_{133} + g_{212}h_{331} + E_3(g_{112}),\\
E_3(h_{331}) &= -h_{132} - h_{113}h_{332},\\
E_1(h_{132}) &= -g_{112} - 3g_{312}h_{113} + 2g_{112}h_{133} + E_3(h_{112}),\\
E_2(h_{132}) &= -g_{212} + 3g_{312}h_{112} + 2g_{212}h_{133} + h_{331}h_{332} + E_3(h_{113}),\\
E_1(h_{332}) &= g_{312} - 2g_{312}h_{133} - 3h_{331} + E_3(h_{132}),\\
E_3(h_{133}) &= -2g_{312}h_{132} + g_{112}h_{332},\\
E_2(g_{312}) &= g_{112}g_{312} + h_{113} - 2h_{112}h_{132} - 2h_{113}h_{133} - g_{112}h_{331} + E_3(g_{212}),\\
E_2(h_{332}) &= 2g_{312}h_{132} + E_3(h_{133}),\\
E_3(h_{132}) &= -g_{312} + 2g_{312}h_{133} - 3h_{331} - g_{212}h_{332}.
\end{aligned}
$$

Proof. Gauss equations (2.4), taken respectively for $(X, Y, Z, W) = (E_1, E_2, E_1, E_2)$ and $(X, Y, Z, W) = (E_1, E_2, E_1, E_3)$ yield

$$
\begin{aligned}
E_1(g_{212}) &= -1 - g_{112}^2 - g_{212}^2 + 2h_{112}^2 + 2h_{113}^2 - 2g_{312}h_{331} - 2h_{331}^2 + E_2(g_{112}),\\
E_1(h_{331}) &= h_{112} + 2h_{113}h_{132} - 2h_{112}h_{133}.
\end{aligned}
$$

Further, the Codazzi equation (2.5), for the triplet $(X, Y, Z) = (E_1, E_2, E_1)$ in the directions of the vector fields E_4, E_5 and E_6 respectively gives

$$
\begin{aligned}
E_2(h_{331}) &= h_{113} - 2h_{112}h_{132} - 2h_{113}h_{133},\\
E_1(h_{113}) &= -3g_{112}h_{112} - 3g_{212}h_{113} - 4h_{132}h_{331} + E_2(h_{112}),\\
E_2(h_{113}) &= -3g_{212}h_{112} + 3g_{112}h_{113} - 2h_{331} + 4h_{133}h_{331} - E_1(h_{112}).
\end{aligned}
$$

The other expressions for the given derivatives are obtained in the similar manner. □

We note that, by taking the derivatives from the Lemma 4.2 not all of the Gauss, Codazzi and Ricci equalities are still satisfied.

Lemma 4.3. *For the coefficients (3.3) of the second fundamental form it holds:*

$$1 - 2h_{132}^2 + 2h_{133} - 2h_{133}^2 = 0, \tag{4.2}$$
$$1 - h_{132}^2 - h_{133}^2 + h_{112}h_{332} = 0, \tag{4.3}$$
$$h_{112} + 2h_{113}h_{132} - 2h_{112}h_{133} - h_{332} - 2h_{133}h_{332} = 0, \tag{4.4}$$
$$h_{132} + h_{113}h_{332} = 0, \tag{4.5}$$
$$h_{132}^2 - 2h_{133} + h_{133}^2 + h_{112}h_{332} = 0. \tag{4.6}$$
$$h_{113} - 2h_{112}h_{132} - 2h_{113}h_{133} + 2h_{132}h_{332} = 0. \tag{4.7}$$

Proof. Using Lemma 4.2, from the Gauss equations (2.4) for $(X, Y, Z, W) = (E_1, E_3, E_1, E_3)$, $(X, Y, Z, W) = (E_2, E_3, E_1, E_3)$, respectively, we obtain

$$
\begin{aligned}
1 - h_{132}^2 - h_{133}^2 + h_{112}h_{332} &= 0,\\
h_{132} + h_{113}h_{332} &= 0,
\end{aligned}
$$

Similarly, from the Codazzi equations (2.5) for the triplets $(X,Y,Z)=(E_1,E_2,E_3)$, $(X,Y,Z)=(E_1,E_3,E_3)$, $(X,Y,Z)=(E_2,E_3,E_3)$, $(X,Y,Z)=(E_2,E_3,E_3)$ in the direction of the normal vector field E_4, respectively, we get

$$\begin{aligned}1-2h_{132}^2+2h_{133}-2h_{133}^2&=0,\\ h_{112}+2h_{113}h_{132}-2h_{112}h_{133}-h_{332}-2h_{133}h_{332}&=0,\\ h_{132}^2-2h_{133}+h_{133}^2+h_{112}h_{332}&=0,\\ h_{113}-2h_{112}h_{132}-2h_{113}h_{133}+2h_{132}h_{332}&=0.\end{aligned}$$

Straightforward computation now shows that the other Gauss, Codazzi and Ricci equations do not yield any new relations. □

Now, let us show that such submanifold does not exist.
If we multiply (4.6) by 2 and add (4.2) we get

$$1-2h_{133}+2h_{112}h_{332}=0, \tag{4.8}$$

then from (4.5) and (4.8) we get

$$\begin{aligned}h_{132}&=-h_{113}h_{332},\\ h_{133}&=\frac{1}{2}+h_{112}h_{332}.\end{aligned}$$

Now equations (4.3), (4.4) and (4.7) simplify to

$$\frac{3}{4}-h_{112}^2h_{332}^2-h_{113}^2h_{332}^2=0, \tag{4.9}$$
$$-2h_{332}(1+h_{112}^2+h_{113}^2+h_{112}h_{332})=0, \tag{4.10}$$
$$2h_{113}h_{332}^2=0. \tag{4.11}$$

From (4.11) it follows that $h_{113}=0$ or $h_{332}=0$. If we assume that $h_{332}=0$, then (4.9) reduces to $\frac{3}{4}=0$, which is a contradiction. Therefore, we may take that $h_{113}=0$, $h_{332}\neq 0$ and from (4.9) we get $|h_{112}h_{332}|=\frac{\sqrt{3}}{2}$. If we replace it in (4.10), we get

$$-2h_{332}\left(1\pm\frac{\sqrt{3}}{2}+h_{112}^2\right)=0. \tag{4.12}$$

Since $h_{332}\neq 0$ and $\left(1\pm\frac{\sqrt{3}}{2}\right)+h_{112}^2>0$, we obtain a contradiction. This completes the proof.

Example 4.1. We present here a family of three-dimensional CR submanifolds that do admit an umbilical section ξ that has non-trivial projections both in the direction of the vector field E_4 and on the distribution $\mathcal{D}_3$.
Namely, the first example of a three-dimensional CR submanifold was given in [17] and further generalized in [14]. In [2] it was shown that for a proper three-dimensional CR submanifold two conditions are equivalent: (a) it is minimal and contained in a totally geodesic hyprsphere; (b) its $\mathcal{D}_1$ and $\mathcal{D}_1^{\perp}$ distributions are totally geodesic, i.e. $h(\mathcal{D}_1,\mathcal{D}_1)=0$ and $h(\mathcal{D}_1^{\perp},\mathcal{D}_1^{\perp})=0$. Further it was shown that they are given by

$$\begin{aligned}F(s,x_1,x_2)&=\cos x_1\cos x_2(\cos(\mu_1 s)e_0+\sin(\mu_1 s)e_4)+\sin x_1\cos x_2(\cos(\mu_2 s)e_1+\sin(\mu_2 s)e_5)\\ &\quad+\sin x_2(\cos(\mu_3 s)e_2+\sin(\mu_3 s)e_6),\quad \mu_1,\mu_2,\mu_3\in R,\quad \mu_1+\mu_2+\mu_3=0,\quad \mu_1^2+\mu_2^2+\mu_3^2\neq 0,\end{aligned}$$

where $e_0,\dots,e_6$ is a standard G_2 basis of the space $\mathbb{R}^7$. In particular, these submanifolds are given in [14] with additional one particular example. For these examples it holds that $h(E_i,E_j)\neq 0$ only for $i=1,2$ and $j=3$, and the normal section ξ orthogonal to $h(E_1,E_3)$ and $h(E_2,E_3)$ has non-vanishing components both in direction of E_4 and $\mathcal{D}_3$, see [2]. Then, ξ is trivially an umbilical direction with $\lambda=0$.

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