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A New Multiple Fixed Point Theorem with Applications

Svetlin G.Georgiev[a], Karima Mebarki[b]

[a]*Department of Differential Equations, Faculty of Mathematics and Informatics, University of Sofia, Sofia, Bulgaria.*
[b]*Laboratory of Applied Mathematics, Faculty of Exact Sciences, Bejaia University, 06000 Bejaia, Algeria.*

Abstract

The purpose of this work is to establish an extension of a Bai-Ge type multiple fixed point theorem for a sum of two operators. The arguments are based upon recent fixed point index theory in cones of Banach spaces for a k-set contraction perturbed by an expansive operator. As illustration, our approach is applied to prove the existence of at least three nontrivial nonnegative solutions for a class eigenvalue three-point BVPs for a class of fourth order ordinary differential equations (ODEs for short).

Keywords: fixed point index, cone, sum of operators, positive solution.
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1. Introduction

In recent decades, there has been enormous interest in the development of the fixed point theory due to many applications. The existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. The main tools used are fixed-point theorems in cones of Banach spaces. The most well-known version of Leggett-Williams fixed point theorem [11] provides conditions which ensure the existence of at least three fixed points in cones of Banach spaces of a completely continuous operator. Some new fixed point theorems based on Leggett-Williams's work were established. For example, the five functional fixed point theorem [2] due to Avery, a generalization of the

Email addresses: svetlingeorgiev1@gmail.com (Svetlin G.Georgiev), mebarqi_karima@hotmail.fr and karima.mebarki@univ-bejaia.dz (Karima Mebarki)

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Leggett-Williams fixed-point theorem [3] due to Bai and Ge, were applied to archive some new existence and multiplicity results.

In [3, Theorem 2.1], Bai and Ge have discussed the existence of at least three positive solutions to the nonlinear operational equation

$$Ax = x \tag{1.1}$$

where $\mathcal{P}$ is a cone in a Banach space $(E, \|.\|)$, A is a completely continuous nonlinear map acting in $\mathcal{P}$. Note that the obtained result can be regarded as an extension of the Leggett-Williams fixed point theorem.

The Bai-Ge fixed point theorem states the following.
Let $r > a > 0$, $L > 0$ be constants, ψ a nonnegative concave functional and α, β nonnegative convex functionals on $\mathcal{P}$. Define the convex sets:

$$\mathcal{P}_r = \{x \in \mathcal{P} : \|x\| < r\},$$

$$\mathcal{P}(\alpha, r; \beta, L) = \{x \in \mathcal{P} : \alpha(x) < r,\ \beta(x) < L\},$$
$$\overline{\mathcal{P}}(\alpha, r; \beta, L) = \{x \in \mathcal{P} : \alpha(x) \leq r,\ \beta(x) \leq L\},$$
$$\mathcal{P}(\alpha, r; \beta, L; \psi, a) = \{x \in \mathcal{P} : \alpha(x) < r,\ \beta(x) < L,\ \psi(x) > a\},$$
$$\overline{\mathcal{P}}(\alpha, r; \beta, L; \psi, a) = \{x \in \mathcal{P} : \alpha(x) \leq r,\ \beta(x) \leq L,\ \psi(x) \geq a\}.$$

The following assumptions about the nonnegative convex functionals α, β will be used:

$(\mathcal{A}_1)$ there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for all $x \in \mathcal{P}$;

$(\mathcal{A}_2)$ $\mathcal{P}(\alpha, r; \beta, L) \neq \emptyset$ for all $r > 0,\ L > 0$.

Lemma 1.1. *Let $r_2 \geq d > c > r_1 > 0$, $L_2 \geq L_1 > 0$ be constants. Assume that α, β are nonnegative continuous convex functionals satisfying $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Let ψ be a nonnegative continuous concave functional on $\mathcal{P}$ such that $\psi(x) \leq \alpha(x)$ for all $x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ and let $A : \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \to \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ be a completely continuous operator. Assume*

$(\mathcal{B}_1)$ *$\{x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) : \psi(x) > c\} \neq \emptyset$, $\psi(Ax) > c$ for all $x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c)$;*

$(\mathcal{B}_2)$ *$\alpha(Ax) < r_1$, $\beta(Ax) < L_1$ for all $x \in \overline{\mathcal{P}}(\alpha, r_1; \beta, L_1)$;*

$(\mathcal{B}_3)$ *$\psi(Ax) > c$ for all $x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c)$; with $\alpha(Ax) > d$.*

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ with

$$x_1 \in \mathcal{P}(\alpha, r_1; \beta, L_1),\ \ x_2 \in \{\, x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) :\ \psi(x) > c\,\}$$

and

$$x_3 \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \setminus (\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) \bigcup \overline{\mathcal{P}}(\alpha, r_1; \beta, L_1)).$$

In this work, instead of equation (1.1), we consider the nonlinear operational equation

$$Tx + Fx = x,\ x \in U \bigcap \Omega, \tag{1.2}$$

where $T : \Omega \subset \mathcal{P} \to E$ is an expansive mapping with constant $h > 1$ and $F : U \subset \mathcal{P} \to E$ is a k-set contraction with $k < h - 1$. We will establish a generalization of Lemma 1.1 for such class of operators. The arguments are based upon of the properties of the generalized fixed point index i_*, developed by Djebali and Mebarki in [6].

The paper is organized as follows. In Section 2, we formulate and prove our main result. In Section 3, we lustrate our main result with an application for existence of at least three nonnegative solutions for a class of eigenvalue three-point BVP for a fourth order ODEs.

2. Main Results

Let X be a real Banach space.

Definition 2.1. *A mapping $K : X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

The concept for l-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness.

Definition 2.2. *Let Ω_X be the class of all bounded sets of X. The Kuratowski measure of noncompactness $\alpha : \Omega_X \to [0, \infty)$ is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad and \quad diam(Y_j) \le \delta, \quad j \in \{1, \ldots, m\}\right\},$$

where $diam(Y_j) = \sup\{\|x - y\|_X : x, y \in Y_j\}$ is the diameter of Y_j, $j \in \{1, \ldots, m\}$.

For the main properties of measure of noncompactness we refer the reader to [4].

Definition 2.3. *A mapping $K : X \to X$ is said to be l-set contraction if it is continuous, bounded and there exists a constant $l \ge 0$ such that*

$$\alpha(K(Y)) \le l\alpha(Y),$$

for any bounded set $Y \subset X$. The mapping K is said to be a strict set contraction if $l < 1$.

Obviously, if $K : X \to X$ is a completely continuous mapping, then K is 0-set contraction (see [7]).

Definition 2.4. *Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X$$

for any $x, y \in X$.

Let $\mathcal{P}$ be a cone in X, $\Omega \subset \mathcal{P}$ and U is a bounded open subset of $\mathcal{P}$. Assume that $T : \Omega \to X$ is an expansive mapping with constant $h > 1$ and $F : \overline{U} \to X$ is a k-set contraction. The operator $(I - T)^{-1}$ is $(h - 1)^{-1}$-Lipschtzian on $(I - T)(\Omega)$. Suppose that $0 \le k < h - 1$,

$$F(\overline{U}) \subset (I - T)(\Omega), \tag{2.1}$$

and

$$x \ne Tx + Fx, \text{ for all } x \in \partial U \bigcap \Omega. \tag{2.2}$$

Then $x \ne (I - T)^{-1}Fx$, for all $x \in \partial U$ and the mapping $(I - T)^{-1}F : \overline{U} \to \mathcal{P}$ is a strict $k(h - 1)^{-1}$-set contraction. Indeed, $(I - T)^{-1}F$ is continuous and bounded; and for any bounded set B in U, we have

$$\alpha(((I - T)^{-1}F)(B)) \le (h - 1)^{-1}\, \alpha(F(B)) \le k(h - 1)^{-1}\alpha(B).$$

The fixed point index $i\,((I - T)^{-1}F, U, \mathcal{P})$ is well defined. Thus we put, see [6],

$$i_*\,(T + F, U \bigcap \Omega, \mathcal{P}) = \begin{cases} i\,(T^{-1}(I - F), U, \mathcal{P}), & \text{if } U \bigcap \Omega = \emptyset \\ 0, & \text{if } U \bigcap \Omega = \emptyset. \end{cases} \tag{2.3}$$

This integer is called *the generalized fixed point index* of the sum $T + F$ on $U \bigcap \Omega$ with respect to the cone $\mathcal{P}$.

Theorem 2.5. *[6, Theorem 2.3]. The fixed point index defined in (2.3) satisfies the following properties:*

(a) (Normalization). If $U = \mathcal{P}_r$, $0 \in \Omega$, *and* $Fx = z_0 \in \mathcal{B}(-T0, (h-1)r) \bigcap \mathcal{P}$ *for all* $x \in \overline{\mathcal{P}_r}$, *then*

$$i_*(T+F, \mathcal{P}_r \bigcap \Omega, \mathcal{P}) = 1.$$

(b) (Additivity). For any pair of disjoint open subsets U_1, U_2 *in* U *such that* $T+F$ *has no fixed point on* $(\overline{U} \setminus (U_1 \bigcup U_2)) \bigcap \Omega$, *we have*

$$i_*(T+F, U \bigcap \Omega, \mathcal{P}) = i_*(T+F, U_1 \bigcap \Omega, \mathcal{P}) + i_*(T+F, U_2 \bigcap \Omega, \mathcal{P}),$$

where $i_*(T+F, U_j \bigcap \Omega, X) := i_*(T+F|_{\overline{U_j}}, U_j \bigcap \Omega, \mathcal{P})$, $\quad j = 1, 2$.

(c) (Homotopy Invariance). The fixed point index $i_*(T + H(t,.), U \bigcap \Omega, \mathcal{P})$ *does not depend on the parameter* $t \in [0,1]$ *whenever*

(i) $H : [0,1] \times \overline{U} \to E$ *is continuous and* $H(t,x)$ *is uniformly continuous in* t *with respect to* $x \in \overline{U}$,

(ii) $H([0,1] \times \overline{U}) \subset (I-T)(\Omega)$,

(iii) $H(t,.) : \overline{U} \to E$ *is a* l*-set contraction with* $0 \leq l < h-1$ *and* l *does not depend on* $t \in [0,1]$,

(iv) $Tx + H(t,x) \neq x$, *for all* $t \in [0,1]$ *and* $x \in \partial U \bigcap \Omega$.

(d) (Solvability). If $i_*(T+F, U \bigcap \Omega, \mathcal{P}) \neq 0$, *then* $T+F$ *has a fixed point in* $U \bigcap \Omega$.

Several conditions allowing computation of the fixed point index are shown. Details can be found in [6]. The following lemmas are fundamental for the proofs of our main results.

Lemma 2.6. *Let* X *be a closed convex subset of a Banach space* E, U *a nonempty bounded open subset of* X *and* Ω *be a subset of* X. *Assume that* $T : \Omega \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F : U \to E$ *is a* k*-set contraction with* $k < h-1$ *such that* $F(\overline{U}) \subset (I-T)(\Omega)$, *and* $Tx + Fx \neq x$, *for all* $x \in U \bigcap \Omega$. *The index* i_* *satisfying the following properties:*
(i) (Excision property). Let $V \subset U$ *be an open subset such that* $T+F$ *has no fixed point in* $(\overline{U} \setminus V) \bigcap \Omega$. *Then*

$$i_*(T+F, U \bigcap \Omega, X) = i_*(T+F, V \bigcap \Omega, X).$$

(ii) (Preservation property). If Y *is a retract of* X *and* $\Omega \subset Y$, *then*

$$i_*(T+F, U \bigcap \Omega, X) = i_*(T+F, U \bigcap \Omega, Y).$$

Proof. Properties (i) and (ii) follow directly from the definition (2.3) and [6, Remark 2.4] and the corresponding properties of the fixed point index for strict set contractions (see [9, Theorem 1.3.5] or [1, 5, 10]). □

Lemma 2.7. *Let* X *be a closed convex subset of a Banach space* E, X_1 *a bounded closed convex subset of* X, Ω *be a subset of* X *and* U *a nonempty bounded open convex subset of* X *with* $U \subset X_1$. *Assume that* $T : \Omega \to E$ *is an expansive mapping with constant* $h > 1$ *and* $F : X_1 \to E$ *is a* k*-set contraction with* $k < h-1$. *If*

$$F(X_1) \subset (I-T)(X_1 \bigcap \Omega), \tag{2.4}$$

and

$$Tx + Fx \neq x, \text{ for all } x \in (X_1 \setminus U) \bigcap \Omega. \tag{2.5}$$

then

$$i_*(T+F, U \bigcap \Omega, X) = 1.$$

In particular, if X *is a nonempty bounded convex closed subset of* E, $\Omega \subset X$, $F : X \to E$ *is a* k*-set contraction with* $k < h-1$ *and* $F(X) \subset (I-T)(\Omega)$, *then*

$$i_*(T+F, X \bigcap \Omega, X) = 1.$$

Proof. The mapping $(I-T)^{-1}F : X_1 \to X$ is a strict $k(h-1)^{-1}$-set contraction and it is readily seen that $(I-T)^{-1}F(X_1) \subset X_1$ such that there is no fixed point of $(I-T)^{-1}F$ in $X_1 \setminus U$. Otherwise, there would exist some $x_0 \in X_1 \setminus U$ such that $x_0 = (I-T)^{-1}Fx_0$. Hence, if $x_0 \in \Omega$, we get a contradiction with the condition (2.5). If not we get a contradiction with $(I-T)^{-1}Fx_0 \in \Omega$. The result then follows from the definition (2.3) of the index i_* and [9, Theorem 1.3.6]. □

Our main result is as follows.

Theorem 2.8. *Let $r_2 \geq d > c > r_1 > 0$, $L_2 \geq L_1 > 0$ be constants, $R > M\max(r_2, L_2)$ and $0 \in \Omega \subset \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$. Assume that α, β are nonnegative convex functionals satisfying $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Let ψ be a nonnegative concave functional on $\mathcal{P}$ such that $\psi(x) \leq \alpha(x)$ for all $x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$. Assume that $T : \Omega \to E$ is an expansive mapping with constant $h > 1$ and $F : \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \to E$ is a k-set contraction with $k < h-1$ such that*

$$F\left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) \subset (I-T)(\overline{\mathcal{P}}\,(\alpha, r_1; \beta, L_1) \bigcap \Omega). \tag{2.6}$$

Suppose that:

$(\mathcal{C}_1)$ *if $x \in \mathcal{P}$ with $\alpha(x) = r_1$, then $\alpha(Tx + Fx) \neq r_1$;*

$(\mathcal{C}_2)$ *if $x \in \mathcal{P}$ with $\beta(x) = L_1$, then $\beta(Tx + Fx) \neq L_1$;*

$(\mathcal{C}_3)$ *there exist $z_0 \in \{x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) : \ \psi(x) > c\}$ such that*

$$z_0 \in \mathcal{B}(-T0, (h-1)R), \ \psi((I-T)^{-1}z_0) > c,$$

and

$$tF\left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) + (1-t)z_0 \subset (I-T)(\Omega), \ \textit{for all } t \in [0,1].$$

$(\mathcal{C}_4)$ *$\psi(Tx + Fx) > c$, $\psi(Tx + z_0) \geq c$ and $\alpha(Tx + z_0) \leq d$ for all $x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) \bigcap \Omega$;*

$(\mathcal{C}_5)$ *$\psi(Tx + Fx) > c$ and $\psi(Tx + z_0) \geq c$ for all $x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) \bigcap \Omega$ with $\alpha(Tx + Fx) > d$.*

Then $T + F$ has at least three fixed points x_1, x_2 and x_3 in $\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \bigcap \Omega$ with

$$x_1 \in \mathcal{P}(\alpha, r_1; \beta, L_1), \quad x_2 \in \{\, x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) : \ \psi(x) > c\,\}$$

and

$$x_3 \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \setminus \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) \bigcup \overline{\mathcal{P}}(\alpha, r_1; \beta, L_1).$$

Proof. We list

$$\begin{aligned} U_1 &= \mathcal{P}(\alpha, r_1; \beta, L_1), \\ U_2 &= \{\, x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) : \ \psi(x) > c\,\}. \end{aligned}$$

By assumptions on α, β and ψ, U_1 and U_2 are disjoint bounded nonempty open subsets of $\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$.

Claim 1. We show that $i_*(T + F, U_1 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) = 1$.

We have $Tx + Fx \neq x$ for all $x \in \partial U_1 \bigcap \Omega$. Otherwise there exist $x_0 \in \partial U_1 \bigcap \Omega$ such that $x_0 = Tx_0 + Fx_0$.

If $\alpha(x_0) = r_1$, by condition $(\mathcal{C}_1)$, we get

$$r_1 = \alpha(x_0) = \alpha(Tx_0 + Fx_0) \neq r_1,$$

which is a contradiction.

If $\beta(x_0) = L_1$, by condition $(\mathcal{C}_2)$ we get

$$L_1 = \beta(x_0) = \beta(Tx_0 + Fx_0) \neq L_1,$$

which is a contradiction.

Therefore, for $X = \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$, $X_1 = \overline{U}_1$ and $U = U_1$, Lemma 2.7 applies and gives the conclusion.

Claim 2. We show that $i_*(T + F, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) = 1$.
It is easy to see that this claim follows from the condition (2.6) and Lemma 2.7.

Claim 3. We show that $i_*(T + F, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) = 1$.
Suppose that $x_0 \in \partial U_2 \bigcap \Omega$ is a fixed point of $T + F$; then there is either
Case (i): $\psi(x_0) = c$ with $\alpha(x_0) > d$, or
Case (ii): $\psi(x_0) = c$ with $x_0 \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c)$.
In Case (i), there is $\alpha(Tx_0 + Fx_0) = \alpha(x_0) > d$, which combined with $(\mathcal{C}_5)$ yields
$\psi(x_0) = \psi(Tx_0 + Fx_0) > c$, it is a contradiction.
In Case (ii), $\psi(x_0) = \psi(Tx_0 + Fx_0) > c$, leading again to a contradiction with $(\mathcal{C}_4)$.
Consequently, the fixed point index $i_* \left(T + F, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right)$ is well defined and satisfying the properties (a)-(d) of Theorem [6, Theorem 2.3] as well as the properties given in Lemma 2.6.
New, we consider the homotopic deformation $H : [0, 1] \times \overline{U}_2 \to E$ defined by

$$H(t, x) = tFx + (1 - t)z_0.$$

The operator H is continuous and uniformly continuous in t for each x and from $(\mathcal{C}_3)$ we easily see that $H([0, 1] \times \overline{U}_2) \subset (I - T)(\Omega)$. Moreover, $H(t, .)$ is a k-set contraction for each t and the mapping $T + H(t, .)$ has no fixed point on $\partial U_2 \bigcap \Omega$. Otherwise, there would exist some $x_0 \in \partial U_2 \bigcap \Omega$ and $t_0 \in [0, 1]$ such that

$$x_0 = Tx_0 + H(t_0, x_0).$$

Since $x_0 \in \partial U_2$, we have $\psi(x_0) = c$, so we may distinguish between two cases:
$\alpha(Tx_0 + Fx_0) > d$ and $\alpha(Tx_0 + Fx_0) \leq d$.
If $\alpha(Tx_0 + Fx_0) > d$, the concavity of ψ and the condition $(\mathcal{C}_5)$ lead to

$$\begin{aligned} c = \psi(x_0) &= \psi\left(Tx_0 + H(t_0, x_0)\right) \\ &= \psi\left(Tx_0 + t_0Fx_0 + (1 - t_0)z_0\right) \\ &\geq t_0\psi\left(Tx_0 + Fx_0\right) + (1 - t_0)\,\psi(Tx_0 + z_0) \\ &> c, \end{aligned}$$

which is a contradiction.
If $\alpha(Tx_0 + Fx_0) \leq d$, the convexity of α and the condition $(\mathcal{C}_4)$ lead to

$$\begin{aligned} \alpha(x_0) &= \alpha\left(Tx_0 + H(t_0, x_0)\right) \\ &= \alpha\left(t_0(Tx_0 + Fx_0) + (1 - t_0)(Tx_0 + z_0)\right) \\ &\leq t_0\alpha\left(Tx_0 + Fx_0\right) + (1 - t_0)\,\alpha(Tx_0 + z_0) \\ &\leq d, \end{aligned}$$

Thus, $x_0 \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) \bigcap \Omega$ and by the condition $(\mathcal{C}_4)$, we get $\psi(Tx_0 + Fx_0) > c$, which implies that $\psi(x_0) > c$ and again we come to a contradiction with $\psi(x_0) = c$.
According to the homotopy invariance of the index i_*,

$$i_*\left(T + F, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) = i_*\left(T + z_0, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right).$$

Since $\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \subset \overline{\mathcal{P}}_R$ is closed and convex, so it is a retract of $\overline{\mathcal{P}}_R$ with $\Omega \subset \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$, by the preservation property of the index i_* in Lemma 2.6, we deduce that

$$i_*\left(T + z_0, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) = i_*\left(T + z_0, U_2 \bigcap \Omega, \overline{\mathcal{P}}_R\right). \tag{2.7}$$

Since $U_2 \subset \mathcal{P}_R$ and $T + z_0$ has no fixed point in $\overline{\mathcal{P}}_R \setminus U_2$ (from the condition $(\mathcal{C}_3)$ we have $(I - T)^{-1}z_0 \in U_2$), by the excision property of the index i_* in Lemma 2.6, we deduce that

$$i_*\left(T + z_0, U_2 \bigcap \Omega, \overline{\mathcal{P}}_R\right) = i_*\left(T + z_0, \mathcal{P}_R \bigcap \Omega, \overline{\mathcal{P}}_R\right). \tag{2.8}$$

Then, our claim follow from (2.7), (2.8), the condition $(\mathcal{C}_3)$ and the normality property of the index i_*.

Claim 4. We show that $i_*(T+F, \left(\mathcal{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{U_1 \bigcup U_2})\right) \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) \neq 0$. From the additivity property of the index i_*, we have

$$\begin{aligned} & i_*\left(T+F, \left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \setminus (\overline{U_1 \bigcup U_2})\right) \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) \\ = \; & i_*\left(T+F, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) \\ & - i_*\left(T+F, U_1 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) - i_*\left(T+F, U_2 \bigcap \Omega, \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\right) \\ = \; & 1 - 1 - 1 = -1. \end{aligned}$$

Consequently, $T+F$ has at least three fixed points x_1, x_2 and x_3 in $\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \bigcap \Omega$ such that

$$x_1 \in U_1 \bigcap \Omega, \quad x_2 \in U_2 \bigcap \Omega$$

and

$$x_3 \in \left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \setminus (\overline{U_1 \bigcup U_2})\right) \bigcap \Omega.$$

□

3. Applications

In this section, we will investigate the following eigenvalue three-point boundary value problem

$$\begin{aligned} u^{(4)} &+ \lambda g(t) f(u) = 0, \quad 0 < t < 1, \\ u(0) &= u'(1) = u''(0) = u''(p) - u''(1) = 0, \end{aligned} \tag{3.1}$$

where

$(\mathcal{D}_1)$ $\lambda > 0$, $\lambda \neq 1$, $0 < p < 1$.

$(\mathcal{D}_2)$ $g \in \mathcal{C}([0,1])$ is a nonnegative function such that

$$0 \leq g(t) \leq N, \quad t \in [0,1],$$

N is a positive constant.

$(\mathcal{D}_3)$ $f \in \mathcal{C}([0,\infty))$ is a nonnegative function such that

$$f(0) = 0, |f(x) - f(y)| \leq b|x-y|, \quad x, y \in [0, \infty),$$

b is a positive constant.

Define

$$G_1(t,s) = \begin{cases} t & \text{if} \quad t \leq s \\ s & \text{if} \quad s \leq t, \end{cases}$$

$$G_2(t,s) = \begin{cases} t & \text{if} \quad t \le s \le p \\ s & \text{if} \quad s \le t \quad \text{and} \quad s \le p \\ \frac{1-s}{1-p} t & \text{if} \quad s \ge p \quad \text{and} \quad t \le s \\ s + \frac{p-s}{1-p} t & \text{if} \quad t \ge s \ge p, \end{cases}$$

$$J(t,s) = \int_0^1 G_1(t,v) G_2(v,s) dv, \quad t, s \in [0,1].$$

We have $J(0,s) = 0$, $s \in [0,1]$, and

$$0 \le G_1(t,s), G_2(t,s) \le 1, \quad t, s \in [0,1].$$

Thus,

$$0 \le J(t,s) \le 1, \quad t, s \in [0,1].$$

In [8], when $\lambda \neq 1$, it is proved that $J(\cdot,\cdot)$ is the Green function for the BVP (3.1). Then the solutions of the BVP (3.1) can be represented in the form

$$u(t) = -\lambda \int_0^1 J(t,s) g(s) f(u(s)) ds, \quad t \in [0,1].$$

In [8] the problem (3.1) is investigated for existence of at least one nonnegative solution. Here we investigate the problem (3.1) for existence of at least three nonnegative solutions.

Let $E = \mathcal{C}([0,1])$ be endowed with the maximum norm. Below, suppose that z_0, r_1, r_2, L_1, L_2, b, N, d, c, m, ϵ and R are positive constants that satisfy the following inequalities

($\mathcal{D}_4$)

$$\begin{aligned} r_2 &= L_2 \ge d > \tfrac{d}{\epsilon} \ge z_0 > c > r_1, \\ m &> 1, \quad r_1 = L_1, \quad \lambda = \tfrac{\epsilon}{\epsilon + 1 - \frac{1}{m}}, \end{aligned} \tag{3.2}$$

$$0 < A_1 < 1, \quad \epsilon > 2, \quad \epsilon(1 - A_1) > 1, \tag{3.3}$$

$$z_0 < (\epsilon(1 - A_1) - 1) R, \quad \tfrac{r_2}{m(\epsilon + 1 - \epsilon A_1)} \le r_1. \tag{3.4}$$

Here $A_1 = bN$.

After the proof of the main result in this section, we will give an example for such constants and functions f and g that satisfy ($\mathcal{D}_1$)-($\mathcal{D}_4$). Our main result in this section is as follows.

Theorem 3.1. *Suppose that* ($\mathcal{D}_1$)-($\mathcal{D}_4$) *hold. Then the BVP* (3.1) *has at least three nonnegative solutions.*

Proof. For $u \in E$, define the operators

$$\begin{aligned} T_1 u(t) &= \int_0^1 J(t,s) g(s) f(u(s)) ds, \\ T_2 u(t) &= -\epsilon u(t) - \epsilon T_1 u(t), \\ T u(t) &= T_2 u(t) - z_0, \\ F_1 u(t) &= \frac{1}{m} u(t), \\ F u(t) &= F_1 u(t) + z_0 \quad t \in [0,1]. \end{aligned}$$

Define the functional $\alpha : E \to \mathbb{R}$ as follows

$$\alpha(u) = \begin{cases} z_0 & \text{if} \quad u(0) = 0 \\ \\ |u(0)| & \text{if} \quad u(0) \neq 0. \end{cases}$$

Next, for $u \in E$, define the functionals

$$\psi(u) = \alpha(u), \quad \beta(u) = \|u\|.$$

Let

$$\mathcal{P} \;=\; \{u \in E : u(t) \geq 0, \quad t \in [0,1]\}.$$

Note that any fixed point $u \in \mathcal{P}$ of the operator $T + F$ is a solution to the eigenvalue BVP (3.1). We have that α and β are convex continuous functionals on $\mathcal{P}$ and ψ is a concave continuous functional on $\mathcal{P}$ and

$$\psi(u) = \alpha(u), \quad \|u\| \leq \max(\alpha(u), \beta(u)) \quad u \in \mathcal{P},$$

i.e., $(\mathcal{A}_1)$ holds. Now, let $r > 0$ and $L > 0$ be arbitrarily chosen. Let also, $\widetilde{L} = \min\{r, L\}$. Then $\frac{\widetilde{L}}{2} \in \mathcal{P}$ and

$$\alpha\left(\frac{\widetilde{L}}{2}\right) = \frac{\widetilde{L}}{2} < r, \quad \beta\left(\frac{\widetilde{L}}{2}\right) = \frac{\widetilde{L}}{2} < L.$$

Therefore $\frac{\widetilde{L}}{2} \in \mathcal{P}(\alpha, r; \beta, L)$ and $\mathcal{P}(\alpha, r; \beta, L) \neq \emptyset$. So, $(\mathcal{A}_2)$ holds. Let $\Omega = \overline{\mathcal{P}}(\alpha, \frac{d}{\epsilon}; \beta, \frac{d}{\epsilon})$. We have that $0 \in \Omega$ and $\Omega \subset \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$. For $u, u_1, u_2 \in \mathcal{P}$, we have

$$\begin{aligned} T_1 u(t) &\leq \int_0^1 J(t,s) g(s) f(u(s)) ds \\ &\leq bN \int_0^1 u(s) ds \\ &\leq bN \|u\| \\ &= A_1 \|u\|, \quad t \in [0,1], \end{aligned}$$

$$\begin{aligned} \|T_1 u_1 - T_1 u_2\| &= \max_{t \in [0,1]} \left| \int_0^1 J(t,s) g(s) (f(u_1(s)) - f(u_2(s))) ds \right| \\ &\leq \int_0^1 g(s) |f(u_1(s)) - f(u_2(s))| ds \\ &\leq bN \int_0^1 |u_1(s) - u_2(s)| ds \\ &\leq A_1 \|u_1 - u_2\|. \end{aligned}$$

1. Let $u_1, u_2 \in \Omega$ be arbitrarily chosen. Then

$$\begin{aligned}
\|Tu_1 - Tu_2\| &= \|\epsilon(u_1 - u_2) + \epsilon(T_1u_1 - T_1u_2)\| \\
&\geq \epsilon\|u_1 - u_2\| - \epsilon\|T_1u_1 - T_1u_2\| \\
&\geq \epsilon\|u_1 - u_2\| - \epsilon A_1\|u_1 - u_2\| \\
&= \epsilon(1 - A_1)\|u_1 - u_2\|.
\end{aligned}$$

   Therefore $T : \Omega \to E$ is an expansive operator with a constant $h = \epsilon(1 - A_1) > 1$.

2. We have $F : \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \to E$ is a completely continuous operator and then it is a 0-set contraction.

3. Let $u \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ be arbitrarily chosen. Then

$$\begin{aligned}
\|(I - T_2)u\| &= \|(1 + \epsilon)u + \epsilon T_1u\| \\
&\geq (1 + \epsilon)\|u\| - \epsilon\|T_1u\| \\
&\geq (\epsilon + 1 - \epsilon A_1)\|u\|
\end{aligned}$$

   and

$$\begin{aligned}
\|(I - T_2)u\| &= \|(1 + \epsilon)u + \epsilon T_1u\| \\
&\leq (\epsilon + 1)\|u\| + \epsilon\|T_1u\| \\
&\leq (\epsilon + 1 + \epsilon A_1)\|u\|.
\end{aligned}$$

   Take

$$u_1 = (I - T_2)^{-1}F_1u.$$

   Then $u_1 \in \mathcal{P}$ and

$$\begin{aligned}
\|u_1\| &= \|(I - T_2)^{-1}F_1u\| \\
&\leq \frac{1}{m(\epsilon + 1 - \epsilon A_1)}\|u\| \\
&\leq \frac{r_2}{m(\epsilon + 1 - \epsilon A_1)} \\
&\leq r_1
\end{aligned}$$

   and

$$\begin{aligned}
(I - T)u_1 &= (I - T_2)u_1 + z_0 \\
&= F_1u + z_0 \\
&= Fu.
\end{aligned}$$

Therefore $u_1 \in \Omega$ and

$$F(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) \subset (I-T)(\overline{\mathcal{P}}(\alpha, r_1; \beta, L_1) \bigcap \Omega).$$

4. Let $u \in \mathcal{P}$. Then

$$\begin{aligned} Tu + Fu &= -\left(\epsilon - \frac{1}{m}\right) u - \epsilon T_1 u, \\ (Tu+Fu)(0) &= -\left(\epsilon - \frac{1}{m}\right) u(0), \end{aligned}$$

$$\begin{aligned} \beta(Tu+Fu) &= \beta\left(-\left(\epsilon - \frac{1}{m}\right) u - \epsilon T_1 u\right) \\ &= \beta\left(\left(\epsilon - \frac{1}{m}\right) u + \epsilon T_1 u\right) \\ &\geq \left(\epsilon - \frac{1}{m}\right) \beta(u). \end{aligned}$$

Let $\alpha(u) = r_1$. If $u(0) = 0$, then

$$\alpha(Tu+Fu) = z_0 > r_1.$$

If $u(0) \neq 0$, then $r_1 = \alpha(u) = |u(0)|$ and

$$\alpha(Tu+Fu) = \left(\epsilon - \frac{1}{m}\right) u(0) = \left(\epsilon - \frac{1}{m}\right) r_1 > r_1.$$

If $\beta(u) = L_1$, then

$$\beta(Tu+Fu) \geq \left(\epsilon - \frac{1}{m}\right) L_1 > L_1.$$

Therefore $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ hold.

5. We have $z_0 \in \{x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) : \psi(x) > c\}$,

$$z_0 < (\epsilon(1-A_1)-1)R \quad \text{and} \quad \psi\left((I-T)^{-1} z_0\right) = \psi(0) = z_0 > c.$$

Let now, $u \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ and $\mu \in [0,1]$ be arbitrarily chosen. Take

$$u_2 = (I-T_2)^{-1}(\mu F_1 u).$$

We have $u_2 \in \mathcal{P}$ and

$$\begin{aligned} \|u_2\| &\leq \frac{\|u\|}{m(\epsilon + 1 - \epsilon A_1)} \\ &\leq \frac{r_2}{m(\epsilon + 1 - \epsilon A_1)} \\ &\leq \frac{d}{\epsilon} \end{aligned}$$

and

$$\begin{aligned}(I-T)u_2 &= (I-T_2)u_2+z_0\\ &= \mu F_1 u+\mu z_0+(1-\mu)z_0\\ &= \mu F u+(1-\mu)z_0.\end{aligned}$$

Therefore $u_2 \in \Omega$ and

$$\mu F(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2))+(1-\mu)z_0 \subset (I-T)(\Omega), \quad \mu \in [0,1].$$

Thus, $(\mathcal{C}_3)$ holds.

6. Let $u \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c)\bigcap \Omega$ be arbitrarily chosen. Then $\alpha(u) \le \frac{d}{\epsilon}$ and

$$\begin{aligned}\psi(Tu+Fu) &= \psi\left(\left(\epsilon-\frac{1}{m}\right)u\right)=\begin{cases}\left(\epsilon-\frac{1}{m}\right)u(0)>c & \text{if} \quad u(0)\neq 0\\ z_0>c & \text{if} \quad u(0)=0\end{cases},\\ \psi(Tu+z_0) &= \psi(T_2u)=\psi\left(\epsilon u\right)=\begin{cases}\epsilon u(0)>c & \text{if} \quad u(0)\neq 0\\ z_0>c & \text{if} \quad u(0)=0\end{cases},\\ \alpha(Tu+z_0) &= \alpha(T_2u)=\alpha(\epsilon u)=\begin{cases}\epsilon u(0)\le d & \text{if} \quad u(0)\neq 0\\ z_0\le d & \text{if} \quad u(0)=0.\end{cases}\end{aligned}$$

Thus, $(\mathcal{C}_4)$ holds.

7. Let $u \in \{\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c)\bigcap \Omega : \alpha(Tu+Fu)>d\}$. Then $\alpha(u) \le \frac{d}{\epsilon}$ and

$$\begin{aligned}\psi(Tu+Fu) &= \psi\left(\left(\epsilon-\frac{1}{m}\right)u\right)=\begin{cases}\left(\epsilon-\frac{1}{m}\right)u(0)>c & \text{if} \quad u(0)\neq 0\\ z_0>c & \text{if} \quad u(0)=0\end{cases},\\ \psi(Tu+z_0) &= \psi(T_2u)=\psi\left(\epsilon u\right)=\begin{cases}\epsilon u(0)>c & \text{if} \quad u(0)\neq 0\\ z_0>c & \text{if} \quad u(0)=0\end{cases},\\ \alpha(Tu+z_0) &= \alpha(T_2u)=\alpha(\epsilon u)=\begin{cases}\epsilon u(0)\le r_2 & \text{if} \quad u(0)\neq 0\\ z_0\le r_2 & \text{if} \quad u(0)=0.\end{cases}\end{aligned}$$

Thus, $(\mathcal{C}_5)$ holds.

Hence and Theorem 2.8, it follows that the problem (3.1) has at least three solutions u_1, u_2, u_3 such that $u_1 \in \mathcal{P}(\alpha, r_1; \beta, L_1)$,

$$u_2 \in \{x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) : \psi(x)>c\}$$

and

$$u_3 \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)\backslash\left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c)\bigcup \overline{\mathcal{P}}(\alpha, r_1; \beta, L_1)\right).$$

This completes the proof. □

Example 3.2. *Let*

$$
\begin{aligned}
r_2 &= L_2 = d = b = 1, \quad N = \frac{1}{2}, \quad \epsilon = \frac{1994}{3},\\
z_0 &= \frac{1}{1994}, \quad R = 1, \quad m = 1994, \quad L_1 = r_1 = \frac{3}{1994 \cdot 10^3},\\
\lambda &= \frac{1994^2}{1994 \cdot 1997 - 3}, \quad c = \frac{3}{4 \cdot 1994}.
\end{aligned}
$$

1. *We have*

$$
\begin{aligned}
\epsilon + 1 - \frac{1}{m} &= \frac{1994}{3} + 1 - \frac{1}{1994} = \frac{1994 \cdot 1997 - 3}{3 \cdot 1994},\\
\frac{\epsilon}{\epsilon + 1 - \frac{1}{m}} &= \frac{\frac{1994}{3}}{\frac{1994 \cdot 1997 - 3}{3 \cdot 1994}} = \frac{1994^2}{1994 \cdot 1997 - 3} = \lambda
\end{aligned}
$$

and $\frac{d}{\epsilon} = \frac{3}{1994}$ *and*

$$
r_2 = L_2 = d > \frac{d}{\epsilon} > z_0 > c > r_1, \quad m > 1, \quad r_1 = L_1,
$$

i.e., (3.2) *holds.*

2. *Note that* $A_1 = bN = \frac{1}{2}$ *and*

$$
0 < A_1 < 1, \quad \epsilon > 2, \quad \epsilon(1 - A_1) = \frac{997}{3} > 1,
$$

i.e., (3.3) *holds.*

3. *We have*

$$
(\epsilon(1 - A_1) - 1)R = \frac{997}{3} - 1 = \frac{994}{3} > z_0
$$

and

$$
\begin{aligned}
\frac{d}{10^3 \epsilon} &= \frac{1}{10^3 \cdot \frac{1994}{3}} = \frac{3}{1994 \cdot 10^3} = r_1,\\
\epsilon + 1 - \epsilon A_1 &= \epsilon(1 - A_1) + 1 = \frac{997}{3} + 1 = \frac{10^3}{3},\\
\frac{r_2}{m(\epsilon + 1 - \epsilon A_1)} &= \frac{1}{1994 \cdot \frac{10^3}{3}} = \frac{3}{1994 \cdot 10^3} = r_1,
\end{aligned}
$$

i.e., (3.4) *holds.*

Now, consider the BVP

$$
\begin{aligned}
u^{(4)} + \tfrac{1994^2}{1994 \cdot 1997 - 3} \left(\tfrac{1}{2(1+t^2)}\right) \left(\tfrac{u}{1+u}\right) &= 0, \quad t \in (0, 1),\\
u(0) = u'(1) = u''(0) = u''\left(\tfrac{1}{2}\right) - u''(1) &= 0.
\end{aligned} \tag{3.5}
$$

Here

$$
\lambda = \frac{1994^2}{1994 \cdot 1997 - 3}, \quad g(t) = \frac{1}{2(1 + t^2)}, \quad f(u) = \frac{u}{1 + u},
$$

$t \in [0,1]$, $u \in [0,\infty)$. *Then*

$$0 \le g(t) \le \frac{1}{2}, \quad t \in [0,1],$$

and

$$f(0) = 0, \quad |f(x_1) - f(x_2)| \le |x_1 - x_2|, \quad x_1, x_2 \in [0,\infty).$$

Hence and Theorem 3.1, it follows that the problem (3.5) *has at least three nonnegative solutions.*

Now, suppose that z_0, r_1, r_2, L_1, L_2, b, N, d, c, m, ϵ and R are positive constants that satisfy the following inequalities

$(\mathcal{D}_5)$

$$\begin{aligned} r_2 &= L_2 \ge d > \tfrac{d}{\epsilon+1} \ge z_0 > c > r_1, \\ 1 &> \epsilon > \tfrac{1}{m}, \quad r_1 = L_1, \quad \lambda = \tfrac{\epsilon}{\epsilon - \frac{1}{m}}, \quad 0 < A_1 < 1, \end{aligned} \tag{3.6}$$

$$z_0 < \epsilon(1 - A_1)R, \quad \tfrac{r_2}{m\epsilon(1-A_1)} \le r_1. \tag{3.7}$$

Here $A_1 = bN$.

Our next main result is as follows.

Theorem 3.3. *Suppose that* $(\mathcal{D}_1)$-$(\mathcal{D}_3)$ *and* $(\mathcal{D}_5)$ *hold. Then the* BVP (3.1)*has at least three nonnegative solutions.*

Proof. For $u \in E$, define the operators

$$\begin{aligned} T_3 u(t) &= (\epsilon+1)u(t) + \epsilon T_1 u(t), \\ T_4 u(t) &= T_3 u(t) - z_0, \\ F_2 u(t) &= -F_1 u(t) + z_0 \quad t \in [0,1], \end{aligned}$$

where F_1 and T_1 are as in the proof of Theorem 3.1. Let $\mathcal{P}$, α, β and ψ are as in the proof of Theorem 3.1. Note that, by the proof of Theorem 3.1, it follows that $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. Also, any fixed point $u \in \mathcal{P}$ of the operator $T_4 + F_2$ is a solution to the eigenvalue BVP (3.1). Let $\Omega = \overline{\mathcal{P}}(\alpha, \frac{d}{1+\epsilon}; \beta, \frac{d}{1+\epsilon})$. We have that $0 \in \Omega$ and $\Omega \subset \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$.

1. Let $u_1, u_2 \in \Omega$ be arbitrarily chosen. Then

$$\begin{aligned} \|T_4 u_1 - T_4 u_2\| &= \|(\epsilon+1)(u_1 - u_2) + \epsilon(T_1 u_1 - T_1 u_2)\| \\ &\ge (\epsilon+1)\|u_1 - u_2\| - \epsilon\|T_1 u_1 - T_1 u_2\| \\ &\ge (\epsilon+1)\|u_1 - u_2\| - \epsilon A_1 \|u_1 - u_2\| \\ &= (\epsilon(1 - A_1) + 1)\|u_1 - u_2\|. \end{aligned}$$

   Therefore $T_4 : \Omega \to E$ is an expansive operator with a constant $h = \epsilon(1 - A_1) + 1 > 1$.

2. We have $F_2 : \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \to E$ is a completely continuous operator and then it is a 0-set contraction.

3. Let $u \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ be arbitrarily chosen. Then

$$\begin{aligned} \|(I - T_3)u\| &= \|\epsilon u + \epsilon T_1 u\| \\ &\geq \epsilon\|u\| - \epsilon\|T_1 u\| \\ &\geq \epsilon(1 - A_1)\|u\| \end{aligned}$$

and

$$\begin{aligned} \|(I - T_3)u\| &= \|\epsilon u + \epsilon T_1 u\| \\ &\leq \epsilon\|u\| + \epsilon\|T_1 u\| \\ &\leq \epsilon(1 + A_1)\|u\|. \end{aligned}$$

Take

$$u_1 = -(I - T_3)^{-1} F_1 u.$$

Then $u_1 \in \mathcal{P}$ and

$$\begin{aligned} \|u_1\| &= \|(I - T_3)^{-1} F_1 u\| \\ &\leq \frac{1}{m\epsilon(1 - A_1)}\|u\| \\ &\leq \frac{r_2}{m\epsilon(1 - A_1)} \\ &\leq r_1 \end{aligned}$$

and

$$\begin{aligned} (I - T_4)u_1 &= (I - T_3)u_1 + z_0 \\ &= -F_1 u + z_0 \\ &= F_2 u. \end{aligned}$$

Therefore $u_1 \in \Omega$ and

$$F_2(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) \subset (I - T_4)(\overline{\mathcal{P}}(\alpha, r_1; \beta, L_1) \bigcap \Omega).$$

4. Let $u \in \mathcal{P}$. Then

$$\begin{aligned} T_4u + F_2u &= \left(\epsilon + 1 - \frac{1}{m}\right) u + \epsilon T_1 u, \\ (T_4u + F_2u)(0) &= \left(\epsilon + 1 - \frac{1}{m}\right) u(0), \\ \beta(T_4u + F_2u) &= \beta\left(\left(\epsilon + 1 - \frac{1}{m}\right) u + \epsilon T_1 u\right) \\ &= \beta\left(\left(\epsilon + 1 - \frac{1}{m}\right) u + \epsilon T_1 u\right) \\ &\geq \left(\epsilon + 1 - \frac{1}{m}\right) \beta(u). \end{aligned}$$

Let $\alpha(u) = r_1$. If $u(0) = 0$, then

$$\alpha(T_4u + F_2u) = z_0 > r_1.$$

If $u(0) \neq 0$, then

$$\alpha(T_4u + F_2u) = \left(\epsilon + 1 - \frac{1}{m}\right) u(0) = \left(\epsilon + 1 - \frac{1}{m}\right) r_1 > r_1.$$

If $\beta(u) = L_1$, then

$$\beta(T_4u + F_2u) \geq \left(\epsilon + 1 - \frac{1}{m}\right) L_1 > L_1.$$

Therefore $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ hold.

5. We have $z_0 \in \{x \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) : \psi(x) > c\}$,

$$z_0 < \epsilon(1 - A_1)R \quad \text{and} \quad \psi\left((I - T_4)^{-1} z_0\right) = \psi(0) = z_0 > c.$$

Let now, $u \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)$ and $\mu \in [0, 1]$ be arbitrarily chosen. Take

$$u_2 = -(I - T_3)^{-1}(\mu F_1 u).$$

We have $u_2 \in \mathcal{P}$ and

$$\begin{aligned} \|u_2\| &\leq \frac{\|u\|}{m\epsilon(1 - A_1)} \\ &\leq \frac{r_2}{m\epsilon(1 - A_1)} \\ &\leq \frac{d}{\epsilon + 1} \end{aligned}$$

and

$$\begin{aligned} (I - T_4)u_2 &= (I - T_3)u_2 + z_0 \\ &= -\mu F_1 u + \mu z_0 + (1 - \mu) z_0 \\ &= \mu F_2 u + (1 - \mu) z_0. \end{aligned}$$

Therefore $u_2 \in \Omega$ and

$$\mu F_2(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2)) + (1-\mu)z_0 \subset (I - T_4)(\Omega), \quad \mu \in [0,1].$$

Thus, $(\mathcal{C}_3)$ holds.

6. Let $u \in \overline{\mathcal{P}}(\alpha, d; \beta, L_2; \psi, c) \bigcap \Omega$ be arbitrarily chosen. Then $\alpha(u) \leq \frac{d}{1+\epsilon}$ and

$$\begin{aligned}
\psi(T_4 u + F_2 u) &= \psi\left(\left(\epsilon + 1 - \frac{1}{m}\right)u\right) = \begin{cases} \left(\epsilon + 1 - \frac{1}{m}\right)u(0) > c & \text{if} \quad u(0) \neq 0 \\ z_0 > c \quad \text{if} \quad u(0) = 0 \end{cases}, \\
\psi(T_4 u + z_0) &= \psi(T_3 u) = \psi\left((1+\epsilon)u\right) = \begin{cases} (\epsilon+1)u(0) > c & \text{if} \quad u(0) \neq 0 \\ z_0 > c \quad \text{if} \quad u(0) = 0 \end{cases}, \\
\alpha(T_4 u + z_0) &= \alpha(T_3 u) = \begin{cases} (\epsilon+1)u(0) \leq d & \text{if} \quad u(0) \neq 0 \\ z_0 \leq d \quad \text{if} \quad u(0) = 0. \end{cases}
\end{aligned}$$

Thus, $(\mathcal{C}_4)$ holds.

7. Let $u \in \{\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) \bigcap \Omega : \alpha(T_4 u + F_2 u) > d\}$. Then $\alpha(u) \leq \frac{d}{1+\epsilon}$ and

$$\begin{aligned}
\psi(T_4 u + F_2 u) &= \psi\left(\left(\epsilon + 1 - \frac{1}{m}\right)u\right) = \begin{cases} \left(\epsilon + 1 - \frac{1}{m}\right)u(0) > c & \text{if} \quad u(0) \neq 0 \\ z_0 > c \quad \text{if} \quad u(0) = 0 \end{cases}, \\
\psi(T_4 u + z_0) &= \psi(T_3 u) = \psi\left((1+\epsilon)u\right) = \begin{cases} (\epsilon+1)u(0) > c & \text{if} \quad u(0) \neq 0 \\ z_0 > c \quad \text{if} \quad u(0) = 0 \end{cases}, \\
\alpha(T_4 u + z_0) &= \alpha(T_3 u) = \begin{cases} (\epsilon+1)u(0) \leq r_2 & \text{if} \quad u(0) \neq 0 \\ z_0 \leq r_2 \quad \text{if} \quad u(0) = 0. \end{cases}
\end{aligned}$$

Thus, $(\mathcal{C}_5)$ holds.

Hence and Theorem 2.8, it follows that the problem (3.1) has at least three solutions u_1, u_2, u_3 such that $u_1 \in \mathcal{P}(\alpha, r_1; \beta, L_1)$,

$$u_2 \in \{x \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) : \psi(x) > c\}$$

and

$$u_3 \in \overline{\mathcal{P}}(\alpha, r_2; \beta, L_2) \backslash \left(\overline{\mathcal{P}}(\alpha, r_2; \beta, L_2; \psi, c) \bigcup \overline{\mathcal{P}}(\alpha, r_1; \beta, L_1)\right).$$

This completes the proof. □

Example 3.4. *Let*

$$\begin{aligned}
r_2 &= L_2 = d = b = 1, \quad N = \epsilon = A_1 = \frac{1}{2}, \quad m = 40, \\
z_0 &= \frac{1}{5}, \quad c = \frac{1}{9}, \quad r_1 = \frac{1}{10}, \quad \lambda = \frac{20}{19}, \quad R = 1.
\end{aligned}$$

Then

$$A_1 = \frac{1}{2}, \quad \epsilon + 1 = \frac{3}{2}, \quad \frac{d}{\epsilon+1} = \frac{2}{3}, \quad \epsilon - \frac{1}{m} = \frac{1}{2} - \frac{1}{40} = \frac{19}{40}.$$

1. *We have*

$$r_2 = L_2 = d > 1 > \frac{2}{3} = \frac{d}{\epsilon+1} > \frac{1}{5} = z_0 > \frac{1}{9} = c > \frac{1}{10} = r_1,$$

and

$$1 > \frac{1}{2} = \epsilon > \frac{1}{40} = \frac{1}{m},$$

$$\frac{\epsilon}{\epsilon - \frac{1}{m}} = \frac{\frac{1}{2}}{\frac{19}{40}} = \frac{20}{19} = \lambda, \quad 0 < A_1 < 1,$$

i.e., (3.6) *holds.*

2. *We have*

$$\epsilon(1-A_1)R = \frac{1}{4} > \frac{1}{5} = z_0,$$

$$m\epsilon(1-A_1) = 40 \cdot \frac{1}{2} \cdot \frac{1}{2} = 10, \quad \frac{r_2}{m\epsilon(1-A_1)} = r_1,$$

i.e., (3.7).

Now, consider the BVP

$$\begin{aligned} u^{(4)} + \tfrac{10}{19}\left(\tfrac{1}{1+t^2}\right)\left(\tfrac{u}{1+u}\right) &= 0, \quad t \in (0,1), \\ u(0) = u'(1) = u''(0) = u''\left(\tfrac{1}{2}\right) - u''(1) &= 0. \end{aligned} \tag{3.8}$$

Here

$$\lambda = \frac{20}{19}, \quad g(t) = \frac{1}{2(1+t^2)}, \quad f(u) = \frac{u}{1+u},$$

$t \in [0,1]$, $u \in [0,\infty)$. *Then*

$$0 \le g(t) \le \frac{1}{2}, \quad t \in [0,1],$$

and

$$f(0) = 0, \quad |f(x_1) - f(x_2)| \le |x_1 - x_2|, \quad x_1, x_2 \in [0,\infty).$$

Hence and Theorem 3.3, it follows that the problem (3.8) *has at least three nonnegative solutions.*

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