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Three closed forms for convolved Fibonacci numbers

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Abstract

In the paper, by virtue of the Faà di Bruno formula and several properties of the Bell polynomials of the second kind, the author computes higher order derivatives of the generating function of convolved Fibonacci numbers and, consequently, derives three closed forms for convolved Fibonacci numbers in terms of the falling and rising factorials, the Lah numbers, the signed Stirling numbers of the first kind, and the golden ratio.

Keywords: closed form; convolved Fibonacci number; Faà di Bruno formula; Bell polynomial of the second kind; higher order derivative; generating function; falling factorial; rising factorial; Lah number; Stirling number of the first kind; golden ratio.
2010 MSC: 05A10, 11B83, 11C20, 11Y55, 26C05.

1. Motivation and main results

The well-known Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \quad n \in \mathbb{N}$$

form a sequence of integers and satisfy the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$. The first eight Fibonacci numbers F_n for $1 \le n \le 8$ are $1, 1, 2, 3, 5, 8, 13, 21$. We can generate all the Fibonacci numbers F_n for $n \ge 1$ by

$$\frac{1}{1-t-t^2} = \sum_{n=0}^{\infty} F_{n+1} t^n = 1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + \cdots, \quad |t| < \frac{\sqrt{5}-1}{2}. \tag{1.1}$$

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We can find the definition of convolved Fibonacci numbers $\mathcal{F}_n(x)$

$$F(t,x)=\left(\frac{1}{1-t-t^2}\right)^x=\sum_{n=0}^{\infty}\mathcal{F}_n(x)\frac{t^n}{n!},\quad x\in\mathbb{R} \tag{1.2}$$

in the papers [2, 3, 6, 14]. It is obvious that $\mathcal{F}_n(1)=n!F_{n+1}$ for $n\ge0$.

Kim and his three coauthors [13, Theorem 6] proved in an elementary fashion by induction and recursion that the family of differential equations

$$\frac{\partial^n F(t,x)}{\partial t^n}=\left[\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}a_i(n)(x)_{n-i}\frac{(1+2t)^{n-2i}}{\left(1-t-t^2\right)^{n-i}}\right]F(t,x) \tag{1.3}$$

have a solution $F(t,x)=\left(\frac{1}{1-t-t^2}\right)^x$, where $\lfloor x\rfloor$ is the floor function whose value equals the largest integer less than or equal to x, the quantity

$$(x)_n=\prod_{\ell=0}^{n-1}(x+\ell)=\begin{cases}x(x+1)\cdots(x+n-1), & n\ge1;\\ 1, & n=0\end{cases}$$

denotes the rising factorial, and

$$a_i(n)=\begin{cases}1, & i=0;\\ 2^i\displaystyle\sum_{k_i=1}^{n-2i+1}\sum_{k_{i-1}=1}^{k_i+1}\cdots\sum_{k_1=1}^{k_2+1}\prod_{\ell=1}^{i}k_\ell, & 1\le i\le\left\lfloor\dfrac{n}{2}\right\rfloor;\\ 0, & i=\dfrac{n+1}{2}\end{cases}$$

for all $n\in\mathbb{N}$. Consequently, the authors [13, Corollary 8] derived that

$$\mathcal{F}_n(x)=\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor}a_i(n)(x)_{n-i}. \tag{1.4}$$

It is clear that computing, remembering, and understanding the above quantity $a_i(n)$ by hands and brains are not an easy thing. Generally speaking, beauty, simplicity, symmetry, and recursion are the best and the hope to find solutions in mathematics.

In this paper, our aims are to supply three alternative families of differential equations for the generating function $F(t,x)$ of convolved Fibonacci numbers $\mathcal{F}_n(x)$ in the form of higher order derivatives and to provide three alternative closed forms for convolved Fibonacci numbers $\mathcal{F}_n(x)$ in terms of the falling factorials $\langle x\rangle_k$, the rising factorials $(x)_k$, the Lah numbers $L(n,k)$, the signed Stirling numbers of the first kind $s(n,k)$, and the golden ratio $\frac{\sqrt5+1}{2}$.

Our main results can be stated as the following theorems.

Theorem 1. *The nth partial derivative with respect to t of the generating function $F(t,x)$ can be computed by*

$$\frac{\partial^n F(t,x)}{\partial t^n}=\left[\frac{1}{\left(\frac{\sqrt5-1}{2}-t\right)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x)_k(x)_{n-k}\left(\frac{\frac{\sqrt5-1}{2}-t}{t+\frac{\sqrt5+1}{2}}\right)^k\right]F(t,x). \tag{1.5}$$

Consequently, convolved Fibonacci numbers $\mathcal{F}_n(x)$ for $n\ge0$ can be expressed as

$$\mathcal{F}_n(x)=\frac{1}{\left(\frac{\sqrt5-1}{2}\right)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{\sqrt5-1}{\sqrt5+1}\right)^k(x)_k(x)_{n-k}. \tag{1.6}$$

Theorem 2. *The nth partial derivative with respect to t of the generating function $F(t,x)$ can be computed by*

$$\frac{\partial^n F(t,x)}{\partial t^n} = \left[\sum_{k=0}^{n}(-1)^k x^k \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\frac{s(\ell,r)s(m,s)}{\bigl(t+\frac{\sqrt5+1}{2}\bigr)^{\ell}\bigl(t-\frac{\sqrt5-1}{2}\bigr)^{m}}\right]F(t,x), \tag{1.7}$$

where $s(n,k)$ are the signed Stirling numbers of the first kind which can be generated by

$$\frac{(\ln(1+x))^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x|<1.$$

Consequently, convolved Fibonacci numbers $\mathcal{F}_n(x)$ for $n\ge0$ can be expressed as

$$\mathcal{F}_n(x) = \sum_{k=0}^{n}(-1)^k x^k \sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}s(\ell,r)s(m,s)\left(\frac{\sqrt5+1}{2}\right)^{m-\ell}. \tag{1.8}$$

Theorem 3. *The nth partial derivative with respect to t of the generating function $F(t,x)$ can be computed by*

$$\begin{aligned}\frac{\partial^n F(t,x)}{\partial t^n} &\\ = \left[\sum_{k=0}^{n}\frac{\langle x\rangle_k}{(\sqrt5)^k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^r\binom{n}{\ell}L(\ell,r)L(m,s)\left(t+\frac{\sqrt5+1}{2}\right)^{s-\ell}\left(t-\frac{\sqrt5-1}{2}\right)^{r-m}\right]F(t,x),\end{aligned} \tag{1.9}$$

where

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}, \quad n\ge k\ge0$$

are the Lah numbers and

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases}x(x-1)\cdots(x-n+1), & n\ge1;\\ 1, & n=0\end{cases}$$

is the falling factorial. Consequently, convolved Fibonacci numbers $\mathcal{F}_n(x)$ for $n\ge0$ can be expressed as

$$\mathcal{F}_n(x) = \sum_{k=0}^{n}\frac{\langle x\rangle_k}{(\sqrt5)^k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}L(\ell,r)L(m,s)\left(\frac{\sqrt5+1}{2}\right)^{s+m-r-\ell}. \tag{1.10}$$

2. Lemmas

In order to verify our main results in Theorems 1 to 3, we need some notions and lemmas below.

In combinatorial mathematics, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge0$. See [5, p. 134, Theorem A]. In terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, the Faà di Bruno formula for computing higher order derivatives of composite functions is described in [5, p. 139, Theorem C] by

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f\circ g(x) = \sum_{k=0}^{n} f^{(k)}(g(x))\,\mathrm{B}_{n,k}\bigl(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\bigr). \tag{2.1}$$

The Bell polynomials of the second kind $\mathrm{B}_{n,k}$ have the following properties.

Lemma 1 ([1, Example 2.6] and [5, p. 136, Eq. [3n]]). *The Bell polynomials of the second kind* $\mathrm{B}_{n,k}$ *satisfy*

$$\begin{aligned}\mathrm{B}_{n,k}&(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1})\\&=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})\,\mathrm{B}_{m,s}(y_1,y_2,\dots,y_{m-s+1}).\end{aligned}\tag{2.2}$$

Lemma 2 ([5, p. 135]). *For* $n\ge k\ge 0$, *we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}),\tag{2.3}$$

where a *and* b *are any complex numbers.*

Lemma 3 ([5, p. 135, Theorem B] and [22, p. 27, eq. (3.1)]). *For* $n\ge k\ge 0$, *we have*

$$\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!)=\binom{n-1}{k-1}\frac{n!}{k!}=L(n,k)\tag{2.4}$$

and

$$\mathrm{B}_{n,k}(0!,1!,2!,\dots,(n-k)!)=(-1)^{n-k}s(n,k).\tag{2.5}$$

3. Proofs of main results

Now we are in a position to prove our main results in Theorems 1 to 3.

Proof of Theorem 1. It is clear that the function $\frac{1}{1-t-t^2}$ can be written as

$$\frac{1}{1-t-t^2}=\frac{1}{\big(t+\frac{\sqrt5+1}{2}\big)\big(\frac{\sqrt5-1}{2}-t\big)}.$$

Hence, it follows that

$$F(t,x)=\bigg(\frac{1}{1-t-t^2}\bigg)^x=\frac{1}{\big(t+\frac{\sqrt5+1}{2}\big)^x\big(\frac{\sqrt5-1}{2}-t\big)^x}$$

and

$$\begin{aligned}\frac{\partial^nF(t,x)}{\partial t^n}&=\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^k}{\partial t^k}\Bigg[\frac{1}{\big(t+\frac{\sqrt5+1}{2}\big)^x}\Bigg]\frac{\partial^{n-k}}{\partial t^{n-k}}\Bigg[\frac{1}{\big(\frac{\sqrt5-1}{2}-t\big)^x}\Bigg]\\&=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(x)_k}{\big(t+\frac{\sqrt5+1}{2}\big)^{x+k}}\frac{(x)_{n-k}}{\big(\frac{\sqrt5-1}{2}-t\big)^{x+n-k}}\\&=\bigg(\frac{1}{1-t-t^2}\bigg)^x\frac{1}{\big(\frac{\sqrt5-1}{2}-t\big)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\Bigg(\frac{\frac{\sqrt5-1}{2}-t}{t+\frac{\sqrt5+1}{2}}\Bigg)^k(x)_k(x)_{n-k}\\&\to\frac{1}{\big(\frac{\sqrt5-1}{2}\big)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bigg(\frac{\sqrt5-1}{\sqrt5+1}\bigg)^k(x)_k(x)_{n-k}\end{aligned}$$

as $t\to0$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. It is clear that we can write

$$\bigg(\frac{1}{1-t-t^2}\bigg)^x=e^{-x\ln(1-t-t^2)}.$$

Applying the Faà di Bruno formula (2.1) to $f(u)=e^{\pm u}$ and

$$u=g(t)=\mp x\ln\bigl(1-t-t^2\bigr)=\mp x\biggl[\ln\biggl(t+\frac{\sqrt{5}\,+1}{2}\biggr)+\ln\biggl(\frac{\sqrt{5}\,-1}{2}-t\biggr)\biggr]$$

and employing the formulas (2.2), (2.3), and (2.5) yield

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl[\biggl(\frac{1}{1-t-t^2}\biggr)^x\biggr]&=\sum_{k=0}^n\frac{\mathrm{d}^k\bigl(e^{\pm u}\bigr)}{\mathrm{d}u^k}\mathrm{B}_{n,k}\Biggl(\mp x\biggl[\frac{0!}{t+\frac{\sqrt{5}+1}{2}}+\frac{0!}{t-\frac{\sqrt{5}-1}{2}}\biggr],\mp x\biggl[\frac{(-1)^1 1!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^2}\\
&\quad+\frac{(-1)^1 1!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^2}\biggr],\mp x\biggl[\frac{(-1)^2 2!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^3}+\frac{(-1)^2 2!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^3}\biggr],\dots,\mp x\biggl[\frac{(-1)^{n-k}(n-k)!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{n-k+1}}+\frac{(-1)^{n-k}(n-k)!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{n-k+1}}\biggr]\Biggr)\\
&=\sum_{k=0}^n(\pm1)^k e^{\pm u}(\mp x)^k(-1)^{n+k}\mathrm{B}_{n,k}\Biggl(\frac{0!}{t+\frac{\sqrt{5}+1}{2}}+\frac{0!}{t-\frac{\sqrt{5}-1}{2}},\frac{1!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^2}\\
&\quad+\frac{1!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^2},\frac{2!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^3}+\frac{2!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^3},\dots,\frac{(n-k)!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{n-k+1}}+\frac{(n-k)!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{n-k+1}}\Biggr)\\
&=(-1)^n\biggl(\frac{1}{1-t-t^2}\biggr)^x\sum_{k=0}^n x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}\Biggl(\frac{0!}{t+\frac{\sqrt{5}+1}{2}},\frac{1!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^2},\frac{2!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^3},\dots,\\
&\quad\frac{(\ell-r)!}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell-r+1}}\Biggr)\mathrm{B}_{m,s}\Biggl(\frac{0!}{t-\frac{\sqrt{5}-1}{2}},\frac{1!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^2},\frac{2!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^3},\dots,\frac{(m-s)!}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m-s+1}}\Biggr)\\
&=(-1)^n\biggl(\frac{1}{1-t-t^2}\biggr)^x\sum_{k=0}^n x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\frac{\mathrm{B}_{\ell,r}(0!,1!,\dots,(\ell-r)!)}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^\ell}\frac{\mathrm{B}_{m,s}(0!,1!,\dots,(m-s)!)}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^m}\\
&=(-1)^n\biggl(\frac{1}{1-t-t^2}\biggr)^x\sum_{k=0}^n x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\frac{(-1)^{\ell-r}s(\ell,r)}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^\ell}\frac{(-1)^{m-s}s(m,s)}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^m}\\
&=\biggl(\frac{1}{1-t-t^2}\biggr)^x\sum_{k=0}^n(-1)^k x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\frac{s(\ell,r)s(m,s)}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^\ell\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^m}
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
\mathcal{F}_n(x)=\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl[\biggl(\frac{1}{1-t-t^2}\biggr)^x\biggr]&=\sum_{k=0}^n(-1)^k x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\frac{s(\ell,r)s(m,s)}{\bigl(\frac{\sqrt{5}+1}{2}\bigr)^\ell\bigl(-\frac{\sqrt{5}-1}{2}\bigr)^m}\\
&=\sum_{k=0}^n(-1)^k x^k\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}s(\ell,r)s(m,s)\biggl(\frac{\sqrt{5}\,+1}{2}\biggr)^m\biggl(\frac{\sqrt{5}\,-1}{2}\biggr)^\ell\\
&=\sum_{k=0}^n(-1)^k x^k\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}s(\ell,r)s(m,s)\biggl(\frac{\sqrt{5}\,+1}{2}\biggr)^{m-\ell}.
\end{aligned}
$$

The proof of Theorem 2 is thus complete. □

Proof of Theorem 3. Let $u=g(t)=\frac{1}{1-t-t^2}$. Then, by the Faà di Bruno formula (2.1),

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl[\biggl(\frac{1}{1-t-t^2}\biggr)^x\biggr]&=\sum_{k=0}^n\frac{\mathrm{d}^k(u^x)}{\mathrm{d}u^k}\mathrm{B}_{n,k}\bigl(g'(t),g''(t),\dots,g^{(n-k+1)}(t)\bigr)\\
&=\sum_{k=0}^n\langle x\rangle_k g^{x-k}(t)\,\mathrm{B}_{n,k}\bigl(g'(t),g''(t),\dots,g^{(n-k+1)}(t)\bigr).
\end{aligned}
$$

Since

$$g(t)=\frac{1}{\sqrt{5}}\left(\frac{1}{t+\frac{\sqrt{5}+1}{2}}-\frac{1}{t-\frac{\sqrt{5}-1}{2}}\right)$$

and

$$g^{(k)}(t)=\frac{k!}{\sqrt{5}}\left[\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{k+1}}-\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{k+1}}\right],$$

by the formulas (2.2), (2.3), and (2.4), we obtain

$$\begin{aligned}
\mathrm{B}_{n,k}\bigl(g'(t),g''(t),\dots,g^{(n-k+1)}(t)\bigr)&=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}\left(\frac{1!}{\sqrt{5}}\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{2}},\right.\\
&\quad\left.\frac{2!}{\sqrt{5}}\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{3}},\dots,\frac{(\ell-r+1)!}{\sqrt{5}}\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell-r+2}}\right)\mathrm{B}_{m,s}\left(-\frac{1!}{\sqrt{5}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{2}},\right.\\
&\quad\left.-\frac{2!}{\sqrt{5}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{3}},\dots,-\frac{(m-s+1)!}{\sqrt{5}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m-s+2}}\right)\\
&=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\left(\frac{1}{\sqrt{5}}\right)^{r}\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}}\mathrm{B}_{\ell,r}(1!,2!,\dots,(\ell-r+1)!)\\
&\quad\times\left(-\frac{1}{\sqrt{5}}\right)^{s}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}\mathrm{B}_{m,s}(1!,2!,\dots,(m-s+1)!)\\
&=\left(\frac{1}{\sqrt{5}}\right)^{k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{s}\binom{n}{\ell}\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}L(\ell,r)L(m,s).
\end{aligned}\tag{3.1}$$

Therefore, it follows that

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\left(\frac{1}{1-t-t^2}\right)^{x}\right]&=g^{x}(t)\sum_{k=0}^{n}\frac{\langle x\rangle_k}{g^{k}(t)}\left(\frac{1}{\sqrt{5}}\right)^{k}\\
&\quad\times\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{s}\binom{n}{\ell}L(\ell,r)L(m,s)\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}\\
&=\left(\frac{1}{1-t-t^2}\right)^{x}\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\bigr)^{k}}(1-t-t^2)^{k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{s}\binom{n}{\ell}L(\ell,r)\\
&\quad\times L(m,s)\frac{1}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}}\frac{1}{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}\\
&=\left(\frac{1}{1-t-t^2}\right)^{x}\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\bigr)^{k}}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{s+k}\binom{n}{\ell}L(\ell,r)\\
&\quad\times L(m,s)\frac{\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{k}\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{k}}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}\\
&=\left(\frac{1}{1-t-t^2}\right)^{x}\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\bigr)^{k}}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{r}\binom{n}{\ell}L(\ell,r)\\
&\quad\times L(m,s)\frac{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{s}\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{r}}{\bigl(t+\frac{\sqrt{5}+1}{2}\bigr)^{\ell}\bigl(t-\frac{\sqrt{5}-1}{2}\bigr)^{m}}\\
&=\left(\frac{1}{1-t-t^2}\right)^{x}\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\bigr)^{k}}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^{r}\binom{n}{\ell}L(\ell,r)
\end{aligned}$$

$$\times L(m,s)\biggl(t+\frac{\sqrt{5}\,+1}{2}\biggr)^{s-\ell}\biggl(t-\frac{\sqrt{5}\,-1}{2}\biggr)^{r-m}$$

and, consequently,

$$\begin{aligned}
\mathcal{F}_n(x)&=\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl[\biggl(\frac{1}{1-t-t^2}\biggr)^x\biggr]\\
&=\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\,\bigr)^k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}L(\ell,r)L(m,s)\biggl(\frac{\sqrt{5}\,+1}{2}\biggr)^{s-\ell}\biggl(\frac{\sqrt{5}\,-1}{2}\biggr)^{r-m}\\
&=\sum_{k=0}^{n}\frac{\langle x\rangle_k}{\bigl(\sqrt{5}\,\bigr)^k}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}L(\ell,r)L(m,s)\biggl(\frac{\sqrt{5}\,+1}{2}\biggr)^{s+m-r-\ell}.
\end{aligned}$$

The differential equations in (1.9) and the formula (1.10) are thus verified. The proof of Theorem 3 is complete. □

4. Remarks

Finally, we list several remarks on our main results and other things.

Remark 1. Because the differential equations in (1.5), (1.7), and (1.9) and the closed forms (1.6), (1.8), and (1.10) are expressed in terms of the falling factorials $\langle x\rangle_k$, the rising factorial $(x)_k$, the Lah numbers $L(n,k)$, the signed Stirling numbers of the first kind $s(n,k)$, and the golden ratio $\frac{\sqrt{5}+1}{2}$, they and their proofs are simpler, more meaningful, more significant, and more computable than (1.3) and (1.4) and their proofs in [13].

Remark 2. By (1.1) and (3.1), we can conclude that

$$\begin{aligned}
&\mathrm{B}_{n,k}(1!F_2,2!F_3,\dots,(n-k+1)!F_{n-k+2})\\
&\quad=\biggl(\frac{1}{\sqrt{5}}\biggr)^k\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}\frac{L(\ell,r)L(m,s)}{\bigl(\frac{\sqrt{5}+1}{2}\bigr)^{\ell+r}\bigl(\frac{\sqrt{5}-1}{2}\bigr)^{m+s}}\\
&\quad=\biggl(\frac{1}{\sqrt{5}}\biggr)^k\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}L(\ell,r)L(m,s)\biggl(\frac{\sqrt{5}\,+1}{2}\biggr)^{s+m-r-\ell}.
\end{aligned}$$

By the way, in the papers [8, 20, 34, 39, 43, 48, 54, 50, 55, 56, 59, 64] and closely-related references therein, there are some new results and applications of special values of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$.

Remark 3. In the papers [10, 12, 58] and closely-related references, there are some new results for the Lah numbers $L(n,k)$. In the papers [16, 22, 23, 25, 26, 36] and references cited therein, there are some new results for the Stirling numbers of the first kind $s(n,k)$.

Remark 4. In [15, 34, 35], the authors have discussed the Cauchy product of central Delannoy numbers and other properties of the Delannoy numbers.

Remark 5. We can generate the Fibonacci polynomials

$$F_n(s)=\frac{1}{2^n}\frac{\bigl(s+\sqrt{4+s^2}\,\bigr)^n-\bigl(s-\sqrt{4+s^2}\,\bigr)^n}{\sqrt{4+s^2}},\quad n\in\mathbb{N}$$

by

$$\frac{1}{1-sz-z^2}=\sum_{n=0}^{\infty}F_{n+1}(s)z^n=1+sz+\bigl(s^2+1\bigr)z^2+\bigl(s^3+2s\bigr)z^3+\cdots.$$

One can define the generalized Fibonacci polynomials $F_n(s,t)$ by the initial values

$$F_0(s,t)=0, \quad F_1(s,t)=1,$$

and the recurrence relation

$$F_n(s,t)=sF_{n-1}(s,t)+tF_{n-2}(s,t), \quad n\ge 2.$$

It is easy to deduce that

$$F_2(s,t)=s, \quad F_3(s,t)=s^2+t, \quad F_4(s,t)=s^3+2st, \quad F_5(s,t)=s^4+3s^2t+t^2.$$

We can generate the generalized Fibonacci polynomials $F_n(s,t)$ for $n\in\mathbb{N}$ by

$$\frac{1}{1-sz-tz^2}=\sum_{n=0}^{\infty}F_{n+1}(s,t)z^n.$$

For more information, please refer to [19, 40, 57] and closely-related references therein.

It is clear that $F_n(s,1)=F_n(s)$ and $F_n(1,1)=F_n(1)=F_n$ for $n\in\mathbb{N}$.

Motivated by the definition (1.2) for convolved Fibonacci numbers $\mathcal{F}_n(x)$, we can introduce two notions, convolved Fibonacci polynomials $\mathcal{F}_{n+1}(s;x)$ and convolved generalized Fibonacci polynomials $\mathcal{F}_{n+1}(s,t;x)$, by

$$\left(\frac{1}{1-sz-z^2}\right)^x=\sum_{n=0}^{\infty}\mathcal{F}_{n+1}(s;x)z^n$$

and

$$\left(\frac{1}{1-sz-tz^2}\right)^x=\sum_{n=0}^{\infty}\mathcal{F}_{n+1}(s,t;x)z^n.$$

For recent information, please refer to [4, 7, 67] and closely related references therein.

Remark 6. The idea of this paper comes from the papers and preprints [9, 11, 17, 18, 26, 27, 28, 29, 30, 31, 32, 36, 37, 38, 41, 42, 43, 46, 47, 49, 51, 52, 53, 60, 61, 62, 63, 65, 66, 68].

Remark 7. This paper is a companion of the electronic preprint [7] whose methods have been applied in [21, 35, 44, 45] and closely related references therein.

Remark 8. This paper is an expanded and revised version of the electronic preprints [24, 33].

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Conflict of interest

The author declares that he has no conflict of interest.

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