Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (6) (2021), 1609 – 1619
DOI : 10.15672/hujms.791344

Research Article

Disjoint and simultaneous hypercyclic Rolewicz-type operators

Nurhan Çolakoğlu[1], Özgür Martin*[2]

[1] *Department of Mathematics, Istanbul Technical University, Maslak, 34469, Istanbul, Turkey*
[2] *Department of Mathematics, Mimar Sinan Fine Arts University, Silahşör Cad. 71, Bomonti Şişli 34380, Istanbul, Turkey*

Abstract

We characterize disjoint hypercyclic and supercyclic tuples of unilateral Rolewicz-type operators on $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$, $p \in [1, \infty)$, which are a generalization of the unilateral backward shift operator. We show that disjoint hypercyclicity and disjoint supercyclicity are equivalent among a subfamily of these operators and disjoint hypercyclic unilateral Rolewicz-type operators always satisfy the Disjoint Hypercyclicity Criterion. We also characterize simultaneous hypercyclic unilateral Rolewicz-type operators on $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$, $p \in [1, \infty)$.

Mathematics Subject Classification (2020). 47A16, 47B37, 46B45

Keywords. hypercyclic vectors, hypercyclic operators, disjoint hypercyclicity, simultaneous hypercyclicity

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, X be a separable and infinite dimensional Banach space over the real or complex scalar field $\mathbb{K}$, and let $\mathcal{B}(X)$ denote the algebra of bounded linear operators on X. An operator $T \in \mathcal{B}(X)$ is called *hypercyclic* if there exists $x \in X$ such that $\{T^n x : n \in \mathbb{N}\}$ is dense in X and such a vector x is said to be a hypercyclic vector for T.

The first example of a hypercyclic operator on a Banach space was given in 1969 by Rolewicz [11], who showed that if B is the unweighted unilateral backward shift on $\ell^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. Recall that B is defined as $Be_n = e_{n-1}$, for $n \geq 2$ and $Be_1 = 0$ where $\{e_j : j \geq 1\}$ is the canonical basis.

One can generalize these operators to *unilateral weighted shifts* by multiplying the shifted vector by a weight sequence $(w_n)_{n\in\mathbb{N}}$ of scalars in $\mathbb{K}$, that is, $B_w e_n = w_n e_{n-1}$, for $n \geq 2$ and $B_w e_1 = 0$. In a fundamental paper in the area, Salas [12] completely characterized the hypercyclic unilateral weighted backward shifts on ℓ^p with $1 \leq p < \infty$ in terms of their weight sequences.

Another way to generalize Rolewicz's operators is to change the way these operators shift the vectors. In 2015, Bongiorno, Darji, and Di Piazza [6] studied the dynamics of

*Corresponding Author.
Email addresses: colakn@itu.edu.tr (N. Çolakoğlu), ozgur.martin@msgsu.edu.tr (Ö. Martin)
Received: 07.09.2020; Accepted: 13.06.2021

operators which they call as the *Rolewicz-type operators*: Let the Rolewicz-type operator $\lambda T_f : X \to X$ be defined as

$$\lambda T_f x = \lambda T_f(x_1, x_2, \ldots) = (\lambda x_{f(1)}, \lambda x_{f(2)}, \ldots),$$

where $\lambda \in \mathbb{K}$, $X = c_0$ or $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, and $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function with $f(1) \neq 1$. It is easy to see that, a Rolewicz-type operator λT_f is hypercyclic if and only if $|\lambda| > 1$.

The aim of this paper is to study joint dynamics of tuples of Rolewicz-type operators on $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$, $1 \leq p < \infty$. In particular, we will characterize disjoint hypercyclic and simultaneous hypercyclic Rolewicz-type operators on these spaces.

Disjointness in hypercyclicity is introduced independently by Bernal [2] and by Bès and Peris [5] in 2007. For $N \geq 2$, operators $T_1, \ldots, T_N \in \mathcal{B}(X)$ are called *disjoint hypercyclic* or *d-hypercyclic* if the direct sum operator $T_1 \oplus \cdots \oplus T_N$ has a hypercyclic vector of the form $(x, \ldots, x) \in X^N$. Such a vector $x \in X$ is called a d-hypercyclic vector for $T_1, \ldots, T_N$. If the set of d-hypercyclic vectors of $T_1, \ldots, T_N$ is dense, then $T_1, \ldots, T_N$ are called to be *densely d-hypercyclic*.

The similar and weaker notion of simultaneous hypercyclicity is introduced and studied by Bernal and Jung [3] in 2018. Among many examples, they gave a characterization for simultaneous hypercyclicity of different powers of weighted shifts.

Definition 1.1. [3, Definition 2.1] For $N \geq 2$, the operators $T_1, \ldots, T_N \in \mathcal{B}(X)$ are called *simultaneously hypercyclic* (or *s-hypercyclic*) if there exists $x \in X$ such that

$$\overline{\{(T_1^n x, \ldots, T_N^n x) : n \in \mathbb{N}\}} \supset \Delta\left(X^N\right),$$

where $\Delta\left(X^N\right) := \{(x, x, \ldots, x) : x \in X\}$ denotes the diagonal of X^N. Such a vector x is said to be a *s-hypercyclic vector* of $T_1, \ldots, T_N$. If the set of s-hypercyclic vectors of $T_1, \ldots, T_N$ is dense in X, then $T_1, \ldots, T_N \in \mathcal{B}(X)$ are called as *densely s-hypercyclic*.

In Section 2, we will charcterize disjoint hypercyclic and disjoint supercyclic (see the definition in the beginning of Section 2) Rolewicz-type operators on c_0 and $\ell^p(\mathbb{N})$, $1 \leq p < \infty$. In Section 3, we will give a characterization for simultaneous hypercyclic Rolewicz-type operators.

For more on disjoint hypercyclic and disjoint supercyclic weighted shifts, one can see [4, 5, 9, 10]. For more on hypercyclic operators and chaotic linear dynamics, one can see the books [1] and [8].

In the rest of the Introduction, we introduce the notation and results that we will use throughout the paper.

For any positive integer n, we define $[n] := \{1, 2, \ldots, n\}$, $f(A) := \{f(n) : n \in A\}$ for $A \subset \mathbb{N}$, $f^n := f \circ \ldots \circ f$ which is f composed with itself n many times, and f^{-n} as the inverse of f^n defined on its image $f^n(\mathbb{N})$.

One way to prove the d-hypercyclicity of a tuple of operators is to show that they are d-topologically transitive. $T_1, \ldots, T_N \in \mathcal{B}(X)$ are called as *d-topologically transitive* if for any non-empty open sets $U, V_1, \ldots, V_N \subset X$, there exists a positive integer n such that $U \cap T_1^{-n}(V_1) \cap \ldots \cap T_N^{-n}(V_N) \neq \emptyset$. Bès and Peris [5] proved that operators $T_1, \ldots, T_N$ are densely d-hypercyclic if and only if they are *d-topologically transitive*. In [13], contrary to the single operator case, Sanders and Shkarin showed the existence of d-hypercyclic operators which are not densely d-hypercyclic and, therefore, fail to be d-topologically transitive.

Next, we recall another necessary condition for d-hypercyclicity, a natural extension of the Hypercyclicity Criterion which has played a significant role in linear dynamics.

Definition 1.2. Let (n_k) be a strictly increasing sequence of positive integers. We say that $T_1, \ldots, T_N \in \mathcal{B}(X)$ satisfy the *Disjoint Hypercyclicity Criterion with respect to* (n_k)

provided there exist dense subsets $X_0, X_1, \dots, X_N$ of X and mappings $S_{m,k} : X_m \to X$ with $1 \le m \le N,\ k \in \mathbb{N}$ satisfying

$$\begin{aligned} T_m^{n_k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_0, \\ S_{m,k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_m, \text{ and} \\ (T_m^{n_k} S_{i,k} - \delta_{i,m} Id_{X_m}) &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_m \ (1 \le i \le N). \end{aligned} \tag{1.1}$$

In general, we say that $T_1, \dots, T_N$ satisfy the d-Hypercyclicity Criterion if there exists some sequence (n_k) for which (1.1) is satisfied.

Proposition 1.3. [5, Theorem 2.7] $T_1, T_2, \dots, T_N \in \mathcal{B}(X)$ *satisfy the Disjoint Hypercyclicity Criterion with respect to a sequence* (n_k) *if and only if for each* $r \in \mathbb{N}$*, the direct sum operators* $\bigoplus_{j=1}^r T_1, \dots, \bigoplus_{j=1}^r T_N$ *are d-topologically mixing on* X^N*. In particular, this implies that* $T_1,\dots,$ T_N *are densely d-hypercyclic.*

We now state an analogous criterion for simultaneous hypercyclicity from [3]. Recall that the convex hull conv(A) of a subset A of a vector space X is the smallest convex subset of X containing A.

Definition 1.4. [3, Definition 3.6] Let $(n_k) \subset \mathbb{N}$ be a strictly increasing sequence and $T_j \in \mathcal{B}(X)$ $(j = 1, \dots, N)$. We say that $T_1,\dots,$ T_N satisfy *the Simultaneous Hypercyclicity Criterion* with respect to (n_k) if there are subsets $X_0 \subset X$ and $W_0 \subset X^N$ such that X_0 is dense in X and

$$\overline{W_0} \supset \Delta(X^N)$$

as well as mappings $R_k : W_0 \to X$ $(k \in \mathbb{N})$ such that

(i) $T_j^{n_k} \to 0$ pointwise on X_0 as $k \to \infty$ $(j = 1, \dots, N)$,
(ii) $R_k \to 0$ pointwise on W_0 as $k \to \infty$, and
(iii) for every $w = (w_1, \dots, w_N) \in W_0$ and every $j \in \{1, \dots, N\}$ there is $y_j \in$ conv$(\{w_1 \dots, w_N\})$ such that $T_j^{n_k} R_k w \to y_j$ as $k \to \infty$.

Proposition 1.5. [3, Theorem 3.7] *Let* $T_1, \dots, T_N \in \mathcal{B}(X)$*. If* $T_1,\dots,$ T_N *satisfy the Simultaneous Hypercyclicity Criterion with respect to some* $(n_k) \subset \mathbb{N}$*, then* $T_1,\dots,$ T_N *are densely s-hypercyclic.*

2. Disjoint hypercyclic Rolewicz-type operators

In this section, we give a full characterization for d-hypercyclicity and d-supercyclicity of Rolewicz-type operators and show that these two notions are equivalent among a subfamily of these operators. Recall that $T \in \mathcal{B}(X)$ is called *supercyclic* if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{K} \text{ and } n \in \mathbb{N}\}$ is dense in X. Such a vector x is said to be a *supercyclic vector* for T.

For $N \ge 2$, operators $T_1, \dots, T_N \in \mathcal{B}(X)$ are called *disjoint supercyclic* or *d-supercyclic* if the direct sum operator $T_1 \oplus \cdots \oplus T_N$ has a supercyclic vector of the form $(x, \dots, x) \in X^N$. Such a vector $x \in X$ is called a *d-supercyclic vector* for $T_1, \dots, T_N$. If the set of d-supercyclic vectors of $T_1, \dots, T_N$ is dense, then $T_1, \dots, T_N$ are called to be *densely d-supercyclic.*

Definition 2.1. Let (n_k) be a strictly increasing sequence of positive integers. The operators $T_1, \dots, T_N \in \mathcal{B}(X)$ are said to satisfy the *Disjoint Supercyclicity Criterion* with respect to the sequence (n_k) if there exist dense subsets $X_0, X_1, \dots, X_N$ of X, a sequence (μ_k) in $\mathbb{K}^{\mathbb{N}}$, and mappings $S_{m,k} : X_m \to X$ with $1 \le m \le N$, $k \in \mathbb{N}$ so that for each

$1 \le m \le N$ we have

$$\begin{aligned} \mu_k T_m^{n_k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_0, \\ \frac{1}{\mu_k} S_{m,k} &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_m, \text{ and} \\ (T_m^{n_k} S_{i,k} - \delta_{i,m}\, Id_{X_m}) &\underset{k\to\infty}{\longrightarrow} 0 && \text{pointwise on } X_m \ (1 \le i \le N). \end{aligned} \tag{2.1}$$

Proposition 2.2. [10, Proposition 1.11] *Let $T_1, T_2, \dots, T_N$ be operators on X that satisfy the Disjoint Supercyclicity Criterion with respect to a sequence (n_k). Then, $T_1, \dots, T_N$ are densely d-supercyclic.*

Now, we can give our first characterization.

Theorem 2.3. *Let $\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$ be Rolewicz-type operators on c_0 or $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with $1 < |\lambda_1| \le \dots \le |\lambda_N|$. Then, the following are equivalent:*

(i) *$\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$ are d-supercyclic.*
(ii) *$\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$ are d-hypercyclic.*
(iii) *$\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$ are densely d-hypercyclic.*
(iv) *$\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$ satisfy the Disjoint Hypercyclicity Criterion.*
(v) *For any $k \in \mathbb{N}$ there exists arbitrarily large $n \in \mathbb{N}$ such that*
  (a) *if $1 \le t < s \le N$ with $|\lambda_t| < |\lambda_s|$, then $f_t^n([k]) \cap f_s^n(\mathbb{N}) = \emptyset$, and*
  (b) *if $1 \le t < s \le N$ with $|\lambda_t| = |\lambda_s|$, then $f_t^n([k]) \cap f_s^n(\mathbb{N}) = f_s^n([k]) \cap f_t^n(\mathbb{N}) = \emptyset$.*

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are obvious and (iv) $\Longrightarrow$ (iii) follows from Proposition 1.3.

(i) $\Longrightarrow$ (v):

Let $k \in \mathbb{N}$ and $x = (x_1, x_2, \dots)$ be a d-supercyclic vector for $\lambda_1 T_{f_1}, \dots, \lambda_N T_{f_N}$. Choose $0 \ne \alpha \in \mathbb{K}$ and a large enough $n \in \mathbb{N}$ such that for all $i \in [N]$ we have

$$\|\alpha \lambda_i^n T_{f_i}^n x - i(e_1 + \dots + e_k)\| < \delta \tag{2.2}$$

where $1/2 > \delta > 0$ satisfies

$$\frac{t}{s} - \frac{1}{3N} < \frac{t-\delta}{s+\delta} < \frac{t+\delta}{s-\delta} < \frac{t}{s} + \frac{1}{3N} \tag{2.3}$$

and

$$\frac{\delta}{s-\delta} < \frac{1}{3N} \tag{2.4}$$

for any $1 \le t, s \le N$, and n satisfies

$$\left|\frac{\lambda_s}{\lambda_t}\right|^n \frac{s-\delta}{s+\delta} > 2N \tag{2.5}$$

for any $1 \le t < s \le N$ with $|\lambda_t| < |\lambda_s|$.

Observe that, by (2.2), we have that for all $t \in [N]$ and $j \in [k]$

$$t - \delta < |\alpha \lambda_t^n x_{f_t^n(j)}| < t + \delta \tag{2.6}$$

and for $j > k$

$$|\alpha \lambda_t^n x_{f_t^n(j)}| < \delta. \tag{2.7}$$

Now assume $|\lambda_t| < |\lambda_s|$ for some $1 \le t < s \le N$ and, by way of contradiction, suppose that $f_t^n(j_0) \in f_s^n(\mathbb{N})$ for some $j_0 \in [k]$. We have two cases:

Case 1. Assume $\ell := f_s^{-n}(f_t^n(j_0)) \in [k]$. Then, by (2.3), (2.5), and (2.6),

$$\begin{aligned} N+\frac{1}{3N} > \frac{s+\delta}{t-\delta} > \frac{|\alpha\lambda_s^n x_{f_s^n(j_0)}|}{|\alpha\lambda_t^n x_{f_t^n(j_0)}|} &= \left|\frac{\lambda_s}{\lambda_t}\right|^n \frac{|\alpha\lambda_s^n x_{f_s^n(j_0)}|}{|\alpha\lambda_s^n x_{f_s^n(f_s{}^{-n}(f_t^n(j_0)))}|} \\ &> \left|\frac{\lambda_s}{\lambda_t}\right|^n \frac{|\alpha\lambda_s^n x_{f_s^n(j_0)}|}{|\alpha\lambda_s^n x_{f_s^n(\ell)}|} \\ &> \left|\frac{\lambda_s}{\lambda_t}\right|^n \frac{s-\delta}{s+\delta} > 2N, \end{aligned}$$

which is a contradition.

Case 2. Assume $\ell := f_s^{-n}(f_t^n(j_0)) > k$. Then, by (2.3), (2.4), (2.6), and (2.7),

$$\begin{aligned} \frac{1}{N}-\frac{1}{3N} < \frac{t-\delta}{s+\delta} < \frac{|\alpha\lambda_t^n x_{f_t^n(j_0)}|}{|\alpha\lambda_s^n x_{f_s^n(j_0)}|} &= \left|\frac{\lambda_t}{\lambda_s}\right|^n \frac{|\alpha\lambda_s^n x_{f_s^n(f_s{}^{-n}(f_t^n(j_0)))}|}{|\alpha\lambda_s^n x_{f_s^n(j_0)}|} \\ &< \frac{|\alpha\lambda_s^n x_{f_s^n(\ell)}|}{|\alpha\lambda_s^n x_{f_s^n(j_0)}|} \\ &< \frac{\delta}{s-\delta} < \frac{1}{3N}, \end{aligned}$$

which is again a contradition. Therefore, we can conclude that $f_t^n([k]) \cap f_s^n(\mathbb{N}) = \emptyset$.

Now let $|\lambda_t| = |\lambda_s|$ for some $1 \le t < s \le N$ and, by way of contradiction, assume that $f_t^n(j_0) \in f_s^n(\mathbb{N})$ or $f_s^n(j_0) \in f_t^n(\mathbb{N})$ for some $j_0 \in [k]$. Both of these assumptions will lead us to a similar contradiction so it is enough to consider the first one. Again, we have two cases. Assuming that $\ell := f_s^{-n}(f_t^n(j_0)) \in [k]$, by similar calculations above and by (2.3), (2.5), and (2.6), we have

$$\frac{N-1}{N}+\frac{1}{3N} > \frac{|\alpha\lambda_t^n x_{f_t^n(j_0)}|}{|\alpha\lambda_s^n x_{f_s^n(j_0)}|} > 1-\frac{1}{3N},$$

which gives us a contradition. On the other hand, assuming $\ell := f_s^{-n}(f_t^n(j_0)) > k$, by (2.3), (2.4), (2.6), and (2.7), we have

$$\frac{1}{N}-\frac{1}{3N} < \frac{|\alpha\lambda_t^n x_{f_t^n(j_0)}|}{|\alpha\lambda_s^n x_{f_s^n(j_0)}|} < \frac{1}{3N},$$

which again gives a contradition and finishes the proof of the implication.

(v) $\Longrightarrow$ (iv):

Let $X = \{x^{(k)} : k \in \mathbb{N}\}$ be a countable dense set in $\ell^p(\mathbb{N})$ such that $x^{(k)} \in \operatorname{span}\{e_1, \ldots, e_k\}$. Then for $1 \le j \le N$ and for all $x \in X$, we have $\lambda_j^n T_{f_j}^n x \to 0$ as $n \to \infty$. Now we define $S_j := \frac{1}{\lambda_j} F_{f_j}$ where $F_{f_j} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ is given as

$$F_{f_j}(x_1, x_2, \ldots) := (0, \ldots, 0, x_1, 0, \ldots, 0, x_2, 0, \ldots), \tag{2.8}$$

where x_k is in the $f(k)^{th}$ position, for $k \in \mathbb{N}$. Since $|\lambda_j| > 1$, $S_j x \to 0$ as $n \to \infty$ and $\lambda_j T_{f_j} S_j x = x$ for $1 \le j \le N$ and for all $x \in X$.

Observe that for $1 \le t, s \le N$

$$\lambda_t^n T_{f_t}^n S_s^n e_k = \left(\frac{\lambda_t}{\lambda_s}\right)^n T_{f_t}^n F_{f_s}^n e_k = \begin{cases} \left(\frac{\lambda_t}{\lambda_s}\right)^n e_{f_t^{[-n]}(f_s^{[n]}(k))}, & \text{if } f_s^{[n]}(k) \in \operatorname{Im}(f_t^{[n]}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Now, for each $k \in \mathbb{N}$ we choose a big enough $n_k \in \mathbb{N}$ such that for $1 \le t < s \le N$ with $|\lambda_t| = |\lambda_s|$ we have $\lambda_t^{n_k} T_{f_t}^{n_k} S_s^{n_k} x^{(k)} = \lambda_s^{n_k} T_{f_s}^{n_k} S_t^{n_k} x^{(k)} = 0$ and for $1 \le t < s \le N$ with $|\lambda_t| < |\lambda_s|$ we have $\lambda_s^{n_k} T_{f_s}^{n_k} S_t^{n_k} x^{(k)} = 0$ and

$$\|\lambda_t^{n_k} T_{f_t}^{n_k} S_s^{n_k} x^{(k)}\| \le \left|\frac{\lambda_t}{\lambda_s}\right|^{n_k} \|x^{(k)}\| < \frac{1}{k}.$$

Therefore, it is easy to see that $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ satisfy the Disjoint Hypercyclicity Criterion with respect to the sequence $(n_k)_{k=1}^{\infty}$ where $X_j = X$ and $S_{j,n} = S_j^n$ for $1 \le j \le N$. □

Recall that T_1, T_2 are said to be *d-weakly mixing* if the direct sums $T_1 \oplus T_1, T_2 \oplus T_2$ are d-topologically transitive on $X \times X$.

Example 2.4. Let $T_1 = \lambda B = \lambda T_{f_1}$ and $T_2 = \mu B^2 = \mu T_{f_2}$, where $f_1(n) = n + 1$ and $f_2(n) = n + 2$ for $n \in \mathbb{N}$. The following assertions are equivalent:

(1) the operators T_1 and T_2 are d-hypercyclic;
(2) the operators T_1 and T_2 are d-weakly mixing;
(3) $1 < |\lambda| < |\mu|$.

It is easy to see this using Theorem 2.3 and Proposition 1.3.

Example 2.5. Let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be defined as $f_1(n) = 2n$ and $f_2(n) = 2n+1$ for $n \in \mathbb{N}$. Then, since $f_1^n(\mathbb{N}) \cap f_2^n(\mathbb{N}) = \emptyset$ for all $n \in \mathbb{N}$, $\lambda T_{f_1}, \mu T_{f_2}$ are d-hypercyclic if and only if $|\lambda|, |\mu| > 1$.

Of course, the condition $|\lambda_1|, \ldots, |\lambda_N| > 1$ is not necessary for d-supercyclicity. For non-zero scalars, we can give the following characterization:

Theorem 2.6. *Let $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ be Rolewicz-type operators on c_0 or $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with $0 < |\lambda_1| \le \cdots \le |\lambda_N|$. Then, the following are equivalent:*

(i) *$\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ are d-supercyclic.*
(ii) *$\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ are densely d-supercyclic.*
(iii) *$\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ satisfy the d-Supercyclicity Criterion.*
(iv) *For any $k \in \mathbb{N}$ there exists arbitrarily large $n \in \mathbb{N}$ such that*
  (a) *if $1 \le t < s \le N$ with $|\lambda_t| < |\lambda_s|$, then $f_t^n([k]) \cap f_s^n(\mathbb{N}) = \emptyset$, and*
  (b) *if $1 \le t < s \le N$ with $|\lambda_t| = |\lambda_s|$, then $f_t^n([k]) \cap f_s^n(\mathbb{N}) = f_s^n([k]) \cap f_t^n(\mathbb{N}) = \emptyset$.*

Proof. (i) $\Longrightarrow$ (iv):

The proof of this implication is just like the proof of the implication (i) $\Longrightarrow$ (v) in Theorem 2.3 since the condition $|\lambda_1|, \ldots, |\lambda_N| > 1$ is not used in the proof.

(iv) $\Longrightarrow$ (iii):

Let $\mu = \max\{\frac{1}{|\lambda_1|}, \ldots, \frac{1}{|\lambda_N|}\}$ and $(\mu_k)_{k\in\mathbb{N}} = ((1+\mu)^k)_{k\in\mathbb{N}}$. Let $X = \{x^{(k)} : k \in \mathbb{N}\}$ be a countable dense set in $\ell^p(\mathbb{N})$ such that $x^{(k)} \in \operatorname{span}\{e_1, \ldots, e_k\}$. Then, for $1 \le j \le N$ and for all $x \in X$, we have $\mu_n \lambda_j^n T_{f_j}^n x \to 0$ as $n \to \infty$.

Now we define $S_j := \frac{1}{\lambda_j} F_{f_j}$, where $F_{f_j} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ is defined as in (2.8). Thus, $\frac{1}{\mu_n} S_j^n \to 0$ as $n \to \infty$ and $\lambda_j T_{f_j} S_j x = x$ for $1 \le j \le N$ and for all $x \in X$.

Now, by similar calculations as in the proof of the implication (v) $\Longrightarrow$ (iv) in Theorem 2.3, we can see that for each $k \in \mathbb{N}$ we choose a big enough $n_k \in \mathbb{N}$ such that for $1 \le t < s \le N$ with $|\lambda_t| = |\lambda_s|$, we have $\lambda_t^{n_k} T_{f_t}^{n_k} S_s^{n_k} x^{(k)} = \lambda_s^{n_k} T_{f_s}^{n_k} S_t^{n_k} x^{(k)} = 0$. Also, for $1 \le t < s \le N$ with $|\lambda_t| < |\lambda_s|$, we have $\lambda_s^{n_k} T_{f_s}^{n_k} S_t^{n_k} x^{(k)} = 0$ and $\|\lambda_t^{n_k} T_{f_t}^{n_k} S_s^{n_k} x^{(k)}\| < \frac{1}{k}$.

Therefore, we see that $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ satisfy the Disjoint Supercyclicity Criterion with respect to the sequences $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ and $(\mu_k)_{k=1}^{\infty}$ in $\mathbb{K}$, where $X_j = X$ and $S_{j,n} = S_j^n$ for $1 \le j \le N$.

(iii) $\Longrightarrow$ (ii) follows from Proposition 2.2 and the implication (ii) $\Longrightarrow$ (i) is obvious. □

3. Simultaneous hypercyclic Rolewicz-type operators

In this section, we characterize simultaneous hypercyclic Rolewicz-type operators.

Theorem 3.1. *Let $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ be Rolewicz-type operators on c_0 or $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with $1 < |\lambda_1| \le \cdots \le |\lambda_N|$. Then, the following are equivalent:*

(i) $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ *are s-hypercyclic.*
(ii) $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ *are densely s-hypercyclic.*
(iii) $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$ *satisfy the Simultaneous Hypercyclicity Criterion.*
(iv) *For every* $\varepsilon > 0$ *and for every* $k \in \mathbb{N}$ *there exists arbitrarily large* $n_k \in \mathbb{N}$ *such that*
  (a) *if* $1 \leq t < s \leq N$ *with* $|\lambda_t| < |\lambda_s|$, *then* $f_t^{n_k}([k]) \cap f_s^{n_k}(\mathbb{N}) = \emptyset$.
  (b) *if* $1 \leq t < s \leq N$ *with* $|\lambda_t| = |\lambda_s|$, *then*

$$f_t^{n_k}([k]) \cap f_s^{n_k}(\mathbb{N} \setminus [k]) = f_s^{n_k}([k]) \cap f_t^{n_k}(\mathbb{N} \setminus [k]) = \emptyset$$

  and one of the following is satisfied
  (b1) $f_t^{n_k}([k]) \cap f_s^{n_k}([k]) = \emptyset$,
  (b2) *for every* $\ell \in f_t^{n_k}([k]) \cap f_s^{n_k}([k])$,

$$f_t^{-n_k}(\ell) = f_s^{-n_k}(\ell)$$

  and

$$\left| \left(\frac{\lambda_t}{\lambda_s} \right)^{n_k} - 1 \right| < \varepsilon.$$

Proof. (i) $\Longrightarrow$ (iv):
Let $\varepsilon > 0$, $k \in \mathbb{N}$ and let $x = (x_1, x_2, \ldots)$ be a s-hypercyclic vector for $\lambda_1 T_{f_1}, \ldots, \lambda_N T_{f_N}$. Choose a large enough n_k such that for $i = 1, 2, \ldots, N$ we have

$$\|\lambda_i^{n_k} T_{f_i}^{n_k} x - \sum_{j=1}^{k} j\, e_j\| < \delta, \tag{3.1}$$

where $0 < \delta < \frac{1}{k+2}$ and for all $t, s \in \{1, 2, \ldots, N\}$, n_k satisfies

$$\left(\frac{k + \delta}{1 - \delta} \right)^2 < \left| \frac{\lambda_s}{\lambda_t} \right|^{n_k} \tag{3.2}$$

if $|\lambda_t| < |\lambda_s|$.

Observe that, by (3.1), we have that for $i = 1, 2, \ldots, N$ and $j \in [k]$

$$j - \delta < \left| \lambda_i^{n_k} x_{f_i^{n_k}(j)} \right| < j + \delta \tag{3.3}$$

and for $j > k$

$$\left| \lambda_i^{n_k} x_{f_i^{n_k}(j)} \right| < \delta. \tag{3.4}$$

Now assume $|\lambda_t| < |\lambda_s|$ and, by way of contradiction, suppose that $f_t^{n_k}(j_0) \in f_s^{n_k}(\mathbb{N})$ for some $j_0 \in [k]$. We have two cases:

Case 1. Assume $\ell := f_s^{-n_k}(f_t^{n_k}(j_0)) \in [k]$. Then, by (3.2) and (3.3),

$$\begin{aligned} \frac{k + \delta}{1 - \delta} \geq \frac{j_0 + \delta}{j_0 - \delta} > \frac{\left| \lambda_s^{n_k} x_{f_s^{n_k}(j_0)} \right|}{\left| \lambda_t^{n_k} x_{f_t^{n_k}(j_0)} \right|} &= \left| \frac{\lambda_s}{\lambda_t} \right|^{n_k} \left| \frac{\lambda_s^{n_k} x_{f_s^{n_k}(j_0)}}{\lambda_s^{n_k} x_{f_s^{n_k}(\ell)}} \right| \\ &> \left| \frac{\lambda_s}{\lambda_t} \right|^{n_k} \frac{j_0 - \delta}{\ell + \delta} \\ &> \left(\frac{k + \delta}{1 - \delta} \right)^2 \frac{1 - \delta}{k + \delta} \\ &= \frac{k + \delta}{1 - \delta}, \end{aligned}$$

which is a contradition.

Case 2. Assume $\ell := f_s^{-n_k}(f_t^{n_k}(j_0)) > k$. Then, by (3.3) and (3.4),

$$\frac{1-\delta}{k+\delta} \le \frac{j_0-\delta}{j_0+\delta} < \frac{\left|\lambda_t^{n_k} x_{f_t^{n_k}(j_0)}\right|}{\left|\lambda_s^{n_k} x_{f_s^{n_k}(j_0)}\right|} = \left|\frac{\lambda_t}{\lambda_s}\right|^{n_k} \left|\frac{\lambda_s^{n_k} x_{f_s^{n_k}(\ell)}}{\lambda_s^{n_k} x_{f_s^{n_k}(j_0)}}\right| < \frac{\delta}{j_0-\delta} \le \frac{\delta}{1-\delta},$$

which contradicts $\delta < \frac{1}{k+2}$.

Therefore, we can conclude that $f_t^{n_k}([k]) \cap f_s^{n_k}(\mathbb{N}) = \emptyset$.

Now let $|\lambda_t| = |\lambda_s|$. Note that the reasoning in Case 2 works here too. Consequently, because of the symmetry, we have $f_t^{m_k}([k]) \cap f_s^{n_k}(\mathbb{N} \setminus [k]) = f_s^{n_k}([k]) \cap f_t^{n_k}(\mathbb{N} \setminus [k]) = \emptyset$.

Now, by way of contradiction, assume that $\ell \in f_t^{n_k}([k]) \cap f_s^{n_k}([k])$ and $j_1 = f_t^{-n_k}(\ell) > f_s^{-n_k}(\ell) = j_2$. Then, by (3.3),

$$\frac{j_1-\delta}{j_2+\delta} < \frac{\left|\lambda_t^{n_k} x_{f_t^{n_k}(j_1)}\right|}{\left|\lambda_s^{n_k} x_{f_s^{n_k}(j_2)}\right|} = \left|\frac{\lambda_t}{\lambda_s}\right|^{n_k} \left|\frac{x_\ell}{x_\ell}\right| = 1,$$

which gives us a contradiction. Therefore, we conclude that $f_t^{-n_k}(\ell) = f_s^{-n_k}(\ell)$.

Now we have two cases:

If $f_t^{n_k}([k]) \cap f_s^{n_k}([k]) = \emptyset$ for all $k \in \mathbb{N}$, then (iv)(b1) is satisfied.

Otherwise there exists a k_0 such that

$$f_t^{n_k}([k_0]) \cap f_s^{n_k}([k_0]) \ne \emptyset \quad \text{for all } k \ge k_0. \tag{3.5}$$

If this is not the case, then it is possible to choose a subsequence of (n_k) for which (b1) is satisfied. By (3.5), there exist $j_0 \in [k_0]$ and $(m_k) \subset (n_k)$ such that

$$f_t^{m_k}(j_0) = f_s^{m_k}(j_0) \quad \text{for all } k \ge k_0.$$

By (3.1), it follows that as for all $k \ge k_0$

$$j_0 - \delta < \left|\lambda_t^{m_k} x_{f_t^{m_k}(j_0)}\right| < j_0 + \delta$$

and

$$j_0 - \delta < \left|\lambda_s^{m_k} x_{f_s^{m_k}(j_0)}\right| < j_0 + \delta.$$

Consequently,

$$\left|\left(\frac{\lambda_t}{\lambda_s}\right)^{m_k} - 1\right| = \left|\frac{\lambda_t^{m_k} x_{f_t^{m_k}(j_0)}}{\lambda_s^{m_k} x_{f_s^{m_k}(j_0)}} - 1\right| = \left|\frac{\lambda_t^{m_k} x_{f_t^{m_k}(j_0)} - j_0 + j_0 - \lambda_s^{m_k} x_{f_s^{m_k}(j_0)}}{\lambda_s^{m_k} x_{f_s^{m_k}(j_0)}}\right| < \frac{2\delta}{j_0-\delta}.$$

Then we rename (m_k) as (n_k) and (iv) is satisfied for (n_k).

(iv) $\implies$ (iii):

We will give the proof for two Rolewicz-type operators $\lambda_1 T_{f_1}$, $\lambda_2 T_{f_2}$ for brevity. The general case follows similarly.

If (iv)(a) is satisfied, then, from Theorem 2.3, it follows that $\lambda_1 T_{f_1}$, $\lambda_2 T_{f_2}$ satisfy the Disjoint Hypercyclicity Criterion and consequently, by [3, Remark 3.8.3], they satisfy the Simultaneous Hypercyclicity Criterion.

If conditions of (iv)(b) are satisfied for some strictly increasing sequence (m_k), let $X := \ell^p(\mathbb{N})$ and X_0 be the set of finitely supported sequences in X. It follows that $\overline{X_0} = X$. Let $W_0 := \Delta(X_0^2) \subset X^2$. Note that $\overline{W_0} = \overline{\Delta(X_0^2)} = \Delta(X^2)$. Now we want to show that $\lambda_1 T_{f_1}$, $\lambda_2 T_{f_2}$ satisfy the conditions of the Simultaneous Hypercyclicity Criterion (Definition 1.4).

It is clear that $(\lambda_i T_{f_i})^{m_k} \to 0$ pointwise on X_0 as $k \to \infty$ for $i = 1, 2$. Define, for any $k \in \mathbb{N}$, the mapping $R_k : W_0 \to X$ as follows. Let $A_k := \{j \in [k] : f_1^{m_k}(j) = f_2^{m_k}(j)\}$.

If $x = (x_1, x_2, \dots, x_N, 0, 0 \dots) \in X_0$ and $w = (x, x) \in W_0$, then

$$R_k w := \begin{cases} \sum_{j=1}^{N} \frac{x_j}{\lambda_1^{m_k}} e_{f_1^{m_k}(j)} + \sum_{\substack{j=1 \\ j \notin A_k}}^{N} \frac{x_j}{\lambda_2^{m_k}} e_{f_2^{m_k}(j)}, & k \geq N \\ 0, & k < N \end{cases}.$$

Since $1 < |\lambda_1| = |\lambda_2|$, $R_k \to 0$ pointwise on W_0 as $k \to \infty$.

Note that

$$(\lambda_1 T_{f_1})^{m_k} R_k w = \sum_{j=1}^{N} \frac{x_j}{\lambda_1^{m_k}} \lambda_1^{m_k} e_j = \sum_{j=1}^{N} x_j = x,$$

so $(\lambda_1 T_{f_1})^{m_k} R_k w \to x$ as $k \to \infty$.

Note also that

$$(\lambda_2 T_{f_2})^{m_k} R_k w = \sum_{\substack{j=1 \\ j \in A_k}}^{N} \frac{x_j}{\lambda_1^{m_k}} \lambda_2^{m_k} e_j + \sum_{\substack{j=1 \\ j \notin A_k}}^{N} \frac{x_j}{\lambda_2^{m_k}} \lambda_2^{m_k} e_j = \sum_{\substack{j=1 \\ j \in A_k}}^{N} x_j \left(\frac{\lambda_2}{\lambda_1}\right)^{m_k} e_j + \sum_{\substack{j=1 \\ j \notin A_k}}^{N} x_j e_j.$$

If (b1) is satisfied then $A_k = \emptyset$ for all $k \in \mathbb{N}$ and $(\lambda_2 T_{f_2})^{m_k} R_k w = x$. Otherwise

$$(\lambda_2 T_{f_2})^{m_k} R_k w \to x,$$

since $\left(\frac{\lambda_1}{\lambda_2}\right)^{m_k} \to 1$, as $k \to \infty$.

Therefore, $\lambda_1 T_{f_1}$, $\lambda_2 T_{f_2}$ satisfy the Simultaneous Hypercyclicity Criterion.

(iii) $\Longrightarrow$ (ii): This is true by Proposition 1.5.

(ii) $\Longrightarrow$ (i): This is obvious. □

Example 3.2. Consider the Rolewicz-type operators $T_1 = \lambda T_f$ and $T_2 = \lambda e^{2\pi i \alpha} T_f$, where $\lambda > 1$ and $\alpha \in \mathbb{R}$. If α is rational, then there are infinitely many $n \in \mathbb{N}$ such that $e^{2\pi i \alpha n} = 1$. On the other hand, it is well known that the set $\{e^{2\pi i \alpha n} : n \in \mathbb{N}\}$ is dense in the unit circle $\{z : |z| = 1\} \subset \mathbb{C}$ if and only if α is irrational. Therefore, for any α, there exists an increasing sequence (n_k) of natural numbers such that $e^{2\pi i \alpha n_k} \to 1$ as $k \to \infty$. Therefore, T_1, T_2 are s-hypercyclic. On the other hand, using Theorem 2.3, it is easy to see that these operators are not d-hypercyclic.

Corollary 3.3. [3, Proposition 5.3] *For $1 \leq \ell \leq N$, let $\lambda_\ell \in \mathbb{K}$ and $r_\ell \in \mathbb{N}$ with $r_1 \leq r_2 \leq \dots \leq r_N$. Then, $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ is s-hypercyclic on c_0 or $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, if and only if*

(1) *$1 < |\lambda_j|$ for all $j \in \{1, \dots, N\}$,*
(2) *$|\lambda_j| < |\lambda_{j+1}|$ for all $j \in \{1, \dots, N-1\} \backslash A$,*
(3) *$|\lambda_j| = |\lambda_{j+1}|$ for all $j \in A$,*

where $A = \{j \in \{1, \dots, N-1\} : r_j = r_{j+1}\}$.

Proof. Observe that $\lambda_\ell B^{r_\ell} = \lambda_\ell T_{f_\ell}$ where $f_\ell(n) = n + r_\ell$ for $1 \leq \ell \leq N$. First, assume that the conditions (1), (2), and (3) hold. For any $j \in \{1, \dots, N-1\} \backslash A$ and $k \in \mathbb{N}$, let $n_k = k$. Then, for any $m \in [k]$ we have

$$f_j^{n_k}(m) = m + k r_j \leq k + k r_j = k(r_j + 1) \leq k r_{j+1} \notin \{n + k r_{j+1} : n \geq 1\} = f_{j+1}^{n_k}(\mathbb{N}).$$

Also, for $j \in A$, as argued in Example 3.2, there exists (n_k) such that $(\lambda_{j+1}/\lambda_j)^{n_k} \to 1$. Thus, conditions of Theorem 3.1 are satisfied and $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ are s-hypercyclic.

Now assume that $\lambda_1 B^{r_1}, \dots, \lambda_N B^{r_N}$ are s-hypercyclic. In particular each $\lambda_j B^{r_j}$ is hypercyclic and $|\lambda_j| = |\lambda_j| \|B^{r_j}\| = \|\lambda_j B^{r_j}\| > 1$. Using Theorem 3.1, it is easy to see the rest of the conditions are satisfied. □

One can generalize Rolewicz-type operators further by multiplying the shifted vector with a weight sequence:

Definition 3.4. Let w be a weight sequence and let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map with $f(1) \neq 1$. The unilateral pseudo-shift $T_{f,w}$ on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \le p < \infty$, is given by

$$T_{f,w}(\sum_{j=1}^{\infty} x_j e_j) = \sum_{j=1}^{\infty} w_{f(j)} x_{f(j)} e_j,$$

where $\{e_j : j \ge 1\}$ is the canonical basis.

In 2000, Grosse-Erdmann [7] studied the chaotic dynamics of pseudo-shifts. Wang and Zhou [16], in 2018, characterized d-hypercyclicity of the tuples of pseudo-shifts of the form $T_{f,w_1}, \dots, T_{f,w_N}$ which have the same inducing maps. In 2019, Wang and Liang [15] characterized d-supercyclicity of the tuples of pseudo-shifts of the same form. Wang, Chen and Zhou [14], also in 2019, characterized d-hypercyclicity and d-supercyclicity of tuples of pseudo-shifts of the form $T^{r_1}_{f_1,w_1}, \dots, T^{r_N}_{f_N,w_N}$ where powers are pairwise distinct. Observe that none of these families cover the tuples of Rolewicz-type operators that we study in this paper.

We finish the paper with the following open question.

Question 3.5. Which pseudo-shifts (raised to the same power) are disjoint hypercyclic or simultaneous hypercyclic on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \le p < \infty$?

Acknowledgment. The first author was partially supported by Istanbul Technical University Scientific Research Project [grant no. TAB-2017-40552]. The second author was partially supported by Mimar Sinan Fine Arts University Scientific Research Project [grant no. 2016-18].

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