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An Examination on the Striction Curves in terms of Special Ruled Surfaces

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Abstract. In this paper, we firstly express ruled surfaces drawn by Frenet and Darboux vectors of Bertrand mate depending on Bertrand curve. Then, the tangent vectors of the striciton curves on these surfaces are calculated. Finally, we give some results with these vectors.

1. Introduction and Preliminaries

Many results on ruled surfaces have been obtained by mathematicians (see [1, 5, 9, 11, 12]). In [11], authors examine spatial quaternionic ruled surfaces. Another study, authors express some results about Bertrand offsets in Minkowski space [5]. A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation

$$\varphi(s,v) = \alpha(s) + ve(s) \tag{1}$$

where α base curve and e generator vector [3]. The striction curve is given by [3]

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s). \tag{2}$$

The notion of Bertrand curves was discovered by J. Bertrand in 1850. There are many studies on the Bertrand curve Bertrand curves in different areas. In [6], authors examine the Bertrand curves in the Euclidean 4-space as quaternionic. J. Monterde characterize Bertrand curves defined from Salkowski curves [10].
Let α be a unit speed curve in E^3, and $\{V_1(s), V_2(s), V_3(s)\}$ denote the Frenet frame of α. The Frenet formulas are given by

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where k_1 and k_2 denote the curvature and the torsion of α, respectively. On the other hand, the Darboux vector is [2]

$$D(s) = k_2(s)V_1(s) + k_1(s)V_3(s), \tag{3}$$

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The modified Darboux vector [4]

$$\tilde{D}(s) = \frac{k_2(s)}{k_1(s)}(s)V_1(s) + V_3(s). \tag{4}$$

Let α and α^* be the unit speed two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s), V_2^*(s), V_3^*(s)$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector of the curve α is linearly dependent on the principal normal vector of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ are called Bertrand pair and α^* is called Bertrand mate. [3]. The parametrization of Bertrand mate is [3]

$$\alpha^*(s) = \alpha(s) + \lambda V_2(s) \tag{5}$$

Theorem 1.1. *[3] The distance between corresponding points of the Bertrand pair in $\mathbb{E}^3$ is constant.*

Theorem 1.2. *[3]. If $k_2(s) \neq 0$ along $\alpha(s)$, then $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant*

$$\lambda k_1 + \beta k_2 = 1. \tag{6}$$

Theorem 1.3. *[3] Let α and α^* be the unit speed two curves. $\left\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\right\}$ and $\left\{V_1^*, V_2^*, V_3^*, \tilde{D}^*, k_1^*, k_2^*\right\}$ are Frenet-Serret apparatus of the Bertrand curve and the Bertrand mate, respectively. Then, the formulas are given by*

$$V_1^* = \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \quad V_2^* = V_2, \quad V_3^* = \frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}}, \quad \tilde{D}^* = \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)}\tilde{D}.$$

The first and second curvatures of Bertrand mate are given by

$$k_1^* = \frac{\beta k_1 - \lambda k_2}{(\lambda^2 + \beta^2) k_2}, \quad k_2^* = \frac{1}{(\lambda^2 + \beta^2) k_2}.$$

Let $\alpha : I \to \mathbb{E}^3$ be differentiable unit speed curve and let $\{V_1(s), V_2(s), V_3(s), \tilde{D}\}$ be the Frenet-Serret apparatus of this curve. The equations

$$\begin{aligned} \varphi_1(s, u_1) &= \alpha(s) + u_1 V_1(s) \\ \varphi_2(s, u_2) &= \alpha(s) + u_2 V_2(s) \\ \varphi_3(s, u_3) &= \alpha(s) + u_3 V_3(s) \\ \varphi_4(s, u_4) &= \alpha(s) + u_4 \tilde{D}(s) \end{aligned} \tag{7}$$

are the parametrization of the ruled surface which are called tangent ruled surface, normal ruled surface, binormal ruled surface, modified Darboux ruled surface, respectively. For the sake of shortness, we write Frenet ruled surfaces instead of the above all ruled surfaces.

Theorem 1.4. *[8] The tangent vectors of the striction curves on Frenet ruled surfaces are given by the following matrix*

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta \left\|c_2'(s)\right\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\left\|c_2'(s)\right\|} & \frac{k_1 k_2}{\eta \left\|c_2'(s)\right\|} \\ 1 & 0 & 0 \\ \frac{\mu - \mu' - \frac{k_2}{k_1}}{\mu \left\|c_4'(s)\right\|} & 0 & \frac{\mu'}{\mu^2 \left\|c_4'(s)\right\|} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

where $\eta = k_1^2 + k_2^2$, $\mu = \left(\frac{k_2}{k_1}\right)'$.

Definition 1.5. *[9] Let $\alpha^* : I \to \mathbb{E}^3$ be differentiable unit speed curve and let $\{V_1^*(s), V_2^*(s), V_3^*(s), \tilde{D}^*\}$ be the Frenet-Serret apparatus of this curve. The equations*

$$\begin{aligned}
\varphi_1^*(s, w_1) &= \alpha^*(s) + w_1 V_1^*(s) = \alpha + \lambda V_2 + w_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}} \\
\varphi_2^*(s, w_2) &= \alpha^*(s) + w_2 V_2^*(s) = \alpha + (\lambda + w_2) V_2 \\
\varphi_3^*(s, w_3) &= \alpha^*(s) + w_3 V_3^*(s) = \alpha + \lambda V_2 + w_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}} \right) \\
\varphi_4^*(s, w_4) &= \alpha^*(s) + w_4 \tilde{D}^*(s) = \alpha + \lambda V_2 + w_4 \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}
\end{aligned} \tag{8}$$

are the parametrization of the ruled surface which are called ***Bertrandian tangent ruled surface****,* ***Bertrandian normal ruled surface****,* ***Bertrandian binormal ruled surface*** *and* ***Bertrandian modified Darboux ruled surface****, respectively.*

For the sake of shortness, we write Bertrand ruled surfaces instead of the above all ruled surfaces.

Theorem 1.6. *[7] The tangent vectors of striction curves on Bertrand ruled surfaces are given by the following matrix*

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}$$

where

$$a^* = \frac{{k_2^*}^2}{\eta^* \left\| {c_2^*}'(s) \right\|}, \quad b^* = \frac{\left(\frac{k_1^*}{\eta^*} \right)'}{\left\| {c_2^*}'(s) \right\|}, \quad c^* = \frac{k_1^* k_2^*}{\eta^* \left\| {c_2^*}'(s) \right\|}, \quad d^* = \frac{\mu^* - \mu^{*\prime} - \frac{k_2^*}{k_1^*}}{\mu^* \|{c_4^*}'(s)\|} = \frac{-m' - \left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}} \right)' m^2 - m k_2 \sqrt{\lambda^2 + \beta^2}}{-m' \|{c_4^*}'(s)\|},$$

$$e^* = \frac{\mu^{*\prime}}{\mu^{*2} \left\| {c_4^*}'(s) \right\|} = \frac{\left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2 + \beta^2}} \right)' \frac{1}{k_2 \sqrt{\lambda^2 + \beta^2}}}{\left(\frac{-m'}{m^2 k_2 \sqrt{\lambda^2+\beta^2}} \right)^2 \|{c_4^*}'(s)\|}, \quad \eta^* = {k_1^*}^2 + {k_2^*}^2, \quad \mu^* = \left(\frac{k_2^*}{k_1^*} \right)'.$$

2. An Examination on the Striction Curves in terms of Special Ruled Surfaces

In this section Then, the tangent vectors of the striciton curves on Frenet and Bertrandian ruled surfaces are calculated. We give some results with these vectors.

Theorem 2.1. *The relationship between the tangent vectors of the striciton curves on the Frenet and Bertrandian ruled surfaces is*

$$[T]\,[T^*]^{\mathrm{T}} = \frac{1}{\sqrt{\lambda^2 + \beta^2}} \begin{bmatrix} \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ x & a^*x + b^* \sqrt{\lambda^2 + \beta^2} + a^* y & x & d^* x + e^* y \\ \beta & a^*\beta - c^*\lambda & \beta & d^*\beta - e^*\lambda \\ z & a^* z + c^* t & z & d^* z + e^* t \end{bmatrix}$$

where

$$x = \frac{k_2 \big(\beta k_2 + \lambda k_1 \big)}{\eta \left\| c_2'(s) \right\|}, \quad y = \frac{k_2 \big(-\lambda k_2 + \beta k_1 \big)}{\eta \left\| c_2'(s) \right\|}, \quad z = \frac{\big(\mu - \mu' - \frac{k_2}{k_1} \big) \beta + \mu' \lambda}{\mu \left\| c_4'(s) \right\|}, \quad t = \frac{\big(-\mu + \mu' + \frac{k_2}{k_1} \big) \lambda + \mu' \beta}{\mu \left\| c_4'(s) \right\|}.$$

Proof. Let $[T]=[A][V]$ and $[T^*]=[A^*][V^*]$ hence, by using the properties of the matrix, we can write

$$\begin{aligned}
[T][T^*]^{\mathbf{T}} &= [A][V]([A^*][V^*])^{\mathbf{T}} \\
&= [A]\left([V][V^*]^{\mathbf{T}}\right)[A^*]^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{k_2^2}{\eta\left\|c_2'(s)\right\|} & \dfrac{\left(\frac{k_1}{\eta}\right)'}{\left\|c_2'(s)\right\|} & \frac{k_1k_2}{\eta\left\|c_2'(s)\right\|} \\ 1 & 0 & 0 \\ \frac{\mu-\mu'-\frac{k_2}{k_1}}{\mu\left\|c_4'(s)\right\|} & 0 & \frac{\mu'}{\mu^2\left\|c_4'(s)\right\|} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} \right)^{\mathbf{T}} \\
&= \frac{1}{\sqrt{\lambda^2+\beta^2}} \begin{bmatrix} \beta & 0 & -\lambda \\ x & b\sqrt{\lambda^2+\beta^2} & y \\ \beta & 0 & -\lambda \\ z & 0 & t \end{bmatrix} \begin{bmatrix} 1 & a^* & 1 & d^* \\ 0 & b^* & 0 & 0 \\ 0 & c^* & 0 & e^* \end{bmatrix} \\
&= \frac{1}{\sqrt{\lambda^2+\beta^2}} \begin{bmatrix} \beta & a^*\beta-c^*\lambda & \beta & d^*\beta-e^*\lambda \\ x & a^*x+b^*\sqrt{\lambda^2+\beta^2}+a^*y & x & d^*x+e^*y \\ \beta & a^*\beta-c^*\lambda & \beta & d^*\beta-e^*\lambda \\ z & a^*z+c^*t & z & d^*z+e^*t \end{bmatrix} \\
&= \begin{bmatrix} \left\langle T_1,T_1^*\right\rangle & \left\langle T_1,T_2^*\right\rangle & \left\langle T_1,T_3^*\right\rangle & \left\langle T_1,T_4^*\right\rangle \\ \left\langle T_2,T_1^*\right\rangle & \left\langle T_2,T_2^*\right\rangle & \left\langle T_2,T_3^*\right\rangle & \left\langle T_2,T_4^*\right\rangle \\ \left\langle T_3,T_1^*\right\rangle & \left\langle T_3,T_2^*\right\rangle & \left\langle T_3,T_3^*\right\rangle & \left\langle T_3,T_4^*\right\rangle \\ \left\langle T_4,T_1^*\right\rangle & \left\langle T_4,T_2^*\right\rangle & \left\langle T_4,T_3^*\right\rangle & \left\langle T_4,T_4^*\right\rangle \end{bmatrix}.
\end{aligned}$$

□

Corollary 2.2. *There are four pairs of tangent vector fields equal to each other of the striction curves on Frenet and Bertrandian ruled surfaces.*

Proof. Since $\left\langle T_1,T_1^*\right\rangle=\left\langle T_1,T_3^*\right\rangle=\left\langle T_3,T_1^*\right\rangle=\left\langle T_3,T_3^*\right\rangle=\dfrac{\beta}{\sqrt{\lambda^2+\beta^2}}$, it is trivial. □

Corollary 2.3. ***i)****Tangent vectors of striction curves on tangent ruled surface and Bertrandian normal ruled surface are perpendicular if $\beta=\lambda m$ where $m=\beta k_1-\lambda k_2$.*
ii)*Tangent vectors of striction curves on binormal ruled surface and Bertrandian normal ruled surface are perpendicular if $\beta=\lambda m$.*

Proof. **i)** Since $\langle T_1,T_2^*\rangle=\dfrac{a^*\beta-c^*\lambda}{\sqrt{\lambda^2+\beta^2}}$ and $\langle T_1,T_2^*\rangle=0$

$$\begin{aligned}
a^*\beta-c^*\lambda &= 0, \\
\beta-\lambda(\beta k_1-\lambda k_2) &= 0, \\
\beta &= \lambda m,
\end{aligned}$$

this completes the proof.
ii) Since $\langle T_1, T_2^* \rangle = \langle T_3, T_2^* \rangle$, it is trivial. □

Corollary 2.4. ***i)****Tangent vectors of striction curves on tangent ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\Big(\frac{1}{m}\Big)'\Big[\Big(\frac{1}{m}\Big)' - \Big(\frac{1}{m}\Big)'' - \frac{1}{m}\Big]\beta = \Big(\frac{1}{m}\Big)''\lambda.$$

ii)*Tangent vectors of striction curves on binormal ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\Big(\frac{1}{m}\Big)'\Big[\Big(\frac{1}{m}\Big)' - \Big(\frac{1}{m}\Big)'' - \frac{1}{m}\Big]\beta = \Big(\frac{1}{m}\Big)''\lambda.$$

Proof. **i)** Since $\langle T_1, T_4^* \rangle = \dfrac{d^*\beta - e^*\lambda}{\sqrt{\lambda^2+\beta^2}}$ and $\langle T_1, T_4^* \rangle = 0$

$$\begin{aligned} d^*\beta - e^*\lambda &= 0 \\ \Big(\frac{1}{m}\Big)'\Big[\Big(\frac{1}{m}\Big)' &- \Big(\frac{1}{m}\Big)'' - \frac{1}{m}\Big]\beta - \Big(\frac{1}{m}\Big)''\lambda = 0 \\ \Big(\frac{1}{m}\Big)'\Big[\Big(\frac{1}{m}\Big)' &- \Big(\frac{1}{m}\Big)'' - \frac{1}{m}\Big]\beta = \Big(\frac{1}{m}\Big)''\lambda \end{aligned}$$

this completes the proof.
ii) Since $\langle T_1, T_4^* \rangle = \langle T_3, T_4^* \rangle$, it is trivial. □

The following corollaries are obtained similar to Corollary 2.5.

Corollary 2.5. ***i)****Tangent vectors of striction curves on normal ruled surface and Bertrandian tangent ruled surface have orthogonal under the condition $k_2 = 0$.*
ii)*Tangent vectors of striction curves on normal ruled surface and Bertrandian binormal ruled surface are perpendicular if $k_2 = 0$.*

Corollary 2.6. ***i)****Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian tangent ruled surface are perpendicular if*

$$k_1 = \frac{\beta(\frac{k_2}{k_1})'\Big[k_1(\frac{k_2}{k_1})' - k_2\Big]}{(\frac{k_2}{k_1})''\Big[\beta(\frac{k_2}{k_1})' + \lambda\Big]}.$$

ii)*Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian binormal ruled surface are perpendicular if*

$$k_1 = \frac{\beta(\frac{k_2}{k_1})'\Big[k_1(\frac{k_2}{k_1})' - k_2\Big]}{(\frac{k_2}{k_1})''\Big[\beta(\frac{k_2}{k_1})' + \lambda\Big]}.$$

Corollary 2.7. *Tangent vectors of striction curves on normal ruled surface and Bertrandian normal ruled surface are perpendicular if*

$$k_2 = -\frac{m\Big(x+y\Big)}{(\lambda^2+\beta^2)^{\frac{3}{2}}}.$$

Corollary 2.8. *Tangent vectors of striction curves on normal ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\Big(\frac{1}{m}\Big)'\Big[\Big(\frac{1}{m}\Big)' - \Big(\frac{1}{m}\Big)'' - \frac{1}{m}\Big]x = -\Big(\frac{1}{m}\Big)''y.$$

Corollary 2.9. *Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian normal ruled surface are perpendicular if*

$$k_2 = \beta k_1 - \frac{z}{\lambda\left(\lambda \frac{(\mu-\mu'-\frac{k_2}{k_1})}{\mu\|c_4'(s)\|} - \beta \frac{k_1 k_2}{\eta\|c_2'(s)\|}\right)}.$$

Corollary 2.10. *Tangent vectors of striction curves on modified Darboux ruled surface and Bertrandian modified Darboux ruled surface are perpendicular if*

$$\left(\frac{1}{m}\right)'\left[\left(\frac{1}{m}\right)' - \left(\frac{1}{m}\right)'' - \frac{1}{m}\right]z = -\left(\frac{1}{m}\right)'' t.$$

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