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On Connectedness via a G-method and a Hereditary Class

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Abstract

In 2003, Connor and Grosse-Erdmann [1] introduced the definition of G-method by using G-linear functions instead of limit, based on various types of convergence on real numbers. Later on, some mathematicians examined this concept in topological groups. Then new concepts, which were important in topology such as G-sequential compactness and G-sequential connected, were defined and some properties of those concepts are investigated. S. Lin and L. Liu defined G-method notion by taking any set instead of topological group in 2016. In this paper, we give definition of cl_{G^*}-closure which is more general than G-closure of a set with the help of hereditarily class. Then we define the notion of τ_{G^*}-topology and give the concepts of G^*-connected and G^*-component. Besides, we examine the relationship between these concepts and previously given concepts.

Keywords: *hereditary class, G-method, G^*-closed, G^*-connected, G^*-component*
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1. Introduction

The concept of sequential convergence and any concept related to sequential convergence play a crucial role not only in analysis and topology, but also in other disciplines including mathematics such as computer science, information theory, biological science and dynamic systems. In addition to the types of ordinary convergence at a point and uniform convergence, new types of convergence have been studied by many mathematicians. Firtsly, the concept of statistical convergence was given by Zygmund [19] in his article *Trigonometric Series* in 1935. This concept was developed in 1951 by Fast and Steinhaus [6]. Then Di Maio and Kočinac [13] defined the concept of statistical convergence in topological spaces. Besides, in topological spaces, Tang and Lin [17] studied on statistical sequential convergence and statistical Frechet-Uryshon space. In addition to the concepts of convergence and statistical convergence, there are also a wide variety of convergence concepts; for example A-convergence of a matrix method in summability theory, almost convergence in functional analysis, Cesaro convergence in real analysis, $\mathscr{I}$-convergence concepts defined in real analysis with the help of the ideal concept. Based on the various convergence features in real analysis, convergence was expanded by Connor and Grosse-Erdmann [1] in 2003 using the G-linear continuous function instead of the limit. This G-function is also called G-method. With the help of the G-method, the concepts of G-convergence and G-continuity have also been defined and have expanded to some known results in the literature. Later, H. Çakallı [4, 5] studied on this concept in generalized spaces by defining the concepts of G-sequential compactness and G-sequential connected in topological groups, which provides the feature of being the first countable space by using topology and algebra. In addition, in 2014, O. Mucuk [14] also studied on G-sequential open and G-sequential neighborhood notions in the first countable topological groups. In 2016, instead of examining those concepts in the first countable topological groups, Shou Lin and Li Liu [9] defined G-method, G-submethod, G-open cluster, G-neighborhood, G-kernel, G-hull and gave the concepts of G-continuity in an arbitrary set. Besides, with the help of these concepts, they gave the definition of G-topology and examined the relationships and differences between the concept of openness defined in topological spaces and the concept of G-openness. In his studies, published in 2016 and 2017, Liu [10, 11] examined the characteristics of the concepts of G-neighborhood, G-accumulation point, G-boundaries and G-continuity at a point in an arbitrary set. In 2018, Liu [12] was given the definitions of G-kernel open, G-kernel neighborhood and G-kernel accumulation points and the properties of these concepts were examined. In 2019, Wu and Li [18] generalized the properties related to the concepts of G-connectedness, G-hull, G-kernel and studied the concept of G-topological group. In this paper, we give the definition of G^*-closed, which is a more general structure than the G-closed definition of a set using concept of heredity class on a set. Then, we obtained G^*-topology, which is a finer generalized topological structure than G-topology, obtained with the help of G^*-closed sets. With the help of this definition, the concepts of connected and component was examined under a more general framework.

Throughout the paper, X be a set, $s(X)$ denote the set all X-valued sequences, i.e. $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If $f : X \to Y$ is a mapping, then $f(\boldsymbol{x}) = \{f(x_n)\}_{n\in\mathbb{N}}$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in s(X)$. If X is a topological space, the set of all X-valued sequences

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convergent sequences is denoted by $c(X)$, and we put $\lim \boldsymbol{x} = \lim_{n\to\infty} x_n$ for any $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c(X)$. All topological spaces are assumed to satisfy the T_2-separation property.

2. Preliminaries

Definition 2.1. *[2] A generalized topology on a set X is collection μ of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions.*

Definition 2.2. *[8, 3] Let X be a non empty set. Then, a family of sets $\mathscr{H} \subset P(X)$ is said to be an ideal in X if*

(a) $A, B \in \mathscr{H}$ imply $A \cup B \in \mathscr{H}$,

(b) $A \in \mathscr{H}$ and $B \subset A$ imply $B \in \mathscr{H}$.
In this paper, instead of the ideal, we consider a hereditary class, i.e. a class $\emptyset \neq \mathscr{H} \subseteq P(X)$ satisfying (b) only.

Definition 2.3. *[9] Let X be a non empty set.*

(a) A method (G-method) on X is a function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$. A sequences $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ in said to be G-convergent to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x})$=l.

(b) Let X be a topological space.

(i) A method $G : c_G(X) \to X$ is called regular if $c(X) \subset c_G(X)$ and $G(\boldsymbol{x}) = \lim \boldsymbol{x}$ for each $\boldsymbol{x} \in c(X)$.

(ii) A method $G : c_G(X) \to X$ is called subsequential if whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{y} \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{y} = l$.

Definition 2.4. *[9] Let X be a non empty set, G be a method on X and $A \subset X$.*

(a) The G-hull of A is defined as the set $G(\boldsymbol{x}) : \boldsymbol{x} \in s(A) \cap c_G(X)\}$ and G-hull of A is denoted by $[A]_G$.

(b) A is called a G-closed set if whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$.

(c) A is called a G-open set if $X \setminus A$ is a G-closed set.

(d) The G-closure of A is defined as intersection of G-closed sets containing A, and the G-closure of A is denoted by $Cl_G(A)$.

(e) The G-interior of A is defined as union of G-open sets contained in A, and the G-interior of A is denoted by $int_G(A)$.

Proposition 2.5. *[9] Let G be a method on a set X and $A \subset X$. Then,*

$$x \in Cl_G(A) \text{ if and only if } U \text{ of } X \text{ with } x \in int_G(U) \text{ intersects } A.$$

Definition 2.6. *[9] Let X be a set, G be a method on X and $Y \subset X$. Put $c_{G|Y}(Y) = \{\boldsymbol{x} \in s(Y) \cap c_G(X) : G(\boldsymbol{x}) \in Y\}$. The function $G|_Y : c_{G|Y}(Y) \to Y$ is called the submethod of G on the subset Y of X.*

Definition 2.7. *[9] Let G be a method on a set X. Put $\tau_G = \{A \subset X : A \text{ is } G-open \text{ in } X\}$. The family τ_G is called the G-generalized topology on the set X.*

Definition 2.8. *[9] Let G be a method on a set X. The family τ_G is called the G-generalized topology on the set X.*

(a) τ_G is called the $G-topology$ on the set X if it is topology on X.

(b) If X carries a topology τ then (X, τ) is called $G-topologizable$ if τ=τ_G.

Definition 2.9. *[9] Let G be a method on a topological space X.*

(a) X is said to be G-sequential space if every G-closed set in X is closed.

(b) X is said to be G-Fréchet space if $Cl(A) \subset [A]_G$ for each $A \subset X$.

Definition 2.10. *[4] A non-empty subset A of a topological group X is called G-sequentially connected if there are no non-empty, disjoint G-sequentially closed subsets U and V of X meeting A such that $A \subset U \cup V$. Particularly X is called G- sequentially connected, if there are no non-empty, disjoint G-sequentially closed subsets of X whose union is X.*

Definition 2.11. *[4] The largest G-sequentially connected subset containing a point x in topological group X is called G-sequentially connected component of x and denoted by C_x^G.*

3. G^*-closed sets

In this section, we define local function of a set with respect to *hereditary class* and G-method on a arbitrary set X. By using this definition, we give the concept of G^*-closure which is generalization of the concept of G-closure of a set.

Definition 3.1. *Let G be a method, $\mathscr{H}$ be a hereditary class on a set X. For $A \subset X$,*

$$A^{*G}(\mathscr{H},G) = A^{*G} = \{x \in X : U \cap A \notin \mathscr{H} \ for\ every\ U \in \mathscr{U}_G(x)\}$$

is called the local function of A with respect to $\mathscr{H}$ and G where $\mathscr{U}_G(x)=\{U \subset X : x \in U \ and\ U\ is\ G-open\}$.

Proposition 3.2. *Let G be a method, $\mathscr{H}$ and $\mathscr{H}_1$ be hereditary classes on a set X. For $A,B \subset X$,*

(a) $A \subset B$ implies that $A^{*G} \subset B^{*G}$.

(b) $A^{*G} \subset Cl_G(A)$.

(c) A^{*G} is G-closed.

(d) $(A^{*G})^{*G} \subset A^{*G}$.

(e) $(A \cup A^{*G})^{*G} \subset A^{*G}$.

(f) $\mathscr{H} \subset \mathscr{H}_1$ implies that $A^{*G}(\mathscr{H}_1,G) \subset A^{*G}(\mathscr{H},G)$.

Proof.

(a) Let x $\notin B^{*G}$. Then there exists a subset $U \in \mathscr{U}_G(x)$ such that $U \cap B \in \mathscr{H}$. By hypothesis, $A \cap U \subset B \cap U$ so that $A \cap U \in \mathscr{H}$ and x $\notin A^{*G}$.

(b) Let x $\notin Cl_G(A)$. Then there exists a subset U of X such that x $\in int_G(U)$ and $U \cap A = \emptyset \in \mathscr{H}$ and hence $int_G(U) \cap A = \emptyset \in \mathscr{H}$ Since $int_G(U)$ is G-open, x $\notin A^{*G}$.

(c) Let x $\notin A^{*G}$. Then there exists a subset $U \in \mathscr{U}_G(x)$ such that $U \cap A \in \mathscr{H}$. By definition of A^{*G}, we obtain $U \cap A^{*G} = \emptyset$ and x $\notin Cl_G(A^{*G})$ by U=$int_G(U)$. So we get A^{*G}= $Cl_G(A^{*G})$ i.e. A^{*G} is G-closed.

(d) It is obvious from (b) and (c).

(e) Let x $\notin A^{*G}$. Then there exists a subset $U \in \mathscr{U}_G(x)$ such that $U \cap A \in \mathscr{H}$ and $U \cap A^{*G} = \emptyset$. Then $(U \cap A^{*G}) \cup (U \cap$ A) = U $\cap$ $(A^{*G} \cup A)$ $\in \mathscr{H}$. So x $\notin (A \cup A^{*G})^{*G}$.

(f) Let $\mathscr{H} \subset \mathscr{H}_1$ and x $\notin A^{*G}(\mathscr{H},G)$. Then there exists a subset $U \in \mathscr{U}_G(x)$ such that $U \cap A \in \mathscr{H}_1$. Then $U \cap A \in \mathscr{H}$ and we obtain x $\notin$ $A^{*G}(\mathscr{H}_1,G)$. Hence $A^{*G}(\mathscr{H}_1,G) \subset A^{*G}(\mathscr{H},G)$. □

Corollary 3.3. *Let G be a method, and $\mathscr{H}$ be a hereditary class on a set X. For $A,B \subset X$, $(A \cup B)^{*G} \supset A^{*G} \cup B^{*G}$.*

The following examples show that the converse of Corollary 3.3. is not true in general and $(A^{*G})^{*G} \neq A^{*G}$ for A$\subset$ X.

Example 3.4. *Lin and Liu [9] gave the examples as follows. Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology. Put $c_G(X) = s(X)$ and G: $c_G(X) \to X$ is defined by G($\mathbf{x}$)=0 for each $\mathbf{x} \in c_G(X)$. Then for any non-empty subset A of X, A is G-closed if and only if 0 $\in$ A.*
Let $\mathscr{H}$=$\{\emptyset,\{1\},\{2\}\}$, A=$\{1\}$ and B=$\{2\}$. Then $A^{*G} = B^{*G} = \emptyset$. $(A \cup B)^{*G} = \{0\}$. Hence $(A \cup B)^{*G} \neq A^{*G} \cup B^{*G}$.

Example 3.5. *Lin and Liu [9] gave the examples as follows. Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology. Put $c_G(X)$ =$\{\{x_n\}_{n\in\mathbb{N}} \in s(X) : there\ exists\ an\ m \in \mathbb{N}\ such\ that \{x_n - x_{n-1}\}_{n>m}\ is\ aconstant\ sequences\}$. Define G_1: $c_G(X) \to X$ by $G_1(\mathbf{x})$=$\lim_{n\to\infty}(x_{n+1} - x_n)$ for each $\mathbf{x} \in c_{G_1}(X)$.*
Let $Y = \{0,1,2,3,4\}$ and G=$G_1|_Y$. Let $\mathscr{H}$=$\{\emptyset,\{0\},\{1\},\{3\}\}$ and A=$\{1,3\}$. Then A^{*G} =$\{0\}$. But $(A^{*G})^{*G}$= $\emptyset \neq A^{*G}$.

Let G be a method and $\mathscr{H}$ be a *hereditary class* on a set X. For $A \subset X$, $Cl^*_G(A)$ = A $\cup$ A^{*G}.

Proposition 3.6. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. For $A,B \subset X$,*

(a) $A \subset B$ implies that $Cl^*_G(A) \subset Cl^*_G(B)$.

(b) $A \subset Cl^*_G(A)$.

(c) $Cl^*_G(Cl^*_G(A))$= $Cl^*_G(A)$.

Proof.

(a) It is obvious from Proposition 3.2

(c) Proposition 3.2 (e), $Cl^*_G(Cl^*_G(A)) = Cl^*_G(A \cup A^{*G}) = (A \cup A^{*G}) \cup (A \cup A^{*G})^* \subset (A \cup A^{*G}) \cup A^{*G} \subset Cl^*_G(A)$. Then we get $Cl^*_G(Cl^*_G(A))$ = $Cl^*_G(A)$. □

According to Lemma 1.4 of [2] and Proposition 3.6, there is a G-generalized topology $\tau^\star_G$ i.e.

$$\text{A} \in \tau^\star_G \text{ iff A is } \tau^*_G\text{-open iff A is } G^*\text{-open iff X} - \text{A} = Cl^*_G(X - A).$$

and $Cl_G^*(A)$ is the intersection of all G^*-closed sets contained in A.
If $\mathscr{H} = \{\emptyset\}$, A is G^*-open iff A is G-open.

Theorem 3.7. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. Then, $\tau_G \subset \tau_G^\star$.*

Let U $\in \tau_G$. Then X$-$ U is G-closed iff X$-$ U $= Cl_G(X-U)$. Proposition 3.2 (c), we obtain $(X-U)^{*G} \subset$ X$-$U and then X$-$ U = (X$-$ U) $\cup (X-U)^{*G} = Cl_G^*(X-U)$. So X$-$ U is $\tau_G^\star$-closed i.e U $\in \tau_G^\star$.

Corollary 3.8. *Let G be a method $\mathscr{H}$ be a hereditary class on a set X. For $K \subset X$, K is $\tau_G^\star$-closed iff $K^{*G} \subset K$.*

Example 3.9. *We consider at Example 3.4. We obtain that $\{1\}$ and $\{2\}$ are $\tau_G^\star$-closed from the previous Corollary. Then $\tau_G^\star = \{U : 0 \notin U\} \cup \{X, X-\{1\}, X-\{2\}\}$ is G-generalized topology.*

Proposition 3.10. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. Then, the followings are equivalent:*

(a) $Cl_G(A) \subset Cl_G^*(A)$ for each $A \subset X$.

(b) Every G^*-closed set in X is G-closed.

(c) $\tau_G = \tau_G^*$.

Proof. $(a) \Rightarrow (b)$: Let A be a G^*-closed set. Then A=$Cl_G^*(A)$. By condition (a), A= $Cl_G(A)$. Therefore, A is G-closed.
$(b) \Rightarrow (a)$: Let A $\subset$ X. Because $Cl_G^*(A)$ is G^* -closed and by condition (b), $Cl_G^*(A)$ is G -closed. Then we obtain $Cl_G(A) \subset Cl_G^*(A)$ by Definition 2.4 (d).
$(b) \Rightarrow (c)$: Let A $\in \tau_G^\star$. Then X$-$ A $= Cl_G^*(X-A)$ and so X-A is G^*-closed. By condition (b), X-A is G-closed. Then, we obtain $\tau_G = \tau_G^*$ by Theorem 3.7.
$(c) \Rightarrow (b)$: It is obvious. □

Proposition 3.11. *Let G be a method and $\mathscr{H}$ be a hereditary class on a topological space (X, τ). Then, the followings are equivalent:*

(a) $Cl_G^*(A) \subset Cl(A)$ for each $A \subset X$.

(b) Every closed set in X is G^*-closed.

(c) $\tau \subset \tau_G^\star$.

Proof. $(a) \Rightarrow (b)$: Let A be a closed set. Then A=$Cl_G(A)$. By condition (a), A= $Cl_G^*(A)$. Therefore A is G-closed.
$(b) \Rightarrow (a)$: Let A $\subset$ X. Because $Cl(A)$ is closed and by condition (b), $Cl(A)$ is G^*-closed. Then we get $Cl_G^*(A) \subset Cl_G(A)$ since $Cl_G^*(A)$ is the smallest G^*-closed set contained in A.
$(b) \Leftrightarrow (c)$: It is obvious.

□

Lemma 3.12. *Let $\mathscr{H}$ be a hereditary class on a topological space X. Then, the followings are hold.*

(a) If G is a regular method on X and $\mathscr{H}=\{\emptyset\}$, then every G^*-closed in X is sequentially closed.

(b) If G is a subsequential method on X, then every closed set in X is G^*-closed.
As a result, if G is a regular subsequential method and $\mathscr{H}=\{\emptyset\}$ on a first-countable space (X, τ), then $\tau_G^*= \tau$.

Proof.

(a) Let G be a regular method on X and $\mathscr{H}=\{\emptyset\}$. Let A is G^*-closed in X. Then, A is G-closed in X. So A is sequentially closed by Lemma 2.11 (1) of [9].

(b) The proof is obvious to Lemma 2.11 (2) of [9] □

4. G^*-connected

In this section, we introduced concept of G^*-connected on a set X and obtained some basic properties.

Definition 4.1. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. For $A, B \subset X$, A and B are called (G, G^*)-separated iff $Cl_G(A) \cap B = A \cap Cl_G^*(B) = \emptyset$.*

Lemma 4.2. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. If A, $B \subset X$, then the following properties are equivalent:*

(a) A and B are (G, G^*)-separated;

(b) There are G-closed set F and G^*-closed set F' such that $A \subset F \subset X-B$ and $B \subset F' \subset X-A$;

(c) There are G-open set U and G^*-open set U' such that $A \subset U' \subset X-B$ and $B \subset U \subset X-A$.

Proof. $(a) \Rightarrow (b)$: Set $F = Cl_G(A)$, $F' = Cl^*_G(B)$.
$(b) \Rightarrow (c)$: Set $U' = X - F'$, $U = X - F$.
$(c) \Rightarrow (b)$: It is obvious.
$(b) \Rightarrow (a)$: It is clear by the definitions of Cl_G and Cl^*_G. □

Definition 4.3. *A set $T \subset X$ is called G^*-connected if T cannot be written as the union of nonempty (G,G^*)-separated sets. i.e. T is G^*-connected iff $T = U \cup V$, U and V are (G,G^*)-separated imply $U = \emptyset$ or $V = \emptyset$. The space X is said to be G^*-connected iff it is a G^*-connected subset of itself.*

Theorem 4.4. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. Then, the following statements are equivalent:*

(a) The space X is G^*-connected;

(b) If $X = M \cup N$, $M \cap N = \emptyset$, M is G-open and N is G^*-open then either $M = \emptyset$ or $N = \emptyset$.

(c) If $X = K \cup F$, $K \cap F = \emptyset$, K is G-closed and F is G^*-closed then either $K = \emptyset$ or $F = \emptyset$.

(d) If $H \subset X$ is both G-open and G^*-closed then either $H = \emptyset$ or $H = X$.

Proof. $(a) \Rightarrow (b)$: M and N are (G,G^*)-separated by Lemma 4.2.
$(b) \Rightarrow (a)$: It is clear by the definitions of Cl_G and Cl_{G^*}.
$(b) \Rightarrow (c)$: Set M = $X - K$, N = $X - F$. Then it is obvious.
$(c) \Rightarrow (b)$: It is obvious.
$(c) \Rightarrow (d)$: Let H $\subset$ X be both G-open and G^*-closed. Then X$-$H is G-closed. By (c), H= $\emptyset$ or X$-$H = $\emptyset$. So we obtain either H= $\emptyset$ or H = X.
$(d) \Rightarrow (c)$: It is obvious. □

Corollary 4.5. *Let G be a method on a topological space X. Let X be a G-sequential space and $\mathscr{H} = \{\emptyset\}$. If X is connected space, then X is a G^*-connected space.*

Proof. The proof is obvious by Lemma 6.3 (1) of [9]. □

Corollary 4.6. *Let G be a subsequential method on a topological group X. Let X be a G-Fréchet space. X is G-sequentially connected iff X is a G^*-connected space.*

Proof. The proof is obvious by Proposition 5.6 of [9] and by Lemma 3.12. □

Corollary 4.7. *Let G be a regular method on a topological space X. Let X be a sequential space and $\mathscr{H} = \{\emptyset\}$. If X is connected space, then X is a G^*-connected space.*

Proof. The proof is obvious by Lemma 3.12. □

Corollary 4.8. *Let G be a subsequential method on a topological space X and $\mathscr{H}$ be a hereditary class on X. If X is a G^*-connected space then X is a connected space.*

Proof. The proof is obvious by Lemma 3.12. □

Corollary 4.9. *Let G be a regular subsequential method on a first-countable space X and $\mathscr{H} = \{\emptyset\}$. X is a connected space iff X is a G^*-connected space.*

Proof. The proof is obvious by Lemma 3.12. □

Corollary 4.10. *Let G be a method on a topological space X. Let X be G-topologizable and $\mathscr{H} = \{\emptyset\}$. X is a connected space iff X is a G^*-connected space.*

Proof. The proof is obvious by the definition of G-topologizable. □

Example 4.11 shows that the subsequential method condition of the method G in Corollary 4.8 can not be omitted.

Example 4.11. *Let X be the set $\mathbb{Z}$ of all integers endowed with the discrete topology and $\mathscr{H}=\{\emptyset\}$. Put $c_G(X) = s(X)$ and G: $c_G(X) \to X$ is defined by $G(\mathbf{x})$= 0 for each $\mathbf{x} \in c_G(X)$. Then G-sequential space X is G^*-connected but it is not connected.*

Lemma 4.12. *If T is G^*-connected, $T \subset U \cup V$, and U and V are (G,G^*)-separated then T lies entirely within U or V.*

Proof. Clearly T = (T $\cap$ U) $\cup$ (T $\cap$ V). Since U and V are (G,G^*)-separated, then Cl_{G^*}(T $\cap$ U) $\cap$ (T $\cap$ V) = (T $\cap$ U) $\cap$ Cl_G(T $\cap$ V) = $\emptyset$. So T $\cap$ U and T $\cap$ V are G^*-separated. By hypothesis, T $\cap$ U = $\emptyset$ or T $\cap$ V = $\emptyset$. So, T lies entirely within U or V. □

Theorem 4.13. *If S is G^*-connected and $S \subset T \subset Cl_{G^*}(S)$, then T is G^*-connected.*

Proof. Let T = U $\cup$ V such that U and V are (G,G^*)-separated. By Lemma 4.12 and S $\subset$ T, S lies entirely within U or V. Then T $\subset$ $Cl_{G^*}(S)$ $\subset Cl_G(S) \subset Cl_G(U) \subset X - V$ or T $\subset Cl^*_G(S) \subset Cl^*_G(V) \subset X - U$ by Proposition 3.2 So either U = $\emptyset$ or V = $\emptyset$ i.e. T is G^*-connected. □

Proposition 4.14. *If S is G^*-connected then $Cl_{G^*}(S)$ is G^*-connected.*

Proof. The proof is obvious by Theorem 4.13. □

Lemma 4.15. *Let G be a method and $\mathscr{H}$ be a hereditary class on a set X. If $\{S_\lambda : \lambda \in \Lambda\}$ is a nonempty family of G^*-connected of X and for λ, $\lambda' \in \Lambda$, $S_\lambda \cap S_{\lambda'} \neq \emptyset$, then $S = \bigcup_{\lambda\in\Lambda} S_\lambda$ is G^*-connected.*

Proof. Let S = U ∪ V such that U and V are (G,G^*)-separated. By Lemma 4.12 and $S_\lambda \subset$ S, S_λ lies entirely within U or V for $\lambda \in \Lambda$. By hypothesis, there aren't exist $\lambda \in \Lambda$ and $\lambda' \in \Lambda$ such that $S_\lambda \subset$ U and $S_{\lambda'} \subset$ V. So $S_\lambda \subset$ U or $S_\lambda \subset$ V for each $\lambda \in \Lambda$. Then $S_\lambda \subset$ U or $S_\lambda \subset$ V i.e U = ∅ or V = ∅. Hence S is G^*-connected. □

Definition 4.16. *The union of all G^*-connected containing a point x in X is called G^*-connected component of x and denoted by $C_x^{G^*}$.*

Proposition 4.17. *Let G be a regular subsequential method on a first-countable space X, $\mathscr{H} = \{\emptyset\}$ and $x \in X$. $C_x^{G^*}$ is G^*-connected component of x iff $C_x^{G^*}$ is connected component of x.*

Proof. The proof is obvious by Lemma 3.12. □

Proposition 4.18. *Let G be a method on a topological space X. Let X be G-topologizable, $\mathscr{H} = \{\emptyset\}$ and $x \in X$. $C_x^{G^*}$ is G^*-connected component of x iff X is $C_x^{G^*}$ is connected component of x.*

Proof. The proof is obvious by the definition of G-topologizable. □

Proposition 4.19. *Let G be a subsequential method on a topological group X. Let X be a G-Fréchet space, $x \in X$. $C_x^{G^*}$ is G-sequentially connected component of x iff it is a G^*-connected component of x.*

Proof. The proof is obvious by Lemma 3.12. □

Theorem 4.20. *Let $C_x^{G^*}$ be G^*–connected component of x in X. The following statements are equivalent:*

(a) $C_x^{G^*}$ is G^*-connected.

(b) $C_x^{G^*}$ is G^*-closed.

(c) $X = \bigcup_{x\in X} C_x^{G^*}$. For each x,y ∈X, either $C_x^{G^*} \cap C_y^{G^*} = \emptyset$ or $C_x^{G^*} = C_y^{G^*}$

Proof. (a) Since the intersection of all G^*-connected containing a point x is non empty set, $C_x^{G^*}$ is a G^*-connected by Lemma 4.15.
(b) By (a) and Proposition 4.14, $Cl_{G*}(C_x^{G^*})$ is a G^*-connected. By the definition of G^*-connected component, $Cl_{G*}(C_x^{G^*}) \subset C_x^{G^*}$. So, we obtain that $C_x^{G^*}$ is G^*-closed.
(c) It is obvious that $X=\bigcup_{x\in X} C_x^{G^*}$. Let x, y ∈ X. If $C_x^{G^*} \cap C_y^{G^*} \neq \emptyset$, $C_x^{G^*} \cup C_y^{G^*}$ is G^*-connected containing x and y by Lemma 4.15. By (a), $C_x^{G^*} \cup C_y^{G^*} \subset C_x^{G^*}$ and $C_x^{G^*} \cup C_y^{G^*} \subset C_y^{G^*}$. Then, we obtain $C_x^{G^*} = C_y^{G^*}$ □

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