Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (2) (2022), 430 – 442
DOI : 10.15672/hujms.792212

Research Article

On a generalization of C_2-modules

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Abstract

A module M is called a C_{21}-module if, whenever A and B are submodules of M with $A \cong B$, A is nonsingular and B is a direct summand of M, then A is a direct summand of M. Various examples of C_{21}-modules are presented. Some basic properties of these modules are investigated. It is shown that the class of rings R over which every C_{21}-module is a C_2-module is exactly that of right SI-rings. Also, we prove that for a ring R, every R-module has (C_{21}) if and only if R is a right t-semisimple ring.

Mathematics Subject Classification (2020). 16D10, 16D99

Keywords. C_{21}-module, C_2-module, nonsingular module, singular module, SI-ring, t-semisimple ring

1. Introduction

Among many generalizations of (quasi-) injective modules, the notion of continuous modules and its related properties have attracted considerable attention since 1971 (see, for example, [13, 20–22, 25, 27]). Following [21, Definition 2.3], a module M is called continuous if M satisfies the following two conditions:

(C_1): Every submodule of M is essential in a direct summand of M;

(C_2): If a submodule N of M is isomorphic to a direct summand of M, then N is a direct summand of M.

A module M is said to be *extending* if M satisfies the condition (C_1) (see [8]). Also, a module M is called *quasi-continuous* if M is extending and whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M (see [21]). Asgari and Haghany introduced and studied some generalizations of these notions. According to [4, Definition 2.10 and Theorem 2.11], a module M is called *t-extending* if every submodule of M which contains $Z_2(M)$ is essential in a direct summand of M. A module M is called *t-continuous* if M is t-extending and every submodule of M which contains $Z_2(M)$ and is isomorphic to a direct summand of M, is itself a direct summand (see [2]). Also, a module M is called *t-quasi-continuous* if M is t-extending and whenever A and B are

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Received: 08.09.2020; Accepted: 27.09.2021

nonsingular direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M (see [3]). It was shown in [2, Corollary 2.5] that a module M is t-continuous if and only if M is t-extending and every nonsingular submodule of M which is isomorphic to a direct summand of M, is itself a direct summand. Motivated by this result, we introduce and investigate the notion of C_{21}-modules which is a generalization of the notion of C_2-modules. A module M is called a C_{21}-module if every nonsingular submodule of M which is isomorphic to a direct summand of M, is itself a direct summand of M.

Various examples of C_{21}-modules are presented in Section 2. For instance, it is shown that every module M for which $M/Z_2(M)$ is a C_2-module is a C_{21}-module. Also, we provide an example to show that the concept of C_{21}-modules is a proper generalization of that of C_2-modules.

We begin Section 3 by showing that all direct summands of a C_{21}-module inherit the property. On the other hand, some examples are exhibited to prove that the class of C_{21}-modules is not closed under direct sums. Then we investigate some basic properties of C_{21}-modules. Moreover, we shed some light on the endomorphism ring of a hereditary C_{21}-module.

In Section 4, a number of characterizations of classes of rings in terms of C_{21}-modules are provided. Among others, we first investigate the natural question of when every C_{21}-module over a ring R has (C_2). It turns out that this condition is equivalent to the fact that every singular R-module is injective (i.e., R is a right SI-ring). It is also shown that rings over which every module has (C_{21}) are precisely the right t-semisimple rings (i.e., the rings R for which $R/Z_2(R_R)$ is a semisimple ring). Moreover, we prove that a ring R is a right GV-ring (i.e., every singular simple R-module is injective) if and only if every C_{21}-module is simple-direct-injective.

Throughout, all rings have identities and all modules are unital right modules. Let R be a ring. For an R-module M, we denote by $Rad(M)$, $Soc(M)$, $Z(M)$, $Z_2(M)$, and $E(M)$ the Jacobson radical, the socle, the singular submodule of M, the second singular submodule of M, and the injective hull of M, respectively. The notation $N \subseteq M$ means that N is a subset of M and we write $N \leq M$ if N is a submodule of M. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$, we denote the set of rational numbers, the set of integers, and the set of positive integers, respectively. For a prime number p, the Prüfer p-group is denoted by $\mathbb{Z}(p^\infty)$.

2. Examples

Let M be an R-module. Recall that the *singular submodule* $Z(M)$ of M is defined by

$$Z(M) = \{m \in M \mid mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

The *Goldie torsion submodule* $Z_2(M)$ of M (also known as the *second singular submodule* of M) is defined to be the submodule of M which contains $Z(M)$ such that $Z(M/Z(M)) = Z_2(M)/Z(M)$. The module M is called *singular* if $Z(M) = M$ and is called *nonsingular* if $Z(M) = 0$ (equivalently, $Z_2(M) = 0$). The module M is said to be Z_2-*torsion* if $Z_2(M) = M$. Recall that $Z_2(N) = Z_2(M) \cap N$ for every submodule N of M. Recall further that, $M/Z_2(M)$ is a nonsingular module. Moreover, for every class of R-modules M_λ ($\lambda \in \Lambda$), we have $Z(\oplus_{\lambda\in\Lambda} M_\lambda) = \oplus_{\lambda\in\Lambda} Z(M_\lambda)$ and $Z_2(\oplus_{\lambda\in\Lambda} M_\lambda) = \oplus_{\lambda\in\Lambda} Z_2(M_\lambda)$.

Definition 2.1. (i) An R-module M is called a C_{21}-*module* (or has (C_{21})) if every nonsingular submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M.

(ii) The ring R is called a (*left*) *right* C_{21}-*ring* if the (left) right R-module $({}_RR)$ R_R is a C_{21}-module.

In this section we exhibit many examples of C_{21}-modules.

Example 2.2. Let R be a ring and let I be an essential right ideal of R. By [2, Example 2.6(i)], $E \oplus R/I$ is a C_{21}-module for any injective R-module E.

Let M be an indecomposable module. Then clearly M has (C_{21}) if and only if M has no nonzero proper nonsingular submodule isomorphic to M. For example, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ (where p is any prime) has (C_{21}) but the $\mathbb{Z}$-module $\mathbb{Z}$ is not a C_{21}-module. Next, we shed more light on the structure of indecomposable C_{21}-modules.

Proposition 2.3. *The following are equivalent for an indecomposable module M:*

(i) *M is a C_{21}-module;*

(ii) *$Z(M) \neq 0$ or M is a nonsingular C_2-module.*

Proof. (i) ⇒ (ii) This follows from the fact that the class of nonsingular modules is closed under submodules (see [11, Proposition 1.22(a)]).

(ii) ⇒ (i) Assume that $Z(M) \neq 0$. Let N be a nonsingular submodule of M which is isomorphic to a direct summand K of M. Since M is indecomposable and $Z(M) \neq 0$, we have $N = 0$ and hence N is a direct summand of M. Therefore M is a C_{21}-module. □

Proposition 2.4. *The following are equivalent for an indecomposable $\mathbb{Z}$-module M:*

(i) *M is a C_2-module;*

(ii) *M is a C_{21}-module;*

(iii) *$M \cong \mathbb{Z}(p^\infty)$ or $M \cong \mathbb{Z}/p^n\mathbb{Z}$ or $M \cong \mathbb{Q}$, where p is a prime number and n is a positive integer.*

Proof. (i) ⇒ (ii) This is obvious.

(ii) ⇒ (iii) Let $T(M)$ denote the torsion submodule of M. Suppose that $T(M) \neq 0$. Using [14, Theorem 10], we deduce that $M \cong \mathbb{Z}(p^\infty)$ or $M \cong \mathbb{Z}/p^n\mathbb{Z}$ for some prime number p and some positive integer n. Now assume that $T(M) = 0$. Then $qM \cong M$ for any prime number q. Since M is a C_{21}-module, we conclude that $qM = M$ for every prime number q. That is, M is injective. This yields $M \cong \mathbb{Q}$.

(iii) ⇒ (i) This is clear. □

Example 2.5. Let M be a module whose endomorphism ring is a division ring. Then clearly M is indecomposable. Moreover, if N is a submodule of M such that M is isomorphic to N then $N = M$. So M is a C_{21}-module. Many examples belonging to this class of modules are given in [19].

Note that one can easily observe that every module having no nonzero nonsingular direct summands, is a C_{21}-module. Next, we show that this idea provides a rich source of examples of C_{21}-modules.

Example 2.6. (i) Every Z_2-torsion module is a C_{21}-module, since the only nonsingular submodule of a Z_2-torsion module is the zero submodule.

(ii) From (i), it follows that every module M for which $Z(M)$ is essential in M (for instance, M is a singular module) has (C_{21}). In particular, R/I is a C_{21}-R-module for every essential right ideal I of a ring R. Also, for any module M, $E(M)/M$ is a C_{21}-module.

(iii) Let M be a torsion $\mathbb{Z}$-module. Since M is singular, M has (C_{21}) by (ii).

An abelian group G is called *cotorsion* if $Ext(J, G) = 0$ for every torsion-free abelian group J (see [9, p. 232]). An abelian group G is called *algebraically compact* if G is a direct summand in every abelian group H that contains G as a pure subgroup (see [9, p. 159]). This is equivalent to the fact that G is a direct summand of a direct product of cocyclic abelian groups (see [9, Theorem 38.1]). For example, the abelian group $M = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$ (where p is a prime) is algebraically compact. By [9, Proposition 54.1], an abelian group is cotorsion if and only if it is an epimorphic image of an algebraically compact abelian group. A cotorsion reduced abelian group G is called *adjusted* if G has no nonzero torsion-free direct summands (see [9, p. 238]).

It was shown in [9, Theorem 55.5] that any reduced cotorsion abelian group G is the direct sum $G = A \oplus C$ of a torsion-free algebraically compact abelian group A and an adjusted cotorsion abelian group C. Moreover, C is a uniquely determined subgroup of G and $C \cong Ext(\mathbb{Q}/\mathbb{Z}, T(G))$ where $T(G)$ denotes the torsion subgroup of G.

Example 2.7. (i) It is clear that every reduced cotorsion adjusted abelian group is a C_{21}-module. So $Ext(\mathbb{Q}/\mathbb{Z}, T)$ has (C_{21}) for any torsion abelian group T by [9, Lemma 55.4].

(ii) Let T be a reduced unbounded torsion abelian group and let $G = Ext(\mathbb{Q}/\mathbb{Z}, T)$. By [10, p. 186 Example 1], G is an adjusted abelian group whose torsion part is $T(G) = T$. Moreover, T is not a direct summand of G. In particular, G is a mixed abelian group.

(iii) Let p be a prime number and consider the $\mathbb{Z}$-module $M = \prod_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$. Note that M is a reduced module. Indeed, M has no nonzero elements of infinite p-height. Let $T(M)$ denote the torsion submodule of M. Since $M/T(M)$ is not divisible, it follows that M is not adjusted by [12, Proposition 2.2]. On the other hand, by [9, Theorem 55.5], M has an adjusted direct summand N which contains $T(M)$.

Let M be an R-module. It is clear that if M is a C_2-module, then M is a C_{21}-module. Note that the converse holds when M is noncosingular but it is not true, in general, as illustrated in the following two examples.

Example 2.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Clearly, M is a torsion module. So, by Example 2.6(iii), M is a C_{21}-module. On the other hand, consider the element $x = (\bar{0}, \bar{4})$ of M. It is clear that $x\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. So $x\mathbb{Z}$ is isomorphic to the direct summand $\mathbb{Z}/2\mathbb{Z} \oplus 0$ of M. However, $x\mathbb{Z}$ is not a direct summand of M. This implies that M is not a C_2-module.

In the next example, we present a (right) C_{21}-ring which is not a (right) C_2-ring.

Example 2.9. Let p be a prime number and consider the trivial extension $R = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is a faithful module, we have $Z(R) = p\mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ (see [11, p. 37 Exercise 16]). Note that the ideals of R are $0 \oplus N$ and $n\mathbb{Z} \oplus \mathbb{Z}(p^\infty)$, where N is a submodule of $\mathbb{Z}(p^\infty)$ and n is a positive integer. It is easily seen that every nonzero ideal of R is essential in R. So R is a uniform R-module. In particular, R is an indecomposable R-module. Also, since $Z(R)$ is essential in R, R is a Z_2-torsion R-module. Hence R is a (right) C_{21}-ring (see Example 2.6(i)). On the other hand, taking a prime number $q \neq p$ and any element $x \in M$, we can check that $ann_R((q, x)) = 0$. Therefore $(q, x)R \cong R$. It is clear that (q, x) is not invertible in R. This forces $(q, x)R \neq R$. Consequently, R is not a (right) C_2-ring.

The next proposition provides more examples of C_{21}-modules.

Proposition 2.10. *Let M be a module such that $M/Z_2(M)$ is a C_{21}-module (i.e., $M/Z_2(M)$ is a C_2-module). Then M is a C_{21}-module.*

Proof. Let N be a nonsingular submodule of M and let K be a direct summand of M such that $N \cong K$. Then $Z_2(N) = N \cap Z_2(M) = 0$ and $Z_2(K) = K \cap Z_2(M) = 0$. Hence,

$$(N + Z_2(M))/Z_2(M) \cong N/(N \cap Z_2(M)) \cong K/(K \cap Z_2(M)) \cong (K + Z_2(M))/Z_2(M).$$

Since $Z_2(M)$ is fully invariant in M, $(K+Z_2(M))/Z_2(M)$ is a direct summand of $M/Z_2(M)$. As $M/Z_2(M)$ is a nonsingular C_{21}-module, it follows that $(N + Z_2(M))/Z_2(M)$ is a direct summand of $M/Z_2(M)$. Let L be a submodule of M with $Z_2(M) \subseteq L$ and $M/Z_2(M) = ((N + Z_2(M))/Z_2(M)) \oplus (L/Z_2(M))$. Thus $M = N + L$. Moreover, $N \cap L \subseteq Z_2(M) \cap N = 0$. Therefore $M = N \oplus L$. So M has (C_{21}). □

Corollary 2.11. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that $Z_2(M_1) = M_1$ and M_2 is a nonsingular C_{21}-module. Then M is a C_{21}-module.*

Proof. Since $Z(M_2) = 0$, we have $Z_2(M_2) = 0$. Therefore $Z_2(M) = Z_2(M_1) = M_1$. Thus $M/Z_2(M) \cong M_2$ has (C_{21}). So M has (C_{21}) by Proposition 2.10. □

Remark 2.12. Consider the ring R given in Example 2.9. So $R/Z_2(R) = 0$ is a C_2-module, but the R-module R does not have (C_2). This shows that the analogue of Proposition 2.10 for C_2-modules does not hold true in general.

Note that all the modules presented in Example 2.6 are Z_2-torsion. So they are t-continuous. As an application of Proposition 2.10, we get the following three examples. The first one exhibits a C_{21}-module that is not t-continuous.

Example 2.13. Consider the $\mathbb{Z}$-module $M = \prod_{p\in\mathbb{P}} \mathbb{Z}/p\mathbb{Z}$ where $\mathbb{P}$ is the set of all prime numbers. It is easily seen that $Z_2(M) = \oplus_{p\in\mathbb{P}}\mathbb{Z}/p\mathbb{Z}$. Note that $M/Z_2(M)$ is a divisible (injective) $\mathbb{Z}$-module. In particular, $M/Z_2(M)$ is a C_2-module. So M is a C_{21}-module by Proposition 2.10. On the other hand, the $\mathbb{Z}$-module M is not t-continuous by [4, Example 2.16].

Example 2.14. Let R be a right nonsingular ring (i.e., $Z(R_R) = 0$) and let E be an injective module. Let N be a proper submodule of E and set $M = E/N$. By [11, Proposition 1.23(a)], $Z(M/Z(M)) = 0$. This gives $Z_2(M) = Z(M)$. Using [23, Theorem 2.10], it follows that $M/Z_2(M)$ is an injective module. Therefore $M/Z_2(M)$ is a C_{21}-module. From Proposition 2.10, we conclude that M is a C_{21}-module.

Example 2.15. Let R be a right semiartinian ring in which every maximal right ideal is essential (for example, R can be a local semiartinian ring which is not a division ring). Let M be an R-module. Then $Soc(M)$ is essential in M. Moreover, we have $Soc(M) \subseteq Z(M) \subseteq Z_2(M)$. Thus $Z_2(M)$ is an essential submodule of M. This implies that $M/Z_2(M)$ is a singular module and hence $M/Z_2(M)$ is a C_{21}-module. Applying Proposition 2.10, it follows that every R-module is a C_{21}-module.

3. Some properties of C_{21}-modules

In this section we establish some properties of C_{21}-modules. We begin by showing that having (C_{21}) is preserved by direct summands but it is not preserved under direct sums.

Proposition 3.1. *Any direct summand of a C_{21}-module is again a C_{21}-module.*

Proof. Let M be a C_{21}-module and let N be a direct summand of M. Let K and L be two isomorphic nonsingular submodules of N such that K is a direct summand of N. Note that K is a direct summand of M. Then L is a direct summand of M. Hence $M = L \oplus L'$ for some submodule L' of M. By modularity, we have $N = N \cap (L \oplus L') = L \oplus (N \cap L')$. Hence L is a direct summand of N. Therefore N is a C_{21}-module. □

A direct sum of C_{21}-modules (or even C_2-modules) need not be a C_{21}-module as the next two examples show. Note that the first one appeared in [21, Example 2.9] to show that a direct sum of quasi-continuous modules may not be quasi-continuous. Also, this example appeared in [22, p. 170] to show that a direct sum of C_2-modules need not be a C_2-module.

Example 3.2. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and its right ideals $A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$, where F is a field. Clearly, $R_R = A \oplus B$. Since B_R is simple, B_R has (C_2). Moreover, A_R has exactly one proper nonzero submodule $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, and $J(R)$ is not isomorphic to A_R. Thus A_R has (C_2). On the other hand, the R-module R_R is not a C_2-module (see [22, p. 168] or [21, Example 2.9]). In addition, it is well known that R is a right hereditary ring (see for example, [8, Example 13.6]). Therefore R_R is a nonsingular R-module by [11, Proposition 1.27(a)]. Hence the R-module R_R could not be a C_{21}-module.

Example 3.3. Let T be a commutative local ring such that T is a unique factorization domain (UFD). Assume also that T has infinitely many nonassociate prime elements, but T is not a principal ideal domain (for example, T can be the ring of power series in two variables over an infinite field or the polynomial ring $\mathbb{Z}[X]$ in one indeterminate, over the domain $\mathbb{Z}$ of integers). Then there exist two nonassociate prime p and q in T such that $Tp + Tq \neq T$. Let M be the direct sum of all T/pT, p ranging over the primes of T. Consider the trivial extension $R = T \oplus M$ of T by M. From [15, p. 63 Exercise 7], it follows that no nonzero element of R annihilates $R(p, 0) + R(q, 0)$. Using [16, Corollary 2.4], we conclude that R has a finitely generated free R-module F which is not a C_2-module. On the other hand, R is a C_2-ring by [16, p. 285 Question]. Now we claim that $Y = 0 \oplus M$ is an essential ideal in R. To show this, let $\mathfrak{I}$ be an ideal of R such that $\mathfrak{I} \cap Y = 0$. Then there exist an ideal I of T and a T-submodule N of M such that $\mathfrak{I} = I \oplus N$, $IM \subseteq N$ and $\mathfrak{I} \cap Y = 0 \oplus N = 0$. Thus $N = 0$. Moreover, since $IM \subseteq N$, we have $IM = 0$. As T is a UFD, we deduce that $I = 0$ and hence $\mathfrak{I} = 0$. In addition, since $Y^2 = 0$, we obtain $Y \subseteq Z(R)$. This implies that $Z(R)$ is an essential ideal in R. It follows that $R/Z(R)$ is a singular R-module. We thus get $Z_2(R) = R$. So $Z_2(F) = F$ and hence F is a C_{21}-module.

Proposition 3.4. *Let M be a C_{21}-module. Then the following hold:*

(i) *For every direct summands A and B of M such that $A \cap B = 0$ and B is nonsingular, $A \oplus B$ is a direct summand of M.*

(ii) *Assume that $M = A \oplus B$ such that at least one of the submodules A and B is nonsingular. Then for any homomorphism $f : A \longrightarrow B$ such that $Kerf$ is a direct summand of A, Imf is a direct summand of B.*

Proof. (i) Let A and B be two direct summands of M such that $Z(B) = 0$ and $A \cap B = 0$. Then $M = A \oplus L$ for some submodule L of M. Let $\pi : M \longrightarrow L$ be the natural projection map. It follows that $\pi_{/B} : B \longrightarrow \pi(B)$ is an isomorphim. Since M has (C_{21}), $\pi(B)$ is a direct summand of M. Hence $\pi(B)$ is a direct summand of L. Thus $L = \pi(B) \oplus X$ for some $X \leq L$. It follows that $M = (A \oplus \pi(B)) \oplus X = (A \oplus B) \oplus X$. So $A \oplus B$ is a direct summand of M.

(ii) Let $f : A \longrightarrow B$ be a homomorphism such that $Kerf$ is a direct summand of A. Then $A = Kerf \oplus N$ for some submodule N of A. Hence $Imf \cong A/Kerf \cong N$. From the hypothesis, we infer that Imf is nonsingular. Since M has (C_{21}), we conclude that Imf is a direct summand of M. Thus Imf is a direct summand of B. □

The next corollary follows directly from Proposition 3.4(ii).

Corollary 3.5. *Let A and B be submodules of a C_{21}-module M such that $Z(A) = 0$ or $Z(B) = 0$ and $M = A \oplus B$. If $f : A \longrightarrow B$ is a monomorphism, then Imf is a direct summand of B.*

Corollary 3.6. *Let M be a nonsingular R-module such that $M \oplus E(M)$ is a C_{21}-module. Then M is an injective module.*

Proof. Consider the inclusion map $\mu : M \longrightarrow E(M)$. Then, by Corollary 3.5, $\mu(M) = M$ is a direct summand of $E(M)$. This implies that M is injective, as required. □

Let R be a ring. Recall that an R-module M is said to be (*semi*)*hereditary* if every (finitely generated) submodule of M is a projective module. It is well known that projective right modules over a right hereditary ring are hereditary modules (see e.g., [29, 39.16]). Note that for any nonzero element x in a semihereditary R-module M, $ann_R(x)$ is a direct summand of R_R. So every semihereditary module is nonsingular. Next, we will be concerned with the endomorphism ring of a (semi)hereditary C_{21}-module.

Proposition 3.7. *Let M be a C_{21}-module. Assume that one of the following conditions is satisfied:*

(i) *M is a hereditary module.*
(ii) *M is a semihereditary finitely generated module.*
Then $End_R(M)$ is a von Neumann regular ring.

Proof. (i) Suppose M is a hereditary module and let $f \in End_R(M)$. Then Imf is a projective module. Since $M/Kerf \cong Imf$, it follows that $Kerf$ is a direct summand of M. Thus $M = Kerf \oplus L$ for some submodule L of M. Hence $Imf \cong L$. Since M is a C_{21}-module and Imf is nonsingular, we deduce that Imf is a direct summand of M. Therefore $End_R(M)$ is a von Neumann regular ring by [29, 37.7(2)].

(ii) This follows by the same method as in (i). □

The following corollary is a direct consequence of Proposition 3.7.

Corollary 3.8. *The following conditions are equivalent for a ring R:*
(i) *R is a right semihereditary right C_{21}-ring;*
(ii) *R is a von Neumann regular ring.*

The next example shows that the condition "M is a semihereditary module" in the hypothesis of Proposition 3.7 is not superfluous.

Example 3.9. It is clear that the $\mathbb{Z}$-module $M = \mathbb{Z}/4\mathbb{Z}$ is not semihereditary. Moreover, since M is a torsion $\mathbb{Z}$-module, M is a singular module. Hence M is a C_{21}-module. On the other hand, $End_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ is not a von Neumann regular ring.

Recall that a module M is called *regular* if every cyclic submodule of M is a direct summand of M. Equivalently, every finitely generated submodule of M is a direct summand of M (see [26, p. 67]).

Corollary 3.10. *Let M be a hereditary C_{21}-module over a ring R. Then the following implications hold:*
(i) *If M is indecomposable, then $End_R(M)$ is a division ring.*
(ii) *If M has finite uniform dimension, then $End_R(M)$ is a semilocal ring.*
(iii) *If R is a commutative ring and M is a noetherian R-module, then M is a semisimple module.*

Proof. (i) This follows from Proposition 3.7.

(ii) Let $f : M \longrightarrow M$ be a monomorphism. Then $Imf \cong M$. Since M has finite uniform dimension, Imf is an essential submodule of M by [8, 5.8(1)]. Moreover, Imf is a direct summand of M by Proposition 3.7. This yields $Imf = M$. From [7, Proposition 19.5], it follows that $End_R(M)$ is a semilocal ring.

(iii) By Proposition 3.7, $End_R(M)$ is von Neumann regular. Using [28, Corollary 3.10], we see that M is a regular module. Since M is noetherian, it follows that M is a semisimple module. □

The condition "M has (C_{21})" in the hypothesis of Corollary 3.10 is not superfluous. To see this, consider the following example.

Example 3.11. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$ which is not a C_{21}-module by Proposition 2.4. Since every nonzero submodule of M is isomorphic to M, M is a hereditary module. Also, M is an indecomposable noetherian module (hence M has finite uniform dimension). However, $End_{\mathbb{Z}}(M) \cong \mathbb{Z}$ is neither a division ring nor a semilocal ring and M is not semisimple.

4. Rings over which certain modules have (C_{21})

In this section, we characterize some classes of rings in terms of C_{21}-modules. We begin with the following characterization of the class of rings R for which every C_{21}-module is a C_2-module. This result should be contrasted with Examples 2.8 and 2.9.

Recall that a ring R is said to be a *right SI-ring* if every singular right R-module is injective (see [8, p. 160]). A module M is called a *C_3-module* if whenever A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M (see, for example, [21]).

Theorem 4.1. *The following conditions are equivalent for a ring R:*
(i) *R is a right SI-ring;*
(ii) *Every C_{21}-R-module is a C_2-module;*
(iii) *Every C_{21}-R-module is a C_3-module;*
(iv) *Every Z_2-torsion R-module is a C_3-module.*

Proof. (i) $\Rightarrow$ (ii) Suppose that R is a right SI-ring. Let A be an R-module. Since $Z(A)$ is singular, it follows that $Z(A)$ is an injective module. Thus $Z(A)$ is a direct summand of A. Now let M be a C_{21}-R-module. To prove that M is a C_2-module, let N and K be submodules of M such that $N \cong K$ and K is a direct summand of M. Therefore there exist submodules N' and K' of M such that $N = Z(N) \oplus N'$ and $K = Z(K) \oplus K'$. Since $N \cong K$, it follows easily that $Z(N) \cong Z(K)$ and $N' \cong N/Z(N) \cong K/Z(K) \cong K'$. Note that K' is a direct summand of M and N' is nonsingular. Since M has (C_{21}), it follows that N' is a direct summand of M. This implies that $M = N' \oplus L$ for some submodule L of M. So $Z(M) = Z(L) \subseteq L$. As $Z(M)$ is injective, there exists a submodule L' of L such that $L = Z(M) \oplus L'$. Thus $M = N' \oplus Z(M) \oplus L'$. Moreover, since $Z(N)$ is injective, $Z(M) = Z(N) \oplus B$ for some submodule B of M. Consequently, $M = N' \oplus Z(N) \oplus B \oplus L' = N \oplus B \oplus L'$. It follows that M is a C_{21}-module.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear.

(iv) $\Rightarrow$ (i) Let M be a singular R-module. Clearly, M is Z_2-torsion. It is well known that the class of Z_2-torsion modules is closed under essential extensions and direct sums (see [11, p. 37 Exercise 21]). Then $M \oplus E(M)$ is a Z_2-torsion module. By (iv), $M \oplus E(M)$ is a C_3-module. Consider the inclusion map $\mu : M \to E(M)$. Thus, according to [1, Corollary 2.4], we deduce that $\mu(M) = M$ is a direct summand of $E(M)$. Hence M is an injective module. It follows that R is a right SI-ring. □

Remark 4.2. Let R be a ring which is not a right SI-ring. From Theorem 4.1, it follows that R has a C_{21}-module that is not a C_2-module.

It is shown in [5, Theorem 3.2] that for a ring R, $R/Z_2(R_R)$ is a semisimple ring if and only if every nonsingular R-module is injective. Moreover, the authors called a ring R which satisfies these equivalent conditions a *right t-semisimple* ring. In the next result, we determine the class of rings R for which every (nonsingular) R-module has (C_{21}).

Proposition 4.3. *Let R be a ring with $\overline{R} = R/Z_2(R_R)$. Then the following conditions are equivalent:*
(i) *R is a right t-semisimple ring;*
(ii) *Every R-module is a C_{21}-module;*
(iii) *Every nonsingular R-module is a C_{21}-module;*
(iv) *Every R-submodule of $\overline{R} \oplus \overline{R}$ is a C_{21}-module.*

Proof. (i) $\Rightarrow$ (ii) This follows from the definition of C_{21}-modules and the fact that every nonsingular module over a right t-semisimple ring is injective.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i) Let $\overline{I}$ be a right ideal of $\overline{R}$. Thus $\overline{I} \oplus \overline{R}$ being an R-submodule of $\overline{R} \oplus \overline{R}$ is a C_{21}-R-module by (iv). Let $\mu : \overline{I} \to \overline{R}$ be the inclusion map. Note that $\overline{R}$ is nonsingular. Then $\overline{I}$ is a direct summand of $\overline{R}$ by Corollary 3.5. Therefore $\overline{R}$ is a semisimple ring. This completes the proof. □

It is shown in Example 2.6 that every Z_2-torsion module has (C_{21}). Also, in Example 2.13, we provide a C_{21}-module which is not Z_2-torsion. Next, we characterize the class of rings R for which each C_{21}-R-module is Z_2-torsion. It turns out that this class is a subclass of that of t-semisimple rings.

Proposition 4.4. *The following conditions are equivalent for a ring R:*
(i) *R_R is a Z_2-torsion R-module;*
(ii) *Every C_{21}-R-module is Z_2-torsion.*
In this case, every R-module is a C_{21}-module.

Proof. Let us first note that for any module R-homomorphism $f : M \to N$, we have $f(Z_2(M)) \subseteq Z_2(N)$. Let M be an R-module. Given $a \in Z_2(R_R)$, we consider the R-homomorphism $\varphi : R \to aR$ defined by $\varphi(r) = ar$ for all $r \in R$. Then $\varphi(Z_2(R_R)) = aZ_2(R_R) \subseteq Z_2(M)$. It follows that $MZ_2(R_R) \leq Z_2(M)$.

(i) $\Rightarrow$ (ii) Suppose that $Z_2(R) = R$ and let M be an R-module. Then $MZ_2(R_R) = M \subseteq Z_2(M)$. Hence M is a Z_2-torsion module. Therefore M has (C_{21}).

(ii) $\Rightarrow$ (i) Note that $E(R_R)$ is a C_{21}-module. Then $E(R_R)$ is Z_2-torsion. But the class of Z_2-torsion modules is closed under submodules. Thus R_R is Z_2-torsion. □

For a ring R and an R-module M, the (Goldie) *reduced rank* of M (of R) is the uniform dimension of $M/Z_2(M)$ (of $R_R/Z_2(R_R)$) (see for example, [17, Definition 7.34]). The next result shows that the class of rings R for which every direct sum of injective R-modules has (C_{21}) is exactly that of rings having finite reduced rank. Note that the proof of the implication (iii) $\Rightarrow$ (i) of this result is similar to that of [3, Theorem 4.9((1)$\Rightarrow$ (2))], but it is given for completeness.

Proposition 4.5. *The following conditions are equivalent for a ring R:*
(i) *R is of finite reduced rank;*
(ii) *Every direct sum of injective R-modules is a C_{21}-module;*
(iii) *Every direct sum of nonsingular injective R-modules is a C_{21}-module.*

Proof. (i) $\Rightarrow$ (ii) Suppose that R is of finite reduced rank and let M be an R-module which is a direct sum of injective submodules. By [3, Proof of Theorem 4.9((2) $\Rightarrow$ (1))], $M = Z_2(M) \oplus M'$ for some injective submodule M' of M. Clearly, $M/Z_2(M) \cong M'$ is a continuous module. By Proposition 2.10, M is a C_{21}-module.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) Since $R/Z_2(R_R)$ is a right nonsingular ring, it suffices to show that every direct sum of nonsingular injective $R/Z_2(R_R)$-modules is an injective $R/Z_2(R_R)$-module by [11, Theorem 3.17]. Let $M = \oplus_{i\in I} M_i$ be a direct sum of nonsingular injective $R/Z_2(R_R)$-modules M_i $(i \in I)$. Then by [11, p. 48 Exercise 22], each M_i $(i \in I)$ is a nonsingular injective R-module. Since M is nonsingular, so is $E(M)$. Thus, by hypothesis, $M \oplus E(M)$ is a C_{21}-module. Now using Corollary 3.6, it follows that M is an injective R-module. Hence M is an injective $R/Z_2(R_R)$-module (see [11, p. 48 Exercise 22]). □

It is easily seen that every right t-semisimple ring is of finite reduced rank. Using the C_{21} property, we provide in the next theorem a necessary and sufficient condition for a ring of finite reduced rank to be right t-semisimple. We first prove the following lemma.

Lemma 4.6. *Let R be a ring such that every 2-generated R-module is a C_{21}-module or every direct sum of two uniform modules is a C_{21}-module. Then:*
(i) *Every nonsingular uniform R-module is simple and injective.*
(ii) *Every nonsingular R-module having finite uniform dimension is a semisimple injective module.*

Proof. (i) Let U be a nonsingular uniform R-module. Then clearly $E(U)$ is a nonsingular uniform module. Let $0 \neq x \in E(U)$ and take $0 \neq y \in xR$. By hypothesis, $xR \oplus yR$ is a C_{21}-module. Clearly, xR is a nonsingular R-module. So by Corollary 3.5, it follows that yR is a direct summand of xR. Since xR is a uniform module, xR is indecomposable and hence $xR = yR$. This implies that xR is a simple module. Therefore $E(U)$ is a semisimple module. As $E(U)$ is indecomposable, we see that $E(U)$ is a simple module. Thus $U = E(U)$ is a simple injective R-module.

(ii) Let M be a nonsingular module having finite uniform dimension. So there exists a non-negative integer n such that M has an essential submodule $N = \oplus_{i=1}^{n} U_i$ which is a direct sum of uniform submodules U_i $(1 \le i \le n)$ of M. It is clear that each U_i $(1 \le i \le n)$ is a nonsingular R-module. From (i), it follows that each U_i $(1 \le i \le n)$ is a simple injective R-module. Therefore N is semisimple and injective. This implies that N is a direct summand of M and hence $M = N$. This completes the proof. □

Theorem 4.7. *The following conditions are equivalent for a ring R:*

(i) *R is of finite reduced rank and every 2-generated R-module is a C_{21}-module;*

(ii) *R is of finite reduced rank and every direct sum of two uniform R-modules is a C_{21}-module;*

(iii) *R is a right t-semi-simple ring.*

Proof. (i) ⇒ (iii) Since R is of finite reduced rank, the R-module $\overline{R} = R_R/Z_2(R_R)$ has finite uniform dimension. Moreover, $\overline{R}$ is a nonsingular R-module. Then $\overline{R}$ is a semisimple R-module by Lemma 4.6. Hence R is a right t-semi-simple ring.

(iii) ⇒ (i) This follows from Proposition 4.3.

(ii) ⇔ (iii) This follows by similar arguments as in the equivalence (i) ⇔ (iii). □

Proposition 4.8. *Let R be a ring and let $\mathcal{C}$ be a class of R-modules such that $\mathcal{C}$ contains every direct sum of nonsingular injective modules and every direct sum of two uniform modules. Then the following assertions are equivalent:*

(i) *Every module in $\mathcal{C}$ has (C_{21});*

(ii) *R is a right t-semisimple ring.*

Proof. (i) ⇒ (ii) Using Proposition 4.5, it follows that the ring R is of finite reduced rank. Now we infer from Theorem 4.7 that R is a right t-semisimple ring.

(ii) ⇒ (i) By Proposition 4.3. □

Let N be a submodule of a module M. A *complement* of N in M is a submodule K of M maximal with respect to the property $N \cap K = 0$. Recall that a module M is said to be a *C_{11}-module* if every submodule of M has a complement which is a direct summand. By [24, Theorem 2.4], every direct sum of injective modules is a C_{11}-module and every direct sum of two uniform modules is also a C_{11}-module. As an application of Proposition 4.8, one can take the class $\mathcal{C}$ to be the class of C_{11}-modules. So the following corollary is a direct consequence of the preceding proposition.

Corollary 4.9. *The following conditions are equivalent for a ring R:*

(i) *Every C_{11}-module has (C_{21});*

(ii) *R is a right t-semisimple ring.*

Recall that a module M is called regular if every cyclic submodule of M is a direct summand. Following [18], a module M is said to be *d-Rickart* if $Im\varphi$ is a direct summand of M for every endomorphism φ of M.

Next, we provide a characterization in terms of C_{21}-modules for a right semi-hereditary ring to be von Neumann regular.

Proposition 4.10. *The following conditions are equivalent for a right semi-hereditary ring R:*

(i) *Every finitely generated projective R-module is a C_2-module;*

(ii) *Every finitely generated projective R-module is a C_{21}-module;*

(iii) *Every finitely generated projective R-module is a d-Rickart module;*

(iv) *Every finitely generated projective R-module is a regular module;*

(v) *R is a von Neumann regular ring.*

Proof. (i) ⇒ (ii) This is immediate.

(ii) ⇒ (iii) Let M be a finitely generated projective R-module and let f be an endomorphism of M. It is clear that Imf is finitely generated. Then $Imf \oplus M$ is a C_{21}-module. Since R is right semi-hereditary, R is right nonsingular. Hence M is nonsingular by [29, 39.13(2)] and [11, Proposition 1.22(a)]. Using Corollary 3.5, we deduce that Imf is a direct summand of M. Thus M is a d-Rickart module.

(iii) ⇒ (v) By (iii), R_R is a d-Rickart module. So R is a von Neumann regular ring by [18, Remark 2.2].

(v) ⇒ (iv) This follows from [26, Proposition 6.7(4)].

(iv) ⇒ (i) Let M be a finitely generated projective R-module. Let N and K be submodules of M such that $N \cong K$ and K is a direct summand of M. Since K is finitely generated, so is N. Therefore N is a direct summand of M as M is regular. Hence M is a C_2-module. □

Proposition 4.11. *The following assertions are equivalent for a ring R:*

(i) *R is right hereditary and every projective R-module is a C_{21}-module;*

(ii) *R is a semi-simple ring.*

Proof. (i) ⇒ (ii) Let I be a right ideal of R. Since R is right hereditary, I is a projective nonsingular right R-module. By assumption, $I \oplus R_R$ is a C_{21}-module. We infer from Colloary 3.5 that I is a direct summand of R_R. Consequently, R is a semisimple ring.

(ii) ⇒ (i) This is obvious. □

In 2014, Camillo, Ibrahim, Yousif and Zhou [6] introduced and studied the notion of *simple-direct-injective modules* which is another generalization of the notion of C_2-modules. Recall that an R-module M is called simple-direct-injective if, whenever A and B are simple submodules of M with $A \cong B$ and B is a direct summand of M, then A is a direct summand of M. Moreover, a ring R is called a *right generalized V-ring* (or a *right GV-ring*) if every simple R-module is either injective or projective; equivalently, every singular simple R-module is injective.

In the next proposition, we characterize right GV-rings, but first we need the following lemma.

Lemma 4.12. *Let R be a ring. Then every direct sum of a singular R-module and an injective R-module is a C_{21}-module.*

Proof. Let an R-module $N = M \oplus E$ be a direct sum of submodules M and E such that M is singular and E is injective. Note that by [2, Theorem 2.4((1) ⇔ (3))], every direct sum of a Z_2-torsion module and a nonsingular continuous module is t-continuous. Now, since E is injective, $E = Z_2(E) \oplus E'$ for some submodule E' of E such that E' is nonsingular and injective (see [8, 7.11]). Thus $N = (M \oplus Z_2(E)) \oplus E'$. Moreover, it is clear that $M \oplus Z_2(E)$ is Z_2-torsion and E' is continuous. Therefore N is a t-continuous module, and so N is a C_{21}-module by [2, Corollary 2.5]. □

Proposition 4.13. *The following statements are equivalent for a ring R:*

(i) *R is a right GV-ring;*

(ii) *Every C_{21}-module is simple-direct-injective.*

Proof. (i) ⇒ (ii) Let M be a C_{21}-module and let A and B be simple submodules of M with $A \cong B$ and B is a direct summand of M. If A is singular, then, by hypothesis, it is injective. Thus A is a direct summand of M. Now, suppose that A is nonsingular. Since M is a C_{21}-module, A is a direct summand of M. Hence M is simple-direct-injective.

(ii) ⇒ (i) Let M be a singular simple R-module and $E(M)$ be the injective hull of M. Then, by Lemma 4.12, $M \oplus E(M)$ is a C_{21}-module. Therefore, by hypothesis, $M \oplus E(M)$ is simple-direct-injective. Consequently, the inclusion map $i : M \to E(M)$ splits by

[6, Proposition 2.1]. It follows that M is a direct summand of $E(M)$. Hence M is injective and R is a right GV-ring, as required. □

Acknowledgment. The authors would like to thank the referees for helpful comments that improved the paper and for providing them with Lemma 4.12 and Proposition 4.13.

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