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S-δ-CONNECTEDNESS IN S-PROXIMITY SPACES

Beenu SINGH[1] and Davinder SINGH[2]
[1]Department of Mathematics, University of Delhi, New Delhi 110007 INDIA
[2] Department of Mathematics, Sri Aurobindo College, University of Delhi, Delhi 110017 INDIA

Abstract. New types of connectedness in S-proximity spaces, named as an S-δ-connectedness, local S-δ-connectedness are introduced. Also, their interrelationships are studied. In an S-proximity space (X, δ_X), the S-δ-connectedness of a subset U of X with respect to δ_X may not be same as the S-δ-connectedness of U with respect to its subspace proximity δ_U. Further, S-δ-component and S-δ-treelike spaces are also defined and a number of results are given.

1. Introduction

In 1908, Reisz [13] discussed the idea of proximity (now it is called an E-proximity) and although this idea was revived by Wallace [17, 18]. But the real beginning of E-proximity was due to Efremovič [5, 6] who gave axioms of it as a natural generalization of metric space and topological group. Smirnov [14, 15] demonstrated that a completely regular space always has a compatible E-proximity relation and vice versa. Also, he found the relationship between E-proximity space and uniform space. Several generalizations of E-proximity were defined and studied. The notion of Čech proximity spaces was given by E. Čech [2], later elaborated in [10], [11] and [12]. An S-proximity was introduced independently by Krishna Murti [7], Szymanski [16], Wallace [17, 18].

Mrówka *et al.* [9] defined the notion of δ-connectedness in E-proximity spaces and after that in 1987, the concepts of local δ-connectedness, δ-component and δ-quasi components were introduced by Dimitrijević *et al.* [3]. Dimitrijević *et al.* [4] also studied δ-treelike proximity spaces. Recently, Modak *et al.* [8] introduced the weaker form of connectedness (Cl-Cl-connectedness) in topological spaces.

In this paper, we introduce a new type of δ-connectedness (named as S-δ-connectedness) in S-proximity spaces and show that S-δ-connectedness is different

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singhbeenu47@gmail.com-Corresponding author; dstopology@gmail.com
0000-0003-0196-7670; 0000-0002-1446-707X.

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from δ-connectedness [9] in the category of S-proximity spaces. And it become identical in the categories of L-proximity spaces and E-proximity spaces. We give a characterization for an S-proximity space X to be S-δ-connected and several other properties analogous to δ-connectedness are justified. Relation among different types of connectedness are shown. In the last section, S-δ-component, local S-δ-connectedness and S-δ-treelike spaces are defined and their properties are studied.

Throughout this paper, $(A, B) \in \delta$ ($(A, B) \notin \delta$) denotes A, B are near (δ-separated). We write an S-proximity space as X instead of (X, δ) whenever there is no confusion of the S-proximity δ. $Cl_X(.)$ and $int_X(.)$ are used to denote closure and interior, respectively, with respect to topology $\mathcal{T}_\delta$ generated by δ in X.

2. Preliminaries

In this section, we recall some important definitions and results that will be used in subsequent sections.

Definition 1. *[10] For a nonempty set X, a Čech proximity (or basic proximity) on X is a binary relation δ on the power set of X, $\mathcal{P}(X)$, that satisfies the following axioms for all $A, B, C \in \mathcal{P}(X)$:*

(i) *If $(A, B) \in \delta$, then $(B, A) \in \delta$.*
(ii) *If $(A, B) \in \delta$, then $A \neq \phi$ and $B \neq \phi$.*
(iii) *If $A \cap B \neq \phi$, then $(A, B) \in \delta$.*
(iv) *$(A, B \cup C) \in \delta$ if and only if $(A, B) \in \delta$ or $(A, C) \in \delta$.*

The set X together with a Čech proximity δ is called a Čech proximity space (X, δ).

Definition 2. *[10] A Čech proximity space X is called separated if we have $(\{x\}, \{y\}) \in \delta$, then $x = y$ for all $x, y \in X$.*

Definition 3. *[10, 12] For $A, B, C \in \mathcal{P}(X)$, a Čech proximity δ on a set X is:*

(i) *E-proximity if $(A, B) \notin \delta$, then there is some $E \subset X$ with $(A, E) \notin \delta$ and $(X \backslash E, B) \notin \delta$.*
(ii) *L-proximity if $(A, B) \in \delta$ and $(\{b\}, C) \in \delta$ for each $b \in B$, then $(A, C) \in \delta$.*
(iii) *S-proximity if $(\{x\}, B) \in \delta$ and $(\{b\}, C) \in \delta$ for each $b \in B$, then $(x, C) \in \delta$.*

A Čech proximity space (X, δ) is called an E-proximity space (or a L-proximity space, an S-proximity space respectively) if the Čech proximity δ satisfies the E-proximity axiom (or L-proximity axiom, S-proximity axiom respectively.).

Definition 4. *[10, 12] Let (X, δ) be an S-proximity space and $\mathcal{T}$ be a topology on X. Then δ is compatible with $\mathcal{T}$ if and only if the generated topology $\mathcal{T}_\delta$ and $\mathcal{T}$ are equal.*

Definition 5. *[10] Let (X, δ) be a Čech proximity space. Then a subset V of X is said to be a δ-neighbourhood of $U \subset X$ if $(U, X \backslash V) \notin \delta$.*

Definition 6. *[10,12] Let (X,δ) and $(Y,\delta^{'})$ be two E-proximity spaces, a function $f:(X,\delta)\longrightarrow(Y,\delta^{'})$ is δ-continuous (or p-continuous) if for all $A,B\subset X$ such that $(A,B)\in\delta$, implies $(f(A),f(B))\in\delta^{'}$.*

Definition 7. *[9] Let (X,δ) be an E-proximity space. Then X is said to be δ-connected if every δ-continuous map from X to discrete proximity space is constant.*

Theorem 8. *[9] Let (X,δ) be an E-proximity space. Then the following statements are equivalent:*

(i) *X is δ-connected.*
(ii) *$(A,X\backslash A)\in\delta$ for each nonempty subset A with $A\neq X$.*
(iii) *For every δ-continuous real-valued function f, the image $f(X)$ is dense in some interval of $\mathbb{R}$.*
(iv) *If $X=A\cup B$ and $(A,B)\notin\delta$, then either $A=\phi$ or $B=\phi$.*

However, if X is not δ-connected, then by Theorem 8 (iv) we have $X=A\cup B$ with $(A,B)\notin\delta$ where $A,B\subset X$ are nonempty. Here, the pair (A,B) forms a δ-separation for X.

Definition 9. *[3] Let (X,δ) be an E-proximity space and $Y\subset X$. Then Y is δ-connected, if it is δ-connected with respect to the subspace proximity of Y.*

Definition 10. *[3] An E-proximity space X is locally δ-connected if for every point x of X and for every δ-neighborhood U of x, there exists some δ-connected δ-neighborhood V of x such that $x\in V\subset U$.*

Definition 11. *[7,10] Let (X,δ_X) and (Y,δ_Y) be S-proximity spaces. Then a map $f:X\longrightarrow Y$ is said to be S-δ-continuous if $(A,B)\notin\delta_X$ implies $(f(A),f(B))\notin\delta_Y$, for all $A,B\subset X$.*

Definition 12. *[8] Let $(X,\mathcal{T})$ be a topological space. A pair of non-empty subsets A,B of X is called $Cl-Cl$-separated (weak separated) if $Cl(A)\cap Cl(B)=\phi$. A subset U of a space X is said to be $Cl-Cl$-connected (weak connected) if U is not the union of two $Cl-Cl$-separated (weak separated) sets in X.*

Definition 13. *[4] If an E-proximity space X can be written as $X=P\cup Q$ with $(P,Q)\notin\delta$, then the pair (P,Q) is said to be a separation for X and write it as $X=P+Q$. If P contains some set A and Q contains B, then it can be written as $X=P(A)+Q(B)$.*

Definition 14. *[4] Let X be an E-proximity space. Then it is called δ-treelike if it is δ-connected, and for each pair (x,y) of distinct points in X there is a δ-connected set V such that $X\backslash V=P(x)+Q(y)$.*

3. S-δ-connectedness

In this section, we define S-δ-connectedness in S-proximity spaces and give characterizations of it.

Recall that every discrete proximity is an S-proximity and induces the discrete topology.

Definition 15. *An S-proximity space X is said to be S-δ-connected if every S-δ-continuous map from X to discrete space is constant.*

Next, we give a characterization for an S-proximity space to be S-δ-connected.

Theorem 16. *For an S-proximity space X, the following statements are equivalent:*

(i) *X is S-δ-connected.*
(ii) *$(Cl_X(A), X\backslash A) \in \delta$ for all nonempty proper subset A of X.*
(iii) *If $X = P \cup Q$ with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, then either $P = \phi$ or $Q = \phi$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. If $(Cl_X(A), X\backslash A) \notin \delta$ for some nonempty proper subset A of X, then the map $f : X \longrightarrow \{0, 1\}$ defined as $f(A) = \{0\}$ and $f(X\backslash A) = \{1\}$ is non-constant, S-δ-continuous map. Therefore, X is not S-δ-connected.

(ii) $\Rightarrow$ *(iii)*. If $X = P \cup Q$, where P, Q are nonempty subsets such that $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, then $Q = X\backslash P$. Thus, $(Cl_X(P), X\backslash P) \notin \delta$, a contradiction.

(iii) $\Rightarrow$ *(i)*. If X is not S-δ-connected, then the map $f : (X, \delta) \longrightarrow \{0, 1\}$ defined as $f(P) = \{0\}$ and $f(Q) = \{1\}$ is non-constant, surjective, S-δ-continuous map. Therefore, $X = P \cup Q$, where P, Q are nonempty subsets such that $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, a contradiction. □

Definition 17. *Let X be an S-proximity space. A pair (P, Q) of two nonempty subsets of X is said to be S-δ-separated in X if $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$.*

Every S-δ-separated sets are always δ-separated. However, converse need not be true.

Example 18. *Let $X = \mathbb{R}$ be the real line. For $P, Q \subset X$, define a binary relation δ on $\mathcal{P}(X)$ as:*

$$(P, Q) \in \delta \text{ if and only if } (\bar{P} \cap Q) \cup (P \cap \bar{Q}) \neq \phi$$

Here $\bar{P}$ and $\bar{Q}$ denote the closure of P and Q in X, respectively. Then δ is a compatible S-proximity on X which is not an L-proximity. The pair $P = (1, 2)$ and $Q = (2, 3)$ is δ-separated, but not S-δ-separated in X.

Definition 19. *Let (X, δ_X) be an S-proximity space and $U \subset X$. Then U is said to be S-δ-connected in X (that is, with respect to δ_X) if it cannot be written as the union of a pair of two S-δ-separated sets in X. If U is not S-δ-connected, then it is called S-δ-disconnected and the pair of two S-δ-separated sets is called S-δ-separation for U in X.*

By an S-δ-connected subset U of an S-proximity space (X, δ_X), we mean it is an S-δ-connected with respect to δ_X (that is, with respect to the proximity of X not subspace proximity of U).

Since every S-δ-separation for a set always forms δ-separation, therefore every δ-connected set is S-δ-connected. But converse need not be true.

Example 20. *Let $X = \mathbb{R}$ be the real line and δ be a S-proximity on X defined as in Example 18. Let $U = (1, 2) \cup (2, 3)$. Then U is S-δ-connected, but not δ-connected subset of X.*

Thus, S-δ-connectedness is different from δ-connectedness in general. Next, we know that δ-connectedness [9] of a subset U in E-proximity space (X, δ_X) is same as the δ-connectedness of U with respect to subspace proximity δ_U. But, it is not true for the case of an S-δ-connectedness. In Example 20, note that U is S-δ-connected with respect to δ_X, and with respect to δ_U, it is not S-δ-connected as $Cl_U((0, 1)) = (0, 1)$ and $Cl_U((1, 2)) = (1, 2)$ with respect to δ_U. But, if U is S-δ-connected with respect to δ_U, then it is also S-δ-connected with respect to δ_X.

Remark 21. *The notions of δ-connectedness and S-δ-connectedness are equivalent in the category of L-proximity spaces, as for every L-proximity space X, we have $(P, Q) \in \delta$ if and only if $(Cl_X(P), Cl_X(Q) \in \delta$ for all non-empty P, Q in X.*

Since every E-proximity is an L-proximity, so above remark holds for E-proximity spaces.

Recall that if for all $A, B \subset X$, $(A, B) \in \delta_1$ implies $(A, B) \in \delta_2$, then $\delta_1 > \delta_2$.

Corollary 22. *Let δ_1, δ_2 be two S-proximities on X such that $\delta_1 > \delta_2$. If X is S-δ_1-connected, then so is S-δ_2-connected.*

Theorem 23. *Let X be an S-proximity space. Suppose M is an S-δ-connected subset of X and (P, Q) be a pair of S-δ-separated sets in X such that $M \subset P \cup Q$. Then either $M \subset P$ or $M \subset Q$.*

Proof. If possible, $M \nsubseteq P$ and $M \nsubseteq Q$. M is S-δ-connected set such that $M \subset P \cup Q$. Therefore, $M = (M \cap P) \cup (M \cap Q)$. Also by hypothesis $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. If $(Cl_X(P), Q) \notin \delta$, then $(Cl_X(M \cap P), M \cap Q) \notin \delta$. On the other hand, if $(P, Cl_X(Q)) \notin \delta$, then $(Cl_X(M \cap Q), M \cap P) \notin \delta$. Thus, the pair $M \cap P$ and $M \cap Q$ forms an S-δ-separation for X. □

Theorem 24. *Let M, N are two S-δ-connected subsets of an S-proximity space X. If (M, N) is not S-δ-separated, then $M \cup N$ is S-δ-connected in X.*

Proof. Suppose (P, Q) be an S-δ-separation for $M \cup N$. Therefore, $M \cup N = P \cup Q$ where $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Since M and N are S-δ-connected. Thus, by Theorem 23, two case arises:

Case (i). If $M \subset P$ and $N \subset Q$, then $(Cl_X(M), N) \notin \delta$ or $(M, Cl_X(N)) \notin \delta$, because $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Hence, (M, N) is S-δ-separated which is a contradiction.

Case (ii). If $M \subset Q$ and $N \subset P$, then $(Cl_X(N), M) \notin \delta$ or $(N, Cl_X(M)) \notin \delta$, because $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Hence, (M, N) is S-δ-separated which is a contradiction. □

Theorem 25. *Let $\{W_j : j \in J\}$ be a nonempty family of S-δ-connected subsets of an S-proximity space X. If there exists some $j_0 \in J$ such that $(W_{j_0}, W_j) \in \delta$ for all $j \in J$, then $\bigcup_{j\in J} W_j$ is also S-δ-connected in X.*

Proof. If possible, there exists an S-δ-separation (P, Q) such that $\bigcup_{j\in J} W_j = P \cup Q$ with $(Cl(P), Q) \notin \delta$ or $(P, Cl(Q)) \notin \delta$. Therefore, $W_{j_0} \subset P \cup Q$ which implies either $W_{j_0} \subset P$ or $W_{j_0} \subset Q$. If $W_{j_0} \subset P$, then $W_j \subset P$ for all $j \in J$ because $(W_{j_0}, W_j) \in \delta$ for all $j \in J$. Thus $\bigcup_{j\in J} W_j \subset P$ so $Q = \phi$. Similarly, if $W_{j_0} \subset Q$, then $P = \phi$. Thus, $\bigcup_{j\in J} W_j$ is S-δ-connected. □

Corollary 26. *If $\{W_j : j \in J\}$ is a nonempty family of S-δ-connected subsets of an S-proximity space X and $\bigcap_{j\in J} W_j \neq \phi$, then $\bigcup_{j\in J} W_j$ is also S-δ-connected in X.*

Proof. Since $\bigcap_{j\in J} W_j \neq \phi$, therefore $(W_i, W_j) \in \delta$ for all $i, j \in J$. So for some fix $j_0 \in J$, $(W_{j_0}, W_j) \in \delta$ for all $j \in J$. Thus, by Theorem 25, $\bigcup_{j\in J} W_j$ is S-δ-connected in X. □

Corollary 27. *If Y is an S-δ-connected subset of an S-proximity space X, then every subset Z such that $Y \subset Z \subset Cl_X(Y)$ is also S-δ-connected in X.*

Proof. Note that $\{Y \cup \{z\} : z \in Z\}$ is a family of S-δ-connected sets such that Y is near to each of the set. Therefore, by Theorem 25, Z is S-δ-connected. □

Corollary 28. *If an S-proximity space X contains some S-δ-connected dense subset, then X is S-δ-connected.*

Proof. Let Y be an S-δ-connected dense subset of X. Then, $Cl_X(Y) = X$. Therefore, by Corollary 27, X is S-δ-connected. □

Lemma 29. *Let X be an S-proximity space and $\{M_i : i \in I\}$ be a nonempty family of S-δ-connected subsets of X. If M is S-δ-connected in X such that $M \cap M_i \neq \phi$ for all $i \in I$, then $M \cup (\bigcup_{i\in I} M_i)$ is also S-δ-connected in X.*

Proof. By Theorem 25, $(M, M_i) \in \delta$ for all $i \in I$. Hence the proof follows. □

Corollary 30. *In an S-proximity space X, if any two points can be joined by an S-δ-connected subset of X, then X is S-δ-connected.*

Proof. Fix a point x_0 in X and let M_x be an S-δ-connected subset of X which joins x_0 and x. By Lemma 29, $M = \{x_0\}$ and $M \cap M_x \neq \phi$ for all $x \in X$. Thus, $M \cup (\bigcup_{x\in X} M_x) = X$ is S-δ-connected. □

Theorem 31. *The S-δ-continuous image of S-δ-connected space is S-δ-connected.*

Proof. Let $f : (X, \delta) \longrightarrow (Y, \delta')$ be S-δ-continuous, surjective map and X is S-δ-connected space. It is to show that Y is also an S-δ-connected space. On contrary, suppose Y is not S-δ-connected space. So, there exists a pair (P, Q) in Y such that $Y = P \cup Q$ with $(Cl_Y(P), Q) \notin \delta'$ or $(P, Cl_Y(Q)) \notin \delta'$. If $(Cl_Y(P), Q) \notin \delta'$, then $(f^{-1}(Cl_Y(P)), f^{-1}(Q)) \notin \delta$. Since S-δ-continuity of f implies continuity with respect to $\mathcal{T}_\delta$, so $Cl_X(f^{-1}(P)) \subset f^{-1}(Cl_Y(P))$. Thus, $(Cl_X(f^{-1}(P)), f^{-1}(Q)) \notin \delta$. Hence, $(f^{-1}(P), f^{-1}(Q))$ forms an S-δ-separation for X, a contradiction. A similar case for $(P, Cl_Y(Q)) \notin \delta'$. □

As every S-δ-continuous map is continuous, so every weak connected [8] space is S-δ-connected.

Example 32. *The set of rationals $\mathbb{Q}$ is an S-δ-connected in $\mathbb{R}$, but is not weak connected.*

However, compact Hausdorff S-δ-connected space is weak connected as every continuous map with compact Hausdorff domain is S-δ-continuous. Thus, we have the following diagram of implications.

$$\begin{array}{ccc} \delta-\text{connected} & \Longleftarrow & \text{connected} \\ \Downarrow & & \Downarrow \\ S-\delta-\text{connected} & \Longleftarrow & \text{weak connected} \end{array}$$

Following example concludes that a locally δ-connected space may not be an S-δ-connected.

Example 33. *Let $\mathbb{R}$ be the real line and δ be a compatible S-proximity defined as in Example* 18. *Let $X = (-1, 0) \cup (2, 3)$. Then the pair (P, Q) where $P = (-1, 0)$ and $Q = (2, 3)$, is S-δ-separation for X. Therefore X is not S-δ-connected in $\mathbb{R}$, but it is locally δ-connected.*

An S-δ-connected space may not be locally δ-connected.

Example 34. *The closed Topologist's Sine curve $T = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$ with subspace E-proximity induced by $\mathbb{R}^2$ is S-δ-connected in $\mathbb{R}^2$, but not locally δ-connected.*

Theorem 35. *Suppose $\{(X_i, \delta_i) : i \in I\}$ be a nonempty family of S-proximity spaces. Then the product $(X, \delta) = \prod\{(X_i, \delta_i) : i \in I\}$ is S-δ-connected if and only if X_i is S-δ-connected for each $i \in I$.*

Proof. Let $\prod_{i\in I} X_i$ be S-δ-connected. Since S-δ-continuous image of S-δ-connected set is S-δ-connected, therefore X_i is S-δ-connected for each $i \in I$ as projections are S-δ-continuous, surjective maps.

Conversely, assume that each X_i is S-δ-connected. Firstly, take $I = \{1, 2\}$. Then in $X_1 \times X_2$, any two distinct points (x_1, x_2) and (y_1, y_2) can be connected by the S-δ-connected set $(X_1 \times \{x_2\}) \cup (\{y_1\} \times X_2)$. Therefore, $X_1 \times X_2$ is S-δ-connected. Using induction, it can be shown that any finite product of S-δ-connected set is S-δ-connected. Now, for an arbitrary product, choose $x_i \in X_i$ for all $i \in I$. Consider a family $\mathcal{F}$ consisting of all finite subsets of the set I and put $K_F = \prod_{i\in I} L_i$ for all $F \in \mathcal{F}$ with $L_i = X_i$ if $i \in F$ and $L_i = \{x_i\}$ if $i \notin F$. Then, $\{K_F : F \in \mathcal{F}\}$ is a family of S-δ-connected sets by induction hypothesis. Therefore, $K = \bigcup_{F\in\mathcal{F}} K_F$ is S-δ-connected as $\bigcap_{F\in\mathcal{F}} K_F \neq \phi$. Since K is dense in $\prod_{i\in I} X_i$, therefore by Corollary 28, $\prod_{i\in I} X_i$ is S-δ-connected. □

Definition 36. *For given S-proximity spaces (X,δ) and $(Y,\delta^{'})$, S-δ-continuous map $f : X \longrightarrow Y$ is said to be S-δ-monotone if for every $y \in Y$, the pullback $f^{-1}(y)$ is S-δ-connected in X.*

Definition 37. *A map $f : (X,\delta) \longrightarrow (Y,\delta^{'})$ is said to be δ_b-map if for every pair of subsets A, B of Y, the following two axioms hold:*

(i) *If $(Cl_X f^{-1}(A), f^{-1}(B)) \notin \delta$, then $(Cl_Y(A), B) \notin \delta^{'}$.*
(ii) *If $(f^{-1}(A), Cl_X f^{-1}(B)) \notin \delta$, then $(A, Cl_Y(B)) \notin \delta^{'}$.*

Following theorem shows that if a map is S-δ-monotone, surjective, δ_b-map, then inverse image of S-δ-connected set is S-δ-connected.

Theorem 38. *Let $f : (X,\delta) \longrightarrow (Y,\delta^{'})$ be a δ_b-map, S-δ-monotone, surjective map. Then for each S-δ-connected subset U of Y, $f^{-1}(U)$ is S-δ-connected in X.*

Proof. Let $f^{-1}(U)$ be not S-δ-connected. Then, $f^{-1}(U) = P \cup Q$ with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. As f is S-δ-monotone, so for each $y \in U$, $f^{-1}(y)$ is S-δ-connected. Thus, $f^{-1}(y) \subset P$ or $f^{-1}(y) \subset Q$ for all $y \in U$. Now, let us define $A = \{y \in U : f^{-1}(y) \subset P\}$ and $B = \{y \in U : f^{-1}(y) \subset Q\}$. Note that $P = f^{-1}(A)$, $Q = f^{-1}(B)$ and $U = A \cup B$. Since f is δ_b- map with $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$, therefore (A, B) forms an S-δ-separation for U. □

Definition 39. *In an S-proximity space X, a finite sequence $U_1, U_2, \cdots, U_n$ of subsets of X is called an S-δ-chain if $(Cl_X(U_i), U_{i+1}) \in \delta$ and $(U_i, Cl_X(U_{i+1})) \in \delta$ for all $i = 1, 2, \cdots, n-1$.*

A family $\mathcal{F}$ of subsets of X is said to be S-δ-chained in X if for every pair (U, V) of elements of $\mathcal{F}$, there is an S-δ-chain in $\mathcal{F}$ joining U and V.

Theorem 40. *Suppose $\{U_i\}_{i=1}^{n}$ be a finite family of S-δ-connected subsets of an S-proximity space X and forms an S-δ-chain, then $\bigcup_{i=1}^{n} U_i$ is S-δ-connected in X.*

Proof. The Proof follows by induction on n as it holds for $n = 2$ by Theorem 24. □

Theorem 41. *For an S-δ-chained family $\mathcal{F} = \{U_i : i \in I\}$ in X, if each member U_i is S-δ-connected in X, then $U = \bigcup_{i\in I} U_i$ is also S-δ-connected in X.*

Proof. Let $x, y \in U$ be arbitrary. So, there is some $i, j \in I$ such that $x \in U_i$ and $y \in U_j$. Thus by hypothesis, there is an S-δ-chain joining U_i and U_j. Therefore, by Theorem 40, union of all the members of this S-δ-chain is S-δ-connected. Thus, x and y can be joined by an S-δ-connected set. Hence, by Corollary 30, U is S-δ-connected in X. □

Definition 42. *In an S-proximity space X, a cover $\mathcal{F}$ is said to be an S-δ-cover if $(Cl_X(M), N) \in \delta$ and $(M, Cl_X(N)) \in \delta$ for $M, N \subset X$, then there is some $U \in \mathcal{F}$ such that $M \cap U \neq \phi$ and $N \cap U \neq \phi$.*

Theorem 43. *In an S-δ-connected space X, every S-δ-cover is an S-δ-chained family.*

Proof. Assume that $\mathcal{F} = \{U_i : i \in I\}$ be any S-δ-cover of X. Suppose there exist $i, j \in I$ such that U_i and U_j can not be joined by any S-δ-chain in $\mathcal{F}$. Now, consider P as the union of all the members of $\mathcal{F}$ which are joinable with U_i by some S-δ-chain $\mathcal{F}' \subset \mathcal{F}$, and Q as the union of rest of the elements of $\mathcal{F}$. Then note that $X = P \cup Q$. Now it is to show that X is not S-δ-connected, that is, $(Cl_X(P), Q) \notin \delta$ or $(P, Cl_X(Q)) \notin \delta$. Again on the contrary, let $(Cl_X(P), Q) \in \delta$ and $(P, Cl_X(Q)) \in \delta$. Then there exists $U \in \mathcal{F}$ such that $U \cap P \neq \phi$ and $U \cap Q \neq \phi$. Thus, there is some $U_m \subset P$ and $U_n \subset Q$ such that $U \cap U_m \neq \phi$ and $U \cap U_n \neq \phi$. So, U_n can be joined with U_i using an S-δ-chain $\mathcal{F}'' \subset \mathcal{F}$, which is absurd. □

Theorem 44. *Let X be an S-δ-connected, separated S-proximity space. If for some $x \in X$, $X \backslash \{x\} = P \cup Q$ where (P, Q) is S-δ-separated in X, then $(\{x\}, Cl_X(P)) \in \delta$ and $(\{y\}, Cl_X(Q)) \in \delta$.*

Proof. If $(\{x\}, Cl_X(P)) \notin \delta$, then $(\{x\}, P) \notin \delta$. Since pair (P, Q) is S-δ-separated in X and X is separated, therefore it is easy to conclude that X is not S-δ-connected, a contradiction. Similarly, conclude that $(\{y\}, Cl_X(Q)) \in \delta$. □

4. Local S-δ-connectedness

In this section, local S-δ-connectedness is defined and it's several properties are studied.

Definition 45. *The S-δ-component of a subset U in an S-proximity space X is defined as the union of all S-δ-connected subsets of X containing U and it is denoted by $C^*_\delta(U)$.*

Every δ-component is contained in some S-δ-component. Any S-δ-component being union of S-δ-connected sets with nonempty intersection is S-δ-connected. An S-δ-component being a maximal S-δ-connected set is $\mathcal{T}_\delta$-closed.

Analogously, the S-δ-component of a point x can be defined as the union of all S-δ-connected subsets of X containing x. Note that S-δ-components of any two distinct points of X are either same or δ-far sets in X.

In the next theorem, we show that the S-δ-component of product S-proximity is exactly the product of S-δ-components of each S-proximity.

Theorem 46. *Suppose $\{(X_i, \delta_i) : i \in I\}$ be a nonempty family of S-proximity spaces. Then the S-δ-component of the product $(X, \delta) = \prod\{(X_i, \delta_i) : i \in I\}$ coincides with the product $\prod\{C^*_{\delta_i}(x_i) : i \in I\}$ of each S-δ-component of the point $x_i \in X_i$.*

Proof. Let $C^*_\delta(x)$ be the S-δ-component of x in X and for each $i \in I$, $C^*_{\delta_i}(x_i)$ be the S-δ-component of x_i in X_i. Then, $\prod\{C^*_{\delta_i}(x_i) : i \in I\}$ being the product of the S-δ-connected sets is S-δ-connected. Therefore it is contained in $C^*_\delta(x)$. Conversely, for each $i \in I$, $p_i C^*_\delta(x)$ being S-δ-continuous image of S-δ-connected set is S-δ-connected. Therefore, $p_i C^*_\delta(x) \subset C^*_{\delta_i}(x_i)$ for each $i \in I$. Hence, $C^*_\delta(x) \subset \prod\{p_i C^*_\delta(x) : i \in I\} \subset \prod\{C^*_{\delta_i}(x_i) : i \in I\}$. □

Next, we show that S-δ-component is preserved under an S-δ-monotone, surjective, δ_b-map

Theorem 47. *Suppose $f : (X, \delta) \longrightarrow (Y, \delta')$ be S-δ-monotone, surjective and δ_b-map. Then C^* is an S-δ-component of $W \subset Y$ if and only if $f^{-1}(C^*)$ is an S-δ-component of $f^{-1}(W)$.*

Proof. Assume that C^* is S-δ-component of subspace $W \subset Y$. Obviously, $f^{-1}(C^*)$ is S-δ-connected by Theorem 38. Now, suppose there is some S-δ-connected set M in $f^{-1}(W)$ such that $f^{-1}(C^*) \subset M \subset f^{-1}(W)$. Since the map f is surjective, therefore $C^* \subset f(M) \subset W$. As f is S-δ-continuous being S-δ-monotone, so $f(M)$ is S-δ-connected. Thus, $f(M) = C^*$ which implies $f^{-1}(C^*) = M$.

Conversely, let $f^{-1}(C^*)$ be an S-δ-component of $f^{-1}(W)$. Therfore, $f^{-1}(C^*)$ is S-δ-connected subset of $f^{-1}(W)$ and f is S-δ-continuous being S-δ-monotone. Thus, C^* is S-δ-connected subset of W. Now, suppose that N be an S-δ-connected set such that $C^* \subset N \subset W$. Then, $f^{-1}(C^*) \subset f^{-1}(N) \subset f^{-1}(W)$ and $f^{-1}(N)$ is S-δ-connected by Theorem 38. Hence, by hypothesis, $f^{-1}(C^*) = f^{-1}(N)$ which implies $C^* = N$. □

Definition 48. *Let X be an S-proximity space. Then X is locally S-δ-connected at $x \in X$, if every δ-neighbourhood of x contains some S-δ-connected δ-neighbourhood of x. We call X is locally S-δ-connected if it is locally S-δ-connected for all $x \in X$. Further, a subset $Y \subset X$ is locally S-δ-connected if Y is locally S-δ-connected in the subspace S-proximity of X.*

Now, we show that locally S-δ-connectedness and S-δ-connectedness are two independent concepts.

Example 49. (a). *Let X be any discrete proximity space with $|X| \geq 2$. Then X is locally S-δ-connected, but it is not S-δ-connected.*

(b). *Suppose X be an S-proximity space defined as in Example* 33. *Then X is locally S-δ-connected, but not S-δ-connected.*

Example 50. *The closed Topologist's sine curve $T = \{(x, \sin(1/x)) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}$ with subspace E-proximity induced by $\mathbb{R}^2$ is S-δ-connected, but not locally S-δ-connected.*

Example 51. *The subspace $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$ with S-proximity defined as in Example* 18. *Then X is not locally S-δ-connected.*

Theorem 52. *Suppose $x \in P \cap Q$, where P and Q are locally S-δ-connected sets at x. Then $P \cup Q$ is also locally S-δ-connected at x.*

Proof. Let W be a δ-neighbourhood of the point x. Then, $W_P = W \cap P$ and $W_Q = W \cap Q$ are δ-neighbourhoods of the point x in P and Q respectively. Using hypothesis, there exist some S-δ-connected δ-neighbourhoods M_P and M_Q of x such that $M_P \subset W_P$ and $M_Q \subset W_Q$. Then, $x \in M_P \cup M_Q \subset W_P \cup W_Q$ such that $M_P \cup M_Q$ is S-δ-connected set. Also, $(\{x\}, (P \backslash M_P) \cup (Q \backslash M_Q)) \notin \delta$ which implies $(\{x\}, (P \cup Q) \backslash (M_P \cup M_Q) \notin \delta$. Therefore, $M_P \cup M_Q$ is a δ-neighbourhood of x. □

Theorem 53. *If an S-proximity space X is locally S-δ-connected, then S-δ-component of every $\mathcal{T}_\delta$-open subspace of X is $\mathcal{T}_\delta$-open.*

Proof. Assme that X is locally S-δ-connected and W be $\mathcal{T}_\delta$-open subspace in X. Let C^* be an S-δ-component of W. If $y \in C^*$, then $(\{y\}, X \backslash W) \notin \delta$. Therefore W is a δ-neighbourhood of y. Since X is locally S-δ-connected, then there exists an S-δ-connected δ-neighbourhood M of y such that $y \in M \subset W$. But C^* is maximal S-δ-connected set containing y, so $y \in M \subset C^*$. Therefore, C^* is $\mathcal{T}_\delta$-open. □

Corollary 54. *If X is locally S-δ-connected space, then S-δ-components of X are clopen sets in the induced topology $\mathcal{T}_\delta$.*

Corollary 55. *If an S-proximity space X is locally S-δ-connected and compact, then it has at most finite number of S-δ-components.*

Definition 56. *Let U be a subset of an S-proximity space X. Then it is called an S-δ-treelike in X if it is S-δ-connected and for each pair of points $x, y \in U$ there exists an S-δ-connected set $V \subset U$ in X such that $U \backslash V = P \cup Q$ where $x \in P$, $y \in Q$ and the pair (P, Q) is S-δ-separated in X.*

Example 20 shows that there exists an S-δ-treelike S-proximity space which is not δ-treelike [4], and from Example 32 we conclude that there exists an S-δ-treelike S-proximity space which is not treelike [1] (Topologically).

Theorem 57. *If an S-proximity space X is S-δ-treelike, then it is separated.*

Proof. Suppose X is not separated. So, there exist two distinct points x, y in X such that $(\{x\}, \{y\}) \in \delta$. Thus, $\{x, y\}$ is S-δ-connected in X. Since X is an S-δ-treelike space, therefore there exists an S-δ-connected set U in X such that $X \backslash U = P \cup Q$ where $x \in P$, $y \in Q$ and the pair (P, Q) is S-δ-separated in X. Then the pair $P \cap \{x, y\}$ and $Q \cap \{x, y\}$ forms an S-δ-separation for $\{x, y\}$, a contradiction. □

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