Konuralp Journal of Mathematics, 9 (2) (2021) 274-280

Konuralp Journal of Mathematics

Research Paper

https://dergipark.org.tr/en/pub/konuralpjournalmath

e-ISSN: 2147-625X

Oscillation Criteria for Higher Order Neutral Differential Equations with Several Delays

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Abstract

In this paper, we will consider a class of even order neutral differential equations with several delays and adequate conditions are obtained for the oscillation of solutions. Results of this work extended and improved the oscillatory results studied in [20] with several delays. Examples are specified to illustrate the main results.

Keywords: *Oscillation, Neutral, Higher order, Delay.*
2010 Mathematics Subject Classification: *39A10, 39A12*

1. Introduction

This study concerned with the oscillation of solutions to a class of even order neutral differential equations with several delays of the form

$$\frac{d^n}{dt^n}\left[x(t)+p(t)x(\tau(t))\right]+\sum_{j=1}^{m} q_j(t)x(\sigma_j(t))=0, \tag{1.1}$$

where $t \geq t_0$ for some positive constant t_0, m is natural number, and $n \geq 4$ is an even natural number.
We assume without further mention that

(H_1) $p, q_j \in C([t_0,\infty),\mathbb{R})$, $p(t)>0$, $q_j(t)\geq 0$, $q_j(t)$ is not identically zero for large t, $j=1,2,\ldots,m$;
(H_2) τ, $\sigma_j \in C([t_0,\infty),\mathbb{R})$; $\tau(t)\leq t$, $\sigma_j(t)\leq t$ and τ, σ_j are invertible with $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma_j(t)=\infty$; $\quad j=1,2,\ldots,m.$

Let $t_* = \min_{t\in[t_0,\infty)}\left\{\tau(t),\sigma_j(t)\right\}; \quad j=1,2,\ldots,m.$

1.1. Review of Literature

Differential equations play an innovative role in the development of science, machinery and social sciences. Differential equation is an effective tool in mathematics to describe the changes happening in every stages of nature. Numerous phenomena in these branches have mathematical models in expressions of differential equations. The consequence of differential equations lies in the plenty of the incidences and their effectiveness in understanding the sciences. While describing a physical phenomena by differential equations, that the future state of the system is determined by the present circumstances. On various incidents, the present state of a system depends on some previous history. If we consider this approach while modeling, we end up with another class of equations called delay differential equations.

Various techniques appeared for the investigation of solutions of differential equations. Once the existence of a solution for differential equation is established, the next question in the study is: How does solution behave with the growth of time? These constitute the study of asymptotic behavior of solutions of differential equations. The asymptotic behavior of differential equations has been studied by many authors with different methods, see for example [6, 7, 10, 17, 21, 23, 28], and the references cited therein. Asymptotic behavior means behavior of solutions as time variable goes to infinity. This tacitly assumes that solutions can be extended to infinity. The concept of asymptotic behavior of solutions to differential equations is rich and multifaceted, oscillatory behavior is only one of possible types of asymptotic behavior, in this case all solutions have a sequence of zeroes.

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The existence and location of the zeros of solutions of ordinary differential equations are of central significance in the theory of boundary value problems for such equations and the first essential result by Sturm [29] was the celebrated comparison theorem. Protter [26], Clark and Swanson [9] extended comparison theorems for elliptic differential equations. These results were integrated in the monograph [30]. Also this monograph contains the arguments about comparison and oscillation theorems of Reid, Levin, Nehari, Hille, Wintner, Leighton, Hartman, Kneser, Courant and Hanan. The oscillation theory of ordinary differential equations marks its commencement with the manuscript of Sturm [29] in 1836 appeared in which theorems of oscillation and comparison of the solutions of second order linear homogeneous ordinary differential equations were proved. The first oscillation result for differential equations with translated arguments were obtained by Fite [13] in 1921.

In recent years, there has been growing interest in oscillation theory of functional differential equations of neutral type [1, 2, 12, 14, 15, 33, 34] and the references cited therein. To the best of the authors' knowledge, the first step toward a systematic investigation of the second kind was taken by Ruan [27] who studied the existence of non-oscillatory solutions of second-order equations of the form (1.1). Our study here is related to the recent work of T. Li and Y. V. Rogovchenko [20] which is one of the first attempts in a systematic investigation of oscillatory properties of higher order neutral equations. Recently, many results on oscillation of non-neutral differential equations and neutral functional differential equations have been established. Philos [24] established some Philos-type oscillation criteria for a second-order linear differential equation. In [11, 16, 18, 31], the authors gave some sufficient conditions for oscillation of all solutions of second-order half-linear differential equations. Distinctively, Baculíková and Džurina [3] presented some sufficient conditions for oscillation of the second-order differential equations with mixed arguments.

It is known that determination of the signs of the derivatives of the solution is necessary and causes a significant effect before studying the oscillation of delay differential equations. The other essential thing is to establish relationships between derivatives of different orders, which may lead to additional restrictions during the study. In odd-order differential equations, in some cases, it is difficult to find relationships between derivatives of different orders. Therefore, it can very easily be observed that differential equations of odd-order received less attention than differential equations with even-order. The theoretical background of the second order and even order equations are nearly common and in this direction, we can study oscillatory behavior of even order equations.

1.2. Outline of the study

This paper is organized as follows: In Section 1.3, we present the definitions, notations and results that will be needed. In Section 2, we discuss the main results of the problem (1.1). In Section 3, we present two examples to illustrate the main results. The results obtained in this paper expand and improve the results studied in [20]. Finally Section 4, provides conclusion of the present work.

1.3. Preliminaries

In this section, we begin with definitions, notations and well known results which are required throughout this paper.

By a solution of (1.1), we mean a function $x \in C([t_*,\infty),\mathbb{R})$ such that $z \in C^n([t_0,\infty),\mathbb{R})$ and $x(t)$ satisfies (1.1) on $[t_0,\infty)$. In what follows, we suppose that solutions of (1.1) exist and continuable to infinity to the right. Furthermore, we consider only solutions $x(t)$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$ and we assume that Eq. (1.1) possesses such solution. A solution $x(t)$ of (1.1) is said to be oscillatory if it has no last zero, i.e., if $x(t_1) = 0$, then there exist an $t_2 > t_1$ such that $x(t_2) = 0$. Eq. (1.1) itself is said to be oscillatory if every solution of (1.1) is oscillatory. A solution $x(t)$ which is not oscillatory is called non-oscillatory.

Numerous interesting oscillation criteria for the equations of the type (1.1) studied in [4, 5, 19, 32] have been reported in the recent papers under the assumptions that

$$\begin{aligned} &0 \leq p(t) \leq p_0 < \infty, \quad \tau\circ\sigma_j = \sigma_j\circ\tau, \qquad j = 1,2,\dots,m \\ &\tau \in C^1([t_0,\infty),\mathbb{R}) \quad \text{and} \quad \tau'(t) \geq \tau_0 > 0. \end{aligned} \tag{1.2}$$

For our further reference, we denote and assume that τ^{-1} and σ_j^{-1} for the inverse functions of τ and σ_j along with $f_+(t) = \max\{0, f(t)\}$, $Q(t) = \min\{q_j(t), q_j(\tau(t))\}$, $\bar{Q}(t) = \min\left\{q_j(\sigma_j^{-1}(t)), q_j(\sigma_j^{-1}(\tau(t)))\right\}$, $j = 1,2,\dots,m$, and that

$$\begin{aligned} &I_1(t) = t - t_1; \quad t_1 > t_0, \qquad I_i(t) = \int_{t_1}^{t} I_{i-1}(s)\mathrm{d}s; \quad i = 2,3,\dots,n-1, \\ &J_2^*(t) = \int_t^\infty \tau'(u)\int_u^\infty Q(s)\mathrm{d}s\mathrm{d}u, \quad J_i^*(t) = \int_t^\infty \tau'(s)J_{i-1}^*(s)\mathrm{d}s;\ i = 3,4,\dots,n, \\ &Q_{n-1}^*(t) = Q(t)I_{n-1}(\sigma_j(t)), \\ &Q_i^*(t) = \frac{1}{i!}(\tau^{-1}(\sigma_j(t)) - t_1)^i J_{n-1-i}^*(\tau^{-1}(t)); \quad i = 1,2,\dots,n-3, \\ &p_*(t) = \frac{1}{p(\tau^{-1}(t))}\left[1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{(n-1)/l_\star}}{(\tau^{-1}(t))^{(n-1)/l_\star}p(\tau^{-1}(\tau^{-1}(t)))}\right] \end{aligned}$$

and

$$p^*(t) = \frac{1}{p(\tau^{-1}(t))}\left[1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{1/l_\star}}{(\tau^{-1}(t))^{1/l_\star}p(\tau^{-1}(\tau^{-1}(t)))}\right].$$

Theorem 1.1. *([1, Theorem 2.2]) Let conditions (H_1) and (H_2) be satisfied. Suppose further that $\sigma(t) \leq \tau(t)$,*

$$\tau \in C^1([t_0,\infty),\mathbb{R}), \quad \tau'(t) > 0 \quad \text{and} \quad 1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{n-1}}{(\tau^{-1}(t))^{n-1} p(\tau^{-1}(\tau^{-1}(t)))} \geq 0. \tag{1.3}$$

If there exist functions ρ, $\delta \in C^1([t_0,\infty),(0,\infty))$ such that

$$\int^{\infty} \left[\rho(t)q(t)p_*(\sigma(t)) \frac{(\tau^{-1}(\sigma(t)))^{n-1}}{t^{n-1}} - \frac{(n-2)!}{4\lambda_0} \frac{(\rho'_+(t))^2}{t^{n-2}\rho(t)} \right] \mathrm{d}t = \infty$$

for some $\lambda_0 \in (0,1)$ and

$$\int^{\infty} \left[\frac{\delta(t)}{(n-3)!} \int_t^{\infty} (\eta - t)^{n-3} q(\eta) p^*(\sigma(\eta)) \frac{\tau^{-1}(\sigma(\eta))}{\eta} \mathrm{d}\eta - \frac{(\delta'_+(t))^2}{4\delta(t)} \right] \mathrm{d}t = \infty,$$

then Eq. (1.1) is oscillatory.

Theorem 1.2. *([5, Corollary 2.8], [32, Corollary 2.14]) Assume that $0 \leq p(t) \leq p_0 < \infty$, $\tau \in C^1([t_0,\infty),\mathbb{R})$, $\tau'(t) \geq \tau_0 > 0$, and that the assumptions (H_1) and (H_2) hold. If σ is invertible with $\sigma^{-1} \in C^1([t_0,\infty),\mathbb{R})$, $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\sigma(t) < \tau(t)$ and*

$$\frac{\tau_0\sigma_0}{(\tau_0 + p_0)(n-1)!} \liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \bar{Q}(s)s^{n-1}\mathrm{d}s > \frac{1}{e},$$

then Eq. (1.1) is oscillatory.

Theorem 1.3. *([4, Corollary 2]) Suppose that $J_n^*(t_0) = \infty$ and the assumptions (H_1), (H_2) and (1.2) hold. If $\sigma(t) < \tau(t)$ and*

$$\frac{\tau_0}{\tau_0 + p_0} \liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q_i^*(s)\mathrm{d}s > \frac{1}{e}$$

for $i = 1,2,\ldots,n-1$, then Eq. (1.1) is oscillatory.

The objective of this paper is to derive a new oscillation criteria for Eq. (1.1) that improve Theorems 1.1-1.3. In the sequel, all functional inequalities are supposed to hold for all t large enough. Without loss of generality, we deal only with positive solutions of Eq. (1.1), since under our assumptions, if $x(t)$ is a solution, so is $-x(t)$.

2. Main Results

We need the following lemmas to prove our results.

Lemma 2.1. *(Philos [25]). Let $f \in C^n([t_0,\infty),(0,\infty))$. If the derivative $f^{(n)}(t)$ is eventually of one sign for large t, then there exist a $t_x \geq t_0$ and an integer l, $0 \leq l \leq n$ with $n+l$ even for $f^{(n)}(t) \geq 0$, or $n+l$ odd for $f^{(n)}(t) \leq 0$ such that*
$l > 0$ yields $f^{(k)}(t) > 0$; $t \geq t_x$, $k = 0,1,\ldots,l-1$ and
$l \leq n-1$ yields $(-1)^{l+k} f^{(k)}(t) > 0$; $t \geq t_x$, $k = l, l+1,\ldots,n-1$.

Lemma 2.2. *([2], Lemma 2.2.3) Let $f \in C^n([t_0,\infty),(0,\infty))$, $f^{(n)}(t)f^{(n-1)}(t) \leq 0$ for $t \geq t^*$, and assume that $\lim_{t\to\infty} f(t) \neq 0$. Then for every $\lambda \in (0,1)$ there exists a $t_\lambda \in [t_*,\infty)$ such that*

$$f(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}(t)|$$

for all $t \in [t_\lambda,\infty)$.

Lemma 2.3. *([8], Lemma 1) Suppose that the function h satisfies $h^{(i)}(t) > 0$, $i = 0,1,\ldots,k$, and $h^{(k+1)}(t) \leq 0$ eventually. Then for every $l_\star \in (0,1)$,*

$$\frac{h(t)}{h'(t)} \geq \frac{l_\star t}{k}$$

for t large enough.

Lemma 2.4. *Suppose that $x(t)$ is an eventually positive solution of (1.1), let*

$$z(t) = x(t) + p(t)x(\tau(t)).$$

Then there exists a number $t_1 \geq t_0$ such that

$$z(t) > 0, \; z'(t) > 0, \; z^{(n-1)}(t) > 0 \; \text{and} \; z^{(n)}(t) \leq 0, \; t \geq t_1.$$

Proof. The proof is subsequent from Lemma 2.3 in [22] with the application of Lemma 2.1 in the present paper. ☐

The following is the main result of the paper.

Theorem 2.5. *Let conditions (H_1), (H_2) and (1.3) be satisfied. Suppose that there exist functions $\eta_j \in C([t_0,\infty),\mathbb{R})$ and $\xi_j \in C^1([t_0,\infty),\mathbb{R})$, satisfying $\eta_j(t) \le \sigma_j(t)$, $\eta_j(t) < \tau(t)$, $\xi_j(t) \le \sigma_j(t)$, $\xi_j(t) < \tau(t)$, $\xi'{}_j(t) \ge 0$ and $\lim_{t\to\infty}\eta_j(t) = \lim_{t\to\infty}\xi_j(t) = \infty$; $j = 1,2,\ldots,m$. If there exists a constant $\lambda_0 \in (0,1)$ such that the following two first order delay differential equations*

$$y'(t) + \frac{\lambda_0}{(n-1)!}\sum_{j=1}^{m} q_j(t)p_*(\sigma_j(t))(\tau^{-1}(\eta_j(t)))^{n-1}y(\tau^{-1}(\eta_j(t))) = 0 \tag{2.1}$$

and

$$w'(t) + \frac{1}{(n-3)!}\left(\int_t^{\infty}(s-t)^{n-3}\sum_{j=1}^{m} q_j(s)p^*(\sigma_j(s))\mathrm{d}s\right)\tau^{-1}(\xi_j(t))w(\tau^{-1}(\xi_j(t))) = 0 \tag{2.2}$$

are oscillatory, then Eq. (1.1) is oscillatory.

Proof. Assume that Eq. (1.1) is not oscillatory and $x(t)$ be a non-oscillatory solution of it. Without loss of generality, we may assume that $x(t)$ is eventually positive. It follows from (1.1) that

$$z^n(t) = -\sum_{j=1}^{m} q_j(t)x(\sigma_j(t)) \le 0. \tag{2.3}$$

Then, using Lemma 2.4, we conclude that there are two possible cases for the behavior of z and its derivatives for large t:

Case (i) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, $z^{(n-1)}(t) > 0$, $z^{(n)}(t) \le 0$: Then we have

$$\lim_{t\to\infty} z(t) \ne 0,$$

and, by virtue of Lemma 2.2, for every $\lambda \in (0,1)$ and for a large t, we have

$$z(t) \ge \frac{\lambda}{(n-1)!}t^{n-1}z^{(n-1)}(t). \tag{2.4}$$

It follows from the definition of $z(t)$ that

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\left[\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))} - \frac{x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right] \\ &\ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{1}{p(\tau^{-1}(t))}\frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}. \end{aligned} \tag{2.5}$$

Then, by Lemma 2.3,

$$\frac{z(t)}{z'(t)} \ge \frac{l_\star t}{n-1},$$

and we deduce that

$$\left(\frac{z(t)}{t^{(n-1)/l_\star}}\right)' = \frac{tz'(t) - ((n-1)/l_\star)z(t)}{t^{((n-1)/l_\star)+1}} \le 0. \tag{2.6}$$

Hence, $z(t)/t^{(n-1)/l_\star}$ is nonincreasing for sufficiently large t.
Using the condition $\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t))$ and (2.6), we conclude that

$$z(\tau^{-1}(\tau^{-1}(t))) \le \frac{(\tau^{-1}(\tau^{-1}(t)))^{(n-1)/l_\star}}{(\tau^{-1}(t))^{(n-1)/l_\star}}z(\tau^{-1}(t)). \tag{2.7}$$

Using (2.7) and (2.5) we obtain

$$x(t) \ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - \frac{(\tau^{-1}(\tau^{-1}(t)))^{(n-1)/l_\star}}{(\tau^{-1}(t))^{(n-1)/l_\star}p(\tau^{-1}(\tau^{-1}(t)))}\right] = p_*(t)z(\tau^{-1}(t)). \tag{2.8}$$

Subsequently, substituting (2.8) into (2.3) yields

$$z^{(n)}(t) + \sum_{j=1}^{m} q_j(t)p_*(\sigma_j(t))z(\tau^{-1}(\sigma_j(t))) \le 0.$$

Using conditions $\eta_j(t) \le \sigma_j(t)$ and $z'(t) > 0$, we conclude that

$$z^{(n)}(t) + \sum_{j=1}^{m} q_j(t)p_*(\sigma_j(t))z(\tau^{-1}(\eta_j(t))) \le 0. \tag{2.9}$$

It follows from (2.4) and (2.9) that for all $\lambda \in (0,1)$,

$$z^{(n)}(t)+\frac{\lambda}{(n-1)!}\sum_{j=1}^{m}q_j(t)p_*(\sigma_j(t))(\tau^{-1}(\eta_j(t)))^{n-1}z^{(n-1)}(\tau^{-1}(\eta_j(t)))\leq 0.$$

Now, introduce the function $y(t)=z^{(n-1)}(t)$. Clearly, $y(t)$ is a positive solution of the first-order delay differential inequality

$$y'(t)+\frac{\lambda}{(n-1)!}\sum_{j=1}^{m}q_j(t)p_*(\sigma_j(t))(\tau^{-1}(\eta_j(t)))^{n-1}y(\tau^{-1}(\eta_j(t)))\leq 0. \tag{2.10}$$

It follows from [24, Theorem 1] that the associated with (2.10), Eq. (2.1) also has a positive solution for all $\lambda_0 \in (0,1)$, but this contradicts our assumption on Eq. (2.1).

Case (ii) $z(t)>0$, $z^{(k)}(t)>0$, $z^{(k+1)}(t)<0$ for all odd $k\in\{1,2,3,\dots,n-3\}$, $z^{(n-1)}(t)>0$ and $z^{(n)}(t)\leq 0$: By the definition of $z(t)$, (2.5) holds. When $k=1$, we deduce that, for every $l_\star\in(0,1)$, it follows from conditions $z(t)>0$, $z'(t)>0$, $z''(t)<0$ and Lemma 2.3 that

$$z(t)\geq l_\star t z'(t), \tag{2.11}$$

and hence

$$\left(\frac{z(t)}{t^{1/l_\star}}\right)'=\frac{tz'(t)-(1/l_\star)z(t)}{t^{(1/l_\star)+1}}\leq 0 \tag{2.12}$$

eventually. By virtue of condition $\tau^{-1}(t)\leq\tau^{-1}(\tau^{-1}(t))$ and (2.12), we deduce that

$$z(\tau^{-1}(\tau^{-1}(t)))\leq\frac{[\tau^{-1}(\tau^{-1}(t))]^{1/l_\star}}{[\tau^{-1}(t)]^{1/l_\star}}z(\tau^{-1}(t)). \tag{2.13}$$

Substitution of (2.13) into (2.5) yields

$$x(t)\geq\frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1-\frac{(\tau^{-1}(\tau^{-1}(t)))^{1/l_\star}}{(\tau^{-1}(t))^{1/l_\star}p(\tau^{-1}(\tau^{-1}(t)))}\right]=p^*(t)z(\tau^{-1}(t)). \tag{2.14}$$

Substituting now (2.14) into (2.3), we obtain

$$z^{(n)}(t)+\sum_{j=1}^{m}q_j(t)p^*(\sigma_j(t))z(\tau^{-1}(\sigma_j(t)))\leq 0.$$

Since $\xi_j(t)\leq\sigma_j(t)$ and $z'(t)>0$, we also have

$$z^{(n)}(t)+\sum_{j=1}^{m}q_j(t)p^*(\sigma_j(t))z(\tau^{-1}(\xi_j(t)))\leq 0. \tag{2.15}$$

Integrating (2.15) from t to ∞ consecutively $n-2$ times, we deduce that

$$z''(t)+\frac{1}{(n-3)!}\left(\int_t^\infty(s-t)^{n-3}\sum_{j=1}^{m}q_j(s)p^*(\sigma_j(s))\mathrm{d}s\right)z(\tau^{-1}(\xi_j(t)))\leq 0. \tag{2.16}$$

Letting $w(t)=z'(t)$ and using (2.11) in (2.16), we conclude that $w(t)$ is a positive solution of a first-order delay differential inequality

$$w'(t)+\frac{1}{(n-3)!}\left(\int_t^\infty(s-t)^{n-3}\sum_{j=1}^{m}q_j(s)p^*(\sigma_j(s))\mathrm{d}s\right)\tau^{-1}(\xi_j(t))w(\tau^{-1}(\xi_j(t)))\leq 0. \tag{2.17}$$

It follows from [24, Theorem 1] that the associated with (2.17) delay differential Eq. (2.2) also has a positive solution, which again contradicts our assumption on Eq. (2.2). Therefore, Eq. (2.1) is oscillatory.

□

Theorem 2.5 and the oscillation criterion reported by Baculikova and Džurina [3, Lemma 4] imply the following result.

Corollary 2.6. *Let conditions (H_1), (H_2) and (1.3) be satisfied. Suppose that there exist functions η_j and ξ_j, $j=1,2,\dots,m$, as in Theorem 2.5. If*

$$\frac{1}{(n-1)!}\liminf_{t\to\infty}\sum_{j=1}^{m}\int_{\tau^{-1}(\eta_j(t))}^{t}q_j(s)p_*(\sigma_j(s))(\tau^{-1}(\eta_j(s))^{n-1}\mathrm{d}s>\frac{1}{e} \tag{2.18}$$

and

$$\frac{1}{(n-3)!}\liminf_{t\to\infty}\sum_{j=1}^{m}\int_{\tau^{-1}(\xi_j(t))}^{t}\int_s^\infty(u-s)^{n-3}q_j(u)p^*(\sigma_j(u)\mathrm{d}u(\tau^{-1}(\xi_j(s))\mathrm{d}s>\frac{1}{e}, \tag{2.19}$$

then Eq. (1.1) is oscillatory.

Proof. Applying (2.18), (2.19) and the Lemma 4 in [3], we conclude that (2.1) and (2.2) are oscillatory. Hence by Theorem 2.5, Eq. (1.1) is oscillatory. □

3. Examples and Remarks

The following examples illustrate theoretical results obtained in the previous section. Throughout this section we assume that $t \geq 1$ and $q_0 > 0$.

Example 3.1. Consider the fourth-order neutral delay differential equation

$$[x(t)+12x(t/2)]^{(4)}+\frac{q_0}{t^4}x(t/6)+\frac{q_0}{t^4}x(t/8)=0. \tag{3.1}$$

Let $\eta_1(t)=\xi_1(t)=t/6$ and $\eta_2(t)=\xi_2(t)=t/8$. Application of Corollary 2.6 yields that Eq. (3.1) is oscillatory provided that

$$q_0 > \frac{216}{\left(\frac{1}{27}\ln 3+\frac{1}{64}\ln 4\right)e} \approx 2169.336013$$

Remark 3.2. The oscillation criteria for Eq. (3.1) improves some known theorems. Indeed, Theorem 2.4 in [19] cannot be applied to Eq. (3.1) since $t/6$ and $t/8$ are both less than $t/2$ for $t \geq 1$. Let $p_0 = 12$, $\tau_0 = 1/2$ and $\sigma_0 = 6$. Theorem 2.5 ensures the oscillation of Eq. (3.1) for $q_0 > 102400/(e\ln 12) \approx 15159.86718$. Therefore this criterion provides a sharper estimate.

Example 3.3. Consider the fourth-order neutral differential equation

$$[x(t)+12x(t/2)]^{(4)}+\frac{q_0}{t^4}x(5t/6)+\frac{q_0}{t^4}x(7t/8)=0. \tag{3.2}$$

Note that $5t/6$ and $7t/8$ are both greater than $t/2$ on $[1,\infty)$. Observe that in this case Theorem 1.1-1.3 cannot be applied to Eq. (3.2). Let $\eta_1(t)=\xi_1(t)=t/3$ and $\eta_2(t)=\xi_2(t)=t/4$. By Corollary 2.6, Eq. (3.2) is oscillatory for

$$q_0 > \frac{216}{\left(\frac{8}{27}\ln(3/2)+\frac{1}{8}\ln 2\right)e} \approx 384.280372.$$

Remark 3.4. For a class of even-order neutral functional differential equations with several delays, i.e., for Eq. (1.1), we derived two new oscillation results which complement and improve those obtained by Agarwal et al. [1], Baculíková and Džurina [4], Baculíková et al. [3], Liu et al. [19], and Xing et al. [32]. A distinguishing feature of our criteria is that we do not impose specific restrictions on the deviating argument σ_j, that is, σ_j can be delayed, advanced and even change back and forth from advanced to delayed for $j = 1,\dots,m$.

4. Conclusion

This work was improved oscillation criteria studied in [20] with several delays. Lemma 2.3 provides the sharp estimate for the values of $p^*(t)$, $p_*(t)$ and improve the results of main Theorem 2.5. Finally, in Section 3, Couple of fourth order differential equations with several delays discussed and these examples demonstrated the main results of this study.

Acknowledgement

The author would like to thank the referees for the constructive remarks which significantly enhanced the paper.

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