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HYPERSURFACE FAMILIES WITH SMARANDACHE CURVES IN GALILEAN 4-SPACE

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Abstract. In this paper, we study the hypersurface families with Smarandache curves in 4-dimensional Galilean space G_4 and give the conditions for different Smarandache curves to be parameter and the curve which generates the Smarandache curves is geodesic on a hypersurface in G_4. Also, we investigate three types of marching-scale functions for one of these hypersurfaces and construct an example for it.

1. PRELIMINARIES

In physics, geodesics which are defined as a parallel transport of a tangent vector in a linear (affine) connection on the manifold M are very important for general relativity. Because, the geodesic equation which is given with a set of initial conditions is very useful in theoretical foundations of general relativity. Also, in general, relativity gravity can be regarded as not a force but a consequence of a curved spacetime geometry where the source of curvature is the stress–energy tensor. For example, the path of a planet orbiting a star is the projection of a geodesic of the curved four-dimensional spacetime geometry around the star onto three-dimensional space.

Furthermore, as an alternative definition of a geodesic line can be defined as the shortest curve connecting two points on a manifold. A curve $\gamma(\nu)$ on a hypersurface $\varphi(\nu,\mu,\sigma)$ is geodesic iff the normal $N(\nu)$ of the curve $\gamma(\nu)$ and the normal $\eta(\nu,\mu_0,\sigma_0)$ of the hypersurface $\varphi(\nu,\mu,\sigma)$ at any point on the curve $\gamma(\nu)$ are parallel to each other and a curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ is asymptotic iff

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the normal $N(\nu)$ of the curve $\gamma(\nu)$ and the normal $\eta(\nu, \mu_0, \sigma_0)$ of the hypersurface $\varphi(\nu, \mu, \sigma)$ at any point on the curve $\gamma(\nu)$ are perpendicular.

The problem of constructing a family of surfaces from a given spatial geodesic curve firstly has been studied by Wang et al. in 2004 and in that study, the authors have derived a parametric representation for a surface pencil whose members share the same geodesic curve as an isoparametric curve [15]. After this study, in 2008 the generalization of the Wangs' assumption to more general marching-scale functions has been given by Kasap et al [7]. By using these studies, the problem of finding a surface pencil from a given spatial asymptotic curve has been investigated in [4] and the necessary and sufficient condition for the given curve to be the asymptotic curve for the parametric surface has been stated in [1]. Also, the problem of finding a hypersurface family from a given asymptotic curve in R^4 has been handled in [5].

Surfaces with common geodesic and family of surface with a common null geodesic in Minkowski 3-space have been studied in [8] and [13], respectively.

The Galilean space G_3 is a Cayley-Klein space equipped with the metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean space consists of an ordered triple $\{\omega, f, I\}$ in which ω is the ideal (absolute) plane, f is the line (absolute line) in ω and I is the fixed elliptic involution of f.

In the Galilean n-space, there are just two types of vectors. A vector $u = (u_1, u_2, ..., u_n)$ is said to be non-isotropic, if $u_1 \neq 0$ and it is said to be isotropic otherwise.

If $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ are three vectors in Galilean space G_4, then the Galilean scalar product of x and y is given by

$$\langle u, v\rangle = \begin{cases} u_1v_1, \text{ if } u_1 \neq 0 \text{ or } v_1 \neq 0 \\ u_2v_2 + u_3v_3 + u_4v_4, \text{ if } u_1 = 0 \text{ and } v_1 = 0 \end{cases} \tag{1}$$

and for $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$, the Galilean cross product of u, v and w is defined by

$$u \times v \times w = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}. \tag{2}$$

Let γ be an admissible curve of the class C^∞ in G_4, parameterized by the invariant arc-length parameter ν, given by

$$\gamma(\nu) = (\nu, f(\nu), g(\nu), h(\nu)). \tag{3}$$

Then the Frenet frame is

$$\begin{aligned} T(\nu) &= \gamma'(\nu) = (1, f'(\nu), g'(\nu), h'(\nu)), \\ N(\nu) &= \frac{\gamma''(\nu)}{\kappa_1(\nu)} = \frac{1}{\kappa_1(\nu)}(0, f''(\nu), g''(\nu), h''(\nu)), \\ B_1(\nu) &= \frac{1}{\kappa_2(\nu)}\left(0, (\frac{f''(\nu)}{\kappa_1(\nu)})', (\frac{g''(\nu)}{\kappa_1(\nu)})', (\frac{h''(\nu)}{\kappa_1(\nu)})'\right), \\ B_2(\nu) &= \pm T(\nu) \times N(\nu) \times B_1(\nu) \end{aligned} \tag{4}$$

and the first, second and third curvatures of the curve $\gamma(\nu)$ are given by

$$\begin{array}{l} \kappa_1(\nu) = \sqrt{f''(\nu)^2 + g''(\nu)^2 + h''(\nu)^2}, \\ \kappa_2(\nu) = \sqrt{\langle N'(\nu), N'(\nu)\rangle}, \\ \kappa_3(\nu) = \langle B_1'(\nu), B_2(\nu)\rangle\,, \end{array} \tag{5}$$

respectively and where T, N, B_1 and B_2 are called tangent vector, principal normal vector, first binormal vector and second binormal vector of $\gamma(\nu)$. We must note that, throughout this study, we will assume that $\kappa_1(\nu) \neq 0$ and $\kappa_2(\nu) \neq 0$ at everywhere.

Also, Frenet formulas are given by

$$\begin{array}{l} T'(\nu) = \kappa_1(\nu) N(\nu), \\ N'(\nu) = \kappa_2(\nu) B_1(\nu), \\ B_1'(\nu) = -\kappa_2(\nu) N(\nu) + \kappa_3(\nu) B_2(\nu), \\ B_2'(\nu) = -\kappa_3(\nu) B_1(\nu). \end{array} \tag{6}$$

The equation of a hypersurface in G_4 can be given by the parametrization

$$\varphi(\nu,\mu,\sigma) = \left((\varphi(\nu,\mu,\sigma))_1\,, (\varphi(\nu,\mu,\sigma))_2\,, (\varphi(\nu,\mu,\sigma))_3\,, (\varphi(\nu,\mu,\sigma))_4\right), \tag{7}$$

where $(\varphi(\nu,\mu,\sigma))_i \in C^3$, $i = 1,2,3,4$. The normal of this hypersurface is calculated as follows

$$\eta(\nu,\mu,\sigma) = \varphi_\nu \times \varphi_\mu \times \varphi_\sigma, \tag{8}$$

where $\varphi_i = \frac{\partial \varphi(\nu,\mu,\sigma)}{\partial i}$, $i \in \{\nu,\mu,\sigma\}$.

For more information about 4-dimensional Galilean space, we refer to [6], [11], [12], [16], [17] and etc.

If $\gamma(\nu)$ is a isoparametric curve on the hypersurface $\varphi(\nu,\mu,\sigma)$, then there exists a pair of parameters $\mu_0 \in [T_1, T_2]$ and $\sigma_0 \in [M_1, M_2]$, such that $\gamma(\nu) = \varphi(\nu,\mu_0,\sigma_0)$.

If the curve is both an asymptotic and parameter (isoparametric) curve on φ, then it is called isoasymptotic on the hypersurface φ. Similarly, if the curve is both a geodesic and parameter (isoparametric) curve on the hypersurface φ, then it is called isogeodesic on the hypersurface φ.

On the constructions of surface families with common geodesic and asymptotic curves in Galilean Space G_3 and an approach for hypersurface family with common geodesic curve in the Galilean Space G_4 have been handled in [9], [10] and [11], respectively.

On the other hand, the geometry of Smarandache curves has been very popular topic for differential geometers, recently. Let $\gamma(\nu)$ be an admissible curve in G_4 and $\{T, N, B_1, B_2\}$ be its moving Frenet frame. Then TN, TB_1, TB_2, NB_1, NB_2, B_1B_2, TNB_1, TNB_2, TB_1B_2, NB_1B_2 and TNB_1B_2-Smarandache curves are defined by $r_{TN} = \frac{T+N}{\|T+N\|}$, $r_{TB_1} = \frac{T+B_1}{\|T+B_1\|}$, $r_{TB_2} = \frac{T+B_2}{\|T+B_2\|}$, $r_{NB_1} = \frac{N+B_1}{\|N+B_1\|}$, $r_{NB_2} = \frac{N+B_2}{\|N+B_2\|}$, $r_{B_1B_2} = \frac{B_1+B_2}{\|B_1+B_2\|}$, $r_{TNB_1} = \frac{T+N+B_1}{\|T+N+B_1\|}$, $r_{TNB_2} = \frac{T+N+B_2}{\|T+N+B_2\|}$, $r_{TB_1B_2} = \frac{T+B_1+B_2}{\|T+B_1+B_2\|}$, $r_{NB_1B_2} = \frac{N+B_1+B_2}{\|N+B_1+B_2\|}$, $r_{TNB_1B_2} = \frac{T+N+B_1+B_2}{\|T+N+B_1+B_2\|}$, respectively.

The problem of constructing a family of surfaces from a given some special Smarandache asymptotic curves in Euclidean 3-space has been analyzed in [14] and surfaces using Smarandache asymptotic curves in Galilean space have been studied in [2].

In the present study, we investigate the hypersurface families with Smarandache curves in 4-dimensional Galilean space G_4.

2. HYPERSURFACE FAMILIES WITH SMARANDACHE CURVES IN 4-DIMENSIONAL GALILEAN SPACE G_4

Let $\varphi(\nu,\mu,\sigma)$ be a parametric hypersurface which is defined by a given curve $\gamma(\nu)$ as follows

$$\varphi(\nu,\mu,\sigma)=\gamma(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right], \tag{9}$$

where $L_1 \leq \nu \leq L_2$, $T_1 \leq \mu \leq T_2$ and $M_1 \leq \sigma \leq M_2$. Also, $x(\nu,\mu,\sigma)$, $y(\nu,\mu,\sigma)$, $z(\nu,\mu,\sigma)$ and $m(\nu,\mu,\sigma)$ which are the values of the marching-scale functions indicate and the values of these functions are C^1-functions and $\{T,N,B_1,B_2\}$ is the Frenet frame associated with the curve γ in G_4.

Throughout this study, for simplicity, we will denote $x(\nu,\mu,\sigma)=x$, $x(\nu,\mu_0,\sigma_0) = x_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial \nu}=x_\nu$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial \mu}=x_\mu$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial \sigma}=x_\sigma$ and $\frac{\partial x(\nu,\mu,\sigma)}{\partial \nu}|_{(\nu,\mu_0,\sigma_0)}=(x_v)_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial \mu}|_{(\nu,\mu_0,\sigma_0)}=(x_\mu)_0$, $\frac{\partial x(\nu,\mu,\sigma)}{\partial v}|_{(\nu,\mu_0,\sigma_0)}=(x_\sigma)_0$. Similar abbreviations for $y(\nu,\mu,\sigma)$, $z(\nu,\mu,\sigma)$ and $m(\nu,\mu,\sigma)$ will be used, too.

CASE 1.

In this case, by taking the TN-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TN}(\nu,\mu,\sigma)$ which is given with the aid of the TN-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TN}(\nu,\mu,\sigma)=r_{TN}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right]. \tag{10}$$

From (10), we have

$$\begin{aligned} (\varphi_{TN})_\nu &= x_\nu T+(\kappa_1+x\kappa_1+y_\nu-z\kappa_2)N \\ &\quad +(\kappa_2+y\kappa_2+z_\nu-m\kappa_3)B_1+(m_\nu+z\kappa_3)B_2, \\ (\varphi_{TN})_\mu &= x_\mu T+y_\mu N+z_\mu B_1+m_\mu B_2, \\ (\varphi_{TN})_\sigma &= x_\sigma T+y_\sigma N+z_\sigma B_1+m_\sigma B_2, \end{aligned} \tag{11}$$

where we denote $\frac{\partial \varphi_{TN}(\nu,\mu,\sigma)}{\partial \nu}=(\varphi_{TN})_\nu$, $\frac{\partial \varphi_{TN}(\nu,\mu,\sigma)}{\partial \mu}=(\varphi_{TN})_\mu$, $\frac{\partial \varphi_{TN}(\nu,\mu,\sigma)}{\partial \sigma}=(\varphi_{TN})_\sigma$.

Thus, if we use (11) in (8), by obtaining the normal of the hypersurface (10), we can state the following theorem:

Theorem 1. *$\gamma(\nu)$ is not a geodesic curve where TN-Smarandache curve r_{TN} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4.*

Proof. If TN-Smarandache curve is a isoparametric curve on $\varphi_{TN}(\nu,\mu,\sigma)$, then there exists a pair of parameters $\mu=\mu_0$ and $\sigma=\sigma_0$ such that

$$\varphi_{TN}(\nu,\mu_0,\sigma_0)=T(\nu)+N(\nu),$$

that is

$$x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0, \tag{12}$$

where $L_1\leq\nu\leq L_2$, $T_1\leq\mu_0\leq T_2$ and $M_1\leq\sigma_0\leq M_2$. Here we must note that, from (12), we have

$$x_v(\nu,\mu_0,\sigma_0)=y_v(\nu,\mu_0,\sigma_0)=z_v(\nu,\mu_0,\sigma_0)=m_v(\nu,\mu_0,\sigma_0)=0. \tag{13}$$

So, from (2), (11), (12) and (13), the normal of the hypersurface (10) for $\mu=\mu_0$ and $\sigma=\sigma_0$ is obtained as

$$\begin{aligned}\eta_{TN}(\nu,\mu_0,\sigma_0)&=(\eta_{TN})_1(\nu,\mu_0,\sigma_0)T(\nu)+(\eta_{TN})_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+(\eta_{TN})_3(\nu,\mu_0,\sigma_0)B_1(\nu)+(\eta_{TN})_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned} \tag{14}$$

where

$$\begin{aligned}(\eta_{TN})_1(\nu,\mu_0,\sigma_0)&=0,\\(\eta_{TN})_2(\nu,\mu_0,\sigma_0)&=\kappa_2\left((m_\sigma)_0(x_\mu)_0-(x_\sigma)_0(m_\mu)_0\right),\\(\eta_{TN})_3(\nu,\mu_0,\sigma_0)&=\kappa_1\left((m_\mu)_0(x_\sigma)_0-(x_\mu)_0(m_\sigma)_0\right),\\(\eta_{TN})_4(\nu,\mu_0,\sigma_0)&=\kappa_1\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)\\&\quad+\ \kappa_2\left((y_\mu)_0(x_\sigma)_0-(x_\mu)_0(y_\sigma)_0\right).\end{aligned} \tag{15}$$

Also, from the definition of a given curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ to be geodesic, it must be

$$(\eta_{TN})_2(\nu,\mu_0,\sigma_0)\neq 0$$

and

$$(\eta_{TN})_3(\nu,\mu_0,\sigma_0)=(\eta_{TN})_4(\nu,\mu_0,\sigma_0)=0.$$

So, from (15) the curve $\gamma(\nu)$ is a geodesic on the hypersurface $\varphi_{TN}(\nu,\mu,\sigma)$ in G_4 if

$$\begin{cases}x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\\kappa_1=0,\ \kappa_2\neq 0,\\(y_\mu)_0(x_\sigma)_0=(x_\mu)_0(y_\sigma)_0,\ (m_\mu)_0(x_\sigma)_0\neq(x_\mu)_0(m_\sigma)_0\end{cases} \tag{16}$$

satisfied, where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. Since $\kappa_1(v)\neq 0$, the proof completes. □

CASE 2.

Here, by taking the NB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ which is given with the aid of the NB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_2}(\nu,\mu,\sigma)=r_{NB_2}(\nu)+\left[\begin{array}{c}x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\+z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right].$$

If NB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{NB_2}(\nu,\mu_0,\sigma_0) = {} & \left(\eta_{NB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{NB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) \\ & + \left(\eta_{NB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{NB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned} \tag{17}$$

where

$$\begin{aligned}&\left(\eta_{NB_2}\right)_1(\nu,\mu_0,\sigma_0)=0,\\ &\left(\eta_{NB_2}\right)_2(\nu,\mu_0,\sigma_0)=\left(\tfrac{\kappa_2-\kappa_3}{\sqrt{2}}\right)\left((x_\mu)_0\,(m_\sigma)_0-(m_\mu)_0\,(x_\sigma)_0\right),\\ &\left(\eta_{NB_2}\right)_3(\nu,\mu_0,\sigma_0)=0,\\ &\left(\eta_{NB_2}\right)_4(\nu,\mu_0,\sigma_0)=\left(\tfrac{\kappa_3-\kappa_2}{\sqrt{2}}\right)\left((y_\sigma)_0\,(x_\mu)_0-(x_\sigma)_0\,(y_\mu)_0\right).\end{aligned} \tag{18}$$

Theorem 2. *$\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_2\neq\kappa_3,\\ (y_\mu)_0\,(x_\sigma)_0=(x_\mu)_0\,(y_\sigma)_0\,,\quad (m_\mu)_0\,(x_\sigma)_0\neq(x_\mu)_0\,(m_\sigma)_0\end{cases} \tag{19}$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

Proof. If $x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0$ satisfies for a pair of parameters $\mu=\mu_0$ and $\sigma=\sigma_0$ on hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$, then NB_2-Smarandache curve is a isoparametric curve such that

$$\varphi_{NB_2}(\nu,\mu_0,\sigma_0)=\frac{N(\nu)+B_2(\nu)}{\sqrt{2}},$$

where $L_1\leq\nu\leq L_2$, $T_1\leq\mu_0\leq T_2$ and $M_1\leq\sigma_0\leq M_2$.

Also, from the definition of a given curve $\gamma(\nu)$ on the hypersurface $\varphi(\nu,\mu,\sigma)$ to be geodesic where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric, it must be

$$\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0)\neq 0$$

and

$$\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0)=\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0)=0.$$

So, using these conditions with (19) in (18), the proof completes. □

For the purposes of simplification and better analysis, Wang et al. have studied the case when the marching-scale functions can be decomposed into two factors in Euclidean 3-space. The factor-decomposition form possesses an evident advantage: the designer can select different sets of functions to adjust the shape of the surface until they are gratified with the design, and the resulting surface is guaranteed to belong to the isogeodesic surface pencil with the curve as the common geodesic [15]. Also in [3] and [11], the three types of the marching-scale function which have three parameters have been studied in 4-dimensional Galilean and Euclidean spaces, respectively. In this study, we have used the marching-scale functions which have been given in these studies. Now, for simplicity, let we investigate three types of marching-scale functions for this hypersurface.

Marching-scale functions of type 1

Let us choose

$$\begin{cases} x(\nu,\mu,\sigma) = p(\nu)X(\mu,\sigma), \\ y(\nu,\mu,\sigma) = q(\nu)Y(\mu,\sigma), \\ z(\nu,\mu,\sigma) = w(\nu)Z(\mu,\sigma), \\ m(\nu,\mu,\sigma) = l(\nu)M(\mu,\sigma), \end{cases} \tag{20}$$

where $L_1 \leq \nu \leq L_2$, $T_1 \leq \mu \leq T_2$, $M_1 \leq \sigma \leq M_2$; $p(\nu)$, $q(\nu)$, $w(\nu)$, $l(\nu)$, $X(\mu,\sigma)$, $Y(\mu,\sigma)$, $Z(\mu,\sigma)$, $M(\mu,\sigma) \in C^1$ and $p(\nu)$, $q(\nu)$, $w(\nu)$, $l(\nu)$, $\forall \nu \in [L_1, L_2]$ are not identically zero. By using (19), if the conditions

$$\begin{cases} X(\mu_0,\sigma_0) = Y(\mu_0,\sigma_0) = Z(\mu_0,\sigma_0) = M(\mu_0,\sigma_0) = 0, \\ \kappa_2 \neq \kappa_3, \\ \frac{\partial Y(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial X(\mu_0,\sigma_0)}{\partial \sigma} = \frac{\partial X(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial Y(\mu_0,\sigma_0)}{\partial \sigma}, \\ \frac{\partial M(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial X(\mu_0,\sigma_0)}{\partial \sigma} \neq \frac{\partial X(\mu_0,\sigma_0)}{\partial \mu}\frac{\partial M(\mu_0,\sigma_0)}{\partial \sigma}, \end{cases} \tag{21}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

Marching-scale functions of type 2

If we take

$$\begin{cases} x(\nu,\mu,\sigma) = p(\nu,\mu)X(\sigma), \\ y(\nu,\mu,\sigma) = q(\nu,\mu)Y(\sigma), \\ z(\nu,\mu,\sigma) = w(\nu,\mu)Z(\sigma), \\ m(\nu,\mu,\sigma) = l(\nu,\mu)M(\sigma), \end{cases} \tag{22}$$

where $L_1 \leq \nu \leq L_2$, $T_1 \leq \mu \leq T_2$, $M_1 \leq \sigma \leq M_2$ and $p(\nu,\mu)$, $q(\nu,\mu)$, $w(\nu,\mu)$, $l(\nu,\mu)$, $X(\mu)$, $Y(\mu)$, $Z(\mu)$, $M(\mu) \in C^1$, by using (19), if the conditions

$$\begin{cases} p(\nu,\mu_0)X(\sigma_0) = q(\nu,\mu_0)Y(\sigma_0) = w(\nu,\mu_0)Z(\sigma_0) = l(\nu,\mu_0)M(\sigma_0) = 0, \\ \kappa_2 \neq \kappa_3, \\ \frac{\partial q(\nu,\mu_0)}{\partial \mu}Y(\sigma_0)p(\nu,\mu_0)\frac{\partial X(\sigma_0)}{\partial \sigma} = \frac{\partial p(\nu,\mu_0)}{\partial \mu}X(\sigma_0)q(\nu,\mu_0)\frac{\partial Y(\sigma_0)}{\partial \sigma}, \\ \frac{\partial l(\nu,\mu_0)}{\partial \mu}M(\sigma_0)p(\nu,\mu_0)\frac{\partial X(\sigma_0)}{\partial \sigma} \neq \frac{\partial p(\nu,\mu_0)}{\partial \mu}X(\sigma_0)l(\nu,\mu_0)\frac{\partial M(\sigma_0)}{\partial \sigma}, \end{cases} \tag{23}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

Marching-scale functions of type 3

For

$$\begin{cases} x(\nu,\mu,\sigma) = p(\nu,\sigma)X(\mu), \\ y(\nu,\mu,\sigma) = q(\nu,\sigma)Y(\mu), \\ z(\nu,\mu,\sigma) = w(\nu,\sigma)Z(\mu), \\ m(\nu,\mu,\sigma) = l(\nu,\sigma)M(\mu), \end{cases} \tag{24}$$

where $L_1 \le \nu \le L_2$, $T_1 \le \mu \le T_2$, $M_1 \le \sigma \le M_2$ and $p(\nu,\sigma)$, $q(\nu,\sigma)$, $w(\nu,\sigma)$, $l(\nu,\sigma)$, $X(\mu)$, $Y(\mu)$, $Z(\mu)$, $M(\mu) \in C^1$, from (19), if the conditions

$$\begin{cases} p(\nu,\sigma_0)X(\mu_0) = q(\nu,\sigma_0)Y(\mu_0) = w(\nu,\sigma_0)Z(\mu_0) = l(\nu,\sigma_0)M(\mu_0) = 0, \\ \kappa_2 \neq \kappa_3, \\ \frac{\partial Y(\mu_0)}{\partial \mu} q(\nu,\sigma_0) \frac{\partial p(\nu,\sigma_0)}{\partial \sigma} X(\mu_0) = \frac{\partial X(\mu_0)}{\partial \mu} p(\nu,\sigma_0) \frac{\partial q(\nu,\sigma_0)}{\partial \sigma} Y(\mu_0), \\ \frac{\partial M(\mu_0)}{\partial \mu} l(\nu,\sigma_0) \frac{\partial p(\nu,\sigma_0)}{\partial \sigma} X(\mu_0) \neq \frac{\partial X(\mu_0)}{\partial \mu} p(\nu,\sigma_0) \frac{\partial l(\nu,\sigma_0)}{\partial \sigma} M(\mu_0), \end{cases} \tag{25}$$

satisfy, then $\gamma(\nu)$ is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_2}(\nu,\mu,\sigma)$ in G_4. Here, $\mu_0 \in [T_1, T_2], \sigma_0 \in [M_1, M_2]$.

Now, let we construct an example for this hypersurface family.

Example 3. *Let $\gamma(\nu)$ be a curve which is parametrized by*

$$\gamma(\nu) = (\nu, \sin v, \cos v, \nu^3). \tag{26}$$

From (4) and (5), it is easy to obtain that

$$\begin{cases} T(\nu) = (1, \cos v, -\sin v, 3\nu^2), \\ N(\nu) = \frac{1}{\sqrt{1+36\nu^2}}(0, -\sin v, -\cos v, 6v), \\ B_1(\nu) = \frac{1}{\sqrt{(1+36\nu^2)(37+36\nu^2)}}(0, 36v \sin v - (36v^2+1)\cos v, \\ \qquad\qquad 36v\cos v + (36v^2+1)\sin v, 6), \\ B_2(\nu) = \frac{1}{\sqrt{37+36\nu^2}}(0, 6(\cos v + v\sin v), 6(v\cos v - \sin v), 1) \end{cases} \tag{27}$$

and

$$\kappa_1 = \sqrt{1+36\nu^2}, \quad \kappa_2 = \frac{\sqrt{37+36\nu^2}}{1+36\nu^2}, \quad \kappa_3 = \frac{6v\sqrt{1+36\nu^2}}{37+36\nu^2}. \tag{28}$$

Now, if we choose $\sigma_0 = \frac{\pi}{2}$, $\mu_0 = 0$, $x(\nu,\mu,\sigma) = \nu^3(\sigma - \frac{\pi}{2})$, $y(\nu,\mu,\sigma) = \nu\mu^2 \sin\sigma$, $z(\nu,\mu,\sigma) = \cos\sigma$ and $m(\nu,\mu,\sigma) = \nu\mu\sigma^2$, then (26) is a geodesic curve where NB_2-Smarandache curve r_{NB_2} of the curve (26) is isoparametric on the hypersurface

$$\varphi_{NB_2}(\nu,\mu,\sigma) = (\nu^3(\sigma - \frac{\pi}{2}),$$

$$\frac{-\sin v}{\sqrt{1+36\nu^2}} + \frac{6(\cos v + v\sin v)}{\sqrt{(37+36\nu^2)}} + \nu^3(\sigma - \frac{\pi}{2})\cos v - \frac{\nu\mu^2 \sin\sigma \sin v}{\sqrt{1+36\nu^2}}$$

$$+ \frac{\cos\sigma(36v\sin v - (36v^2+1)\cos v)}{\sqrt{(1+36\nu^2)(37+36\nu^2)}} + \frac{6(\cos v + v\sin v)\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}},$$

$$\frac{-\cos v}{\sqrt{1+36\nu^2}} + \frac{6(v\cos v - \sin v)}{\sqrt{(37+36\nu^2)}} - \nu^3(\sigma - \frac{\pi}{2})\sin v - \frac{\nu\mu^2\sin\sigma\cos v}{\sqrt{1+36\nu^2}} \tag{29}$$
$$+\frac{\cos\sigma(36v\cos v + (36v^2+1)\sin v)}{\sqrt{(1+36\nu^2)(37+36\nu^2)}} + \frac{6(v\cos v - \sin v)\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}},$$
$$\frac{6v}{\sqrt{1+36\nu^2}} + \frac{1}{\sqrt{(37+36\nu^2)}} + 3v^5(\sigma - \frac{\pi}{2}) + \frac{6\nu^2\mu^2\sin\sigma}{\sqrt{1+36\nu^2}}$$
$$+\frac{6\cos\sigma}{\sqrt{(1+36\nu^2)(37+36\nu^2)}} + \frac{\nu\mu\sigma^2}{\sqrt{(37+36\nu^2)}}).$$

in G_4.

Different projections from four-space to three-spaces of the hypersurface (29) for $\sigma = \pi/2$ *can be seen in the Fig.1:*

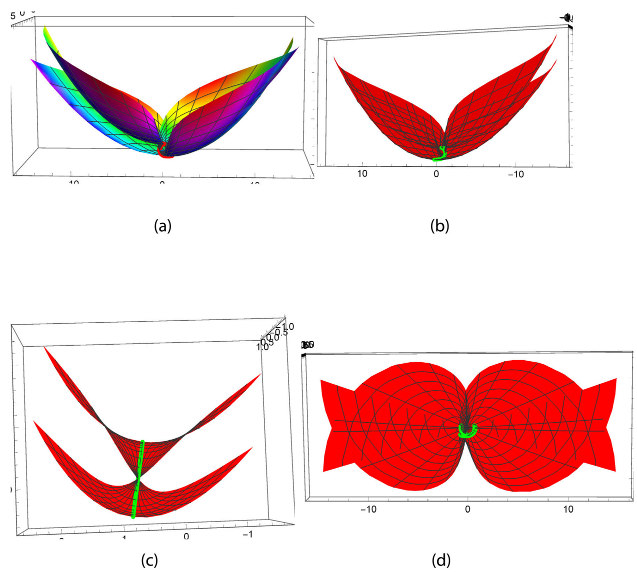

FIGURE 1. Projections of hypersurface family with parameter NB_2-Smarandache curve into $x_2x_3x_4$, $x_1x_3x_4$, $x_1x_2x_4$ and $x_1x_2x_3$-spaces in (a), (b), (c) and (d), respectively.

From now on, we'll give the parametric hypersurfaces given by different Smarandache curves of curve $\gamma(\nu)$ and their normal vector fields. Also, we'll state the theorems which give us the conditions for which $\gamma(\nu)$ is a geodesic curve where

Smarandache curves of the curve $\gamma(\nu)$ is isoparametric on these hypersurfaces. One can prove these theorems and investigate the conditions for different types of marching-scale functions with the similar methods given in the above case.

CASE 3.

Here, by taking the TB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ which is given with the aid of the TB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_1}(\nu,\mu,\sigma)=r_{TB_1}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If TB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TB_1}(\nu,\mu_0,\sigma_0)&=\left(\eta_{TB_1}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TB_1}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\&\quad+\left(\eta_{TB_1}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TB_1}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}\left(\eta_{TB_1}\right)_1(\nu,\mu_0,\sigma_0)&=0,\\ \left(\eta_{TB_1}\right)_2(\nu,\mu_0,\sigma_0)&=\kappa_3\left((x_\sigma)_0\,(z_\mu)_0-(z_\sigma)_0\,(x_\mu)_0\right),\\ \left(\eta_{TB_1}\right)_3(\nu,\mu_0,\sigma_0)&=(\kappa_1-\kappa_2)\left((m_\mu)_0\,(x_\sigma)_0-(x_\mu)_0\,(m_\sigma)_0\right)\\&\quad+\kappa_3\left((x_\mu)_0\,(y_\sigma)_0-(y_\mu)_0\,(x_\sigma)_0\right),\\ \left(\eta_{TB_1}\right)_4(\nu,\mu_0,\sigma_0)&=(\kappa_1-\kappa_2)\left((x_\mu)_0\,(z_\sigma)_0-(z_\mu)_0\,(x_\sigma)_0\right).\end{aligned}$$

Thus,

Theorem 4. *$\gamma(\nu)$ is a geodesic curve where TB_1-Smarandache curve r_{TB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_1=\kappa_2,\ \ \kappa_3\neq 0,\\ (y_\mu)_0\,(x_\sigma)_0=(x_\mu)_0\,(y_\sigma)_0\,,\ \ (z_\mu)_0\,(x_\sigma)_0\neq(x_\mu)_0\,(z_\sigma)_0 \end{array}\right. \tag{30}$$

are satisfied. Here, $\nu\in[L_1,L_2]\,,\ \mu_0\in[T_1,T_2]\,,\ \sigma_0\in[M_1,M_2]\,.$

CASE 4.

Here, by taking the TB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ which is given with the aid of the TB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_2}(\nu,\mu,\sigma)=r_{TB_2}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If TB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TB_2}(\nu,\mu_0,\sigma_0) &= \left(\eta_{TB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{TB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &\quad+\left(\eta_{TB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{TB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}&\left(\eta_{TB_2}\right)_1(\nu,\mu_0,\sigma_0)=0,\\ &\left(\eta_{TB_2}\right)_2(\nu,\mu_0,\sigma_0)=\kappa_3\left((x_\sigma)_0(m_\mu)_0-(m_\sigma)_0(x_\mu)_0\right),\\ &\left(\eta_{TB_2}\right)_3(\nu,\mu_0,\sigma_0)=\kappa_1\left((m_\mu)_0(x_\sigma)_0-(x_\mu)_0(m_\sigma)_0\right),\\ &\left(\eta_{TB_2}\right)_4(\nu,\mu_0,\sigma_0)=\kappa_1\left((x_\mu)_0(z_\sigma)_0-(z_\mu)_0(x_\sigma)_0\right)+\kappa_3\left((y_\sigma)_0(x_\mu)_0-(x_\sigma)_0(y_\mu)_0\right).\end{aligned}$$

For the curve $\gamma(\nu)$ to be a geodesic where TB_2-Smarandache curve r_{TB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1}(\nu,\mu,\sigma)$ in G_4, the following conditions must hold:

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\ \kappa_1=0,\ \kappa_3\neq 0,\\ (y_\mu)_0(x_\sigma)_0=(x_\mu)_0(y_\sigma)_0,\ \ (m_\mu)_0(x_\sigma)_0\neq(x_\mu)_0(m_\sigma)_0 \end{array}\right., \tag{31}$$

where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. But, from our assumption that $\kappa_1(v)\neq 0$, we have a contradiction. So, we have

Theorem 5. *$\gamma(\nu)$ is not a geodesic curve where TB_2-Smarandache curve r_{TB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_2}(\nu,\mu,\sigma)$ in G_4.*

CASE 5.

Here, by taking the NB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ which is given with the aid of the NB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_1}(\nu,\mu,\sigma)=r_{NB_1}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right]. \tag{32}$$

If NB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{NB_1}(\nu,\mu_0,\sigma_0) &= \left(\eta_{NB_1}\right)_1(\nu,\mu_0,\sigma_0)T(\nu)+\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0)N(\nu)\\ &\quad+\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu)+\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}
\left(\eta_{NB_1}\right)_1(\nu,\mu_0,\sigma_0) &= 0, \\
\left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0) &= \tfrac{\kappa_2}{\sqrt{2}}\left((x_\mu)_0\,(m_\sigma)_0 - (m_\mu)_0\,(x_\sigma)_0\right) \\
&\quad - \tfrac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0\,(z_\sigma)_0 - (z_\mu)_0\,(x_\sigma)_0\right), \\
\left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0) &= \tfrac{\kappa_2}{\sqrt{2}}\left((m_\sigma)_0\,(x_\mu)_0 - (x_\sigma)_0\,(m_\mu)_0\right) \\
&\quad + \tfrac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0\,(y_\sigma)_0 - (y_\mu)_0\,(x_\sigma)_0\right), \\
\left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0) &= \tfrac{\kappa_2}{\sqrt{2}}\left((x_\sigma)_0\,(z_\mu)_0 - (z_\sigma)_0\,(x_\mu)_0\right) \\
&\quad + \tfrac{\kappa_2}{\sqrt{2}}\left((x_\sigma)_0\,(y_\mu)_0 - (y_\sigma)_0\,(x_\mu)_0\right).
\end{aligned} \tag{33}$$

Hence,

Theorem 6. *$\gamma(\nu)$ is a geodesic curve where NB_1-Smarandache curve r_{NB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0, \\ \left(\eta_{NB_1}\right)_2(\nu,\mu_0,\sigma_0) \neq 0, \;\; \left(\eta_{NB_1}\right)_3(\nu,\mu_0,\sigma_0) = \left(\eta_{NB_1}\right)_4(\nu,\mu_0,\sigma_0) = 0 \end{cases} \tag{34}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

CASE 6.

Here, by taking the B_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the B_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{B_1B_2}(\nu,\mu,\sigma) = r_{B_1B_2}(\nu) + \left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu) + y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu) + m(\nu,\mu,\sigma)B_2(\nu) \end{array}\right].$$

If B_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu = \mu_0$ and $\sigma = \sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}
\eta_{B_1B_2}(\nu,\mu_0,\sigma_0) &= \left(\eta_{B_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{B_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) \\
&\quad + \left(\eta_{B_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{B_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),
\end{aligned}$$

where

$$\left(\eta_{B_1B_2}\right)_1(\nu,\mu_0,\sigma_0) = 0,$$

$$\left(\eta_{B_1B_2}\right)_2(\nu,\mu_0,\sigma_0) = \frac{\kappa_3}{\sqrt{2}}\left((x_\sigma)_0\left((m_\mu)_0 + (z_\mu)_0\right) - (x_\mu)_0\left((m_\sigma)_0 + (z_\sigma)_0\right)\right),$$

$$\left(\eta_{B_1B_2}\right)_3(\nu,\mu_0,\sigma_0) = \frac{\kappa_2}{\sqrt{2}}\left((x_\mu)_0\,(m_\sigma)_0 - (m_\mu)_0\,(x_\sigma)_0\right) + \frac{\kappa_3}{\sqrt{2}}\left((x_\mu)_0\,(y_\sigma)_0 - (y_\mu)_0\,(x_\sigma)_0\right),$$

$$\left(\eta_{B_1B_2}\right)_4(\nu,\mu_0,\sigma_0) = \frac{\kappa_2}{\sqrt{2}}\left((z_\mu)_0\,(x_\sigma)_0 - (x_\mu)_0\,(z_\sigma)_0\right) - \frac{\kappa_3}{\sqrt{2}}\left((y_\mu)_0\,(x_\sigma)_0 - (x_\mu)_0\,(y_\sigma)_0\right).$$

So, we can state the following Theorem:

Theorem 7. *$\gamma(\nu)$ is a geodesic curve where B_1B_2-Smarandache curve $r_{B_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{B_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0, \\ \kappa_3 \neq 0, \\ (x_\mu)_0 \left((m_\sigma)_0 + (z_\sigma)_0\right) \neq (x_\sigma)_0 \left((m_\mu)_0 + (z_\mu)_0\right), \\ \left(\eta_{B_1B_2}\right)_3 (\nu,\mu_0,\sigma_0) = \left(\eta_{B_1B_2}\right)_4 (\nu,\mu_0,\sigma_0) = 0 \end{cases} \tag{35}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

CASE 7.

Here, by taking the TNB_1-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_1-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TNB_1}(\nu,\mu,\sigma) = r_{TNB_1}(\nu) + \begin{bmatrix} x(\nu,\mu,\sigma)T(\nu) + y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu) + m(\nu,\mu,\sigma)B_2(\nu) \end{bmatrix}.$$

If TNB_1-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ in G_4 for $\mu = \mu_0$ and $\sigma = \sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned} \eta_{TNB_1}(\nu,\mu_0,\sigma_0) &= \left(\eta_{TNB_1}\right)_1 (\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{TNB_1}\right)_2 (\nu,\mu_0,\sigma_0)N(\nu) \\ &\quad + \left(\eta_{TNB_1}\right)_3 (\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{TNB_1}\right)_4 (\nu,\mu_0,\sigma_0)B_2(\nu), \end{aligned}$$

where

$$\begin{aligned} \left(\eta_{TNB_1}\right)_1 (\nu,\mu_0,\sigma_0) &= 0, \\ \left(\eta_{TNB_1}\right)_2 (\nu,\mu_0,\sigma_0) &= \kappa_3 \left((z_\mu)_0 (x_\sigma)_0 - (x_\mu)_0 (z_\sigma)_0\right) - \kappa_2 \left((m_\mu)_0 (x_\sigma)_0 - (x_\mu)_0 (m_\sigma)_0\right), \\ \left(\eta_{TNB_1}\right)_3 (\nu,\mu_0,\sigma_0) &= (\kappa_2 - \kappa_1) \left((x_\mu)_0 (m_\sigma)_0 - (m_\mu)_0 (x_\sigma)_0\right) \\ &\quad + \kappa_3 \left((x_\mu)_0 (y_\sigma)_0 - (y_\mu)_0 (x_\sigma)_0\right), \\ \left(\eta_{TNB_1}\right)_4 (\nu,\mu_0,\sigma_0) &= (\kappa_1 - \kappa_2) \left((x_\mu)_0 (z_\sigma)_0 - (z_\mu)_0 (x_\sigma)_0\right) \\ &\quad + \kappa_2 \left((y_\mu)_0 (x_\sigma)_0 - (x_\mu)_0 (y_\sigma)_0\right). \end{aligned}$$

Hence,

Theorem 8. *$\gamma(\nu)$ is a geodesic curve where TNB_1-Smarandache curve r_{TNB_1} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_1}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0, \\ \left(\eta_{TNB_1}\right)_2 (\nu,\mu_0,\sigma_0) \neq 0, \ \left(\eta_{TNB_1}\right)_3 (\nu,\mu_0,\sigma_0) = \left(\eta_{TNB_1}\right)_4 (\nu,\mu_0,\sigma_0) = 0 \end{cases} \tag{36}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

CASE 8.

Here, by taking the TNB_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TNB_2}(\nu,\mu,\sigma) = r_{TNB_2}(\nu) + \begin{bmatrix} x(\nu,\mu,\sigma)T(\nu) + y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu) + m(\nu,\mu,\sigma)B_2(\nu) \end{bmatrix}.$$

If TNB_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\begin{aligned}\eta_{TNB_2}(\nu,\mu_0,\sigma_0) &= \left(\eta_{TNB_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{TNB_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) \\ &\quad + \left(\eta_{TNB_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{TNB_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),\end{aligned}$$

where

$$\begin{aligned}&\left(\eta_{TNB_2}\right)_1(\nu,\mu_0,\sigma_0) = 0,\\ &\left(\eta_{TNB_2}\right)_2(\nu,\mu_0,\sigma_0) = (\kappa_2-\kappa_3)\left((x_\mu)_0\,(m_\sigma)_0 - (m_\mu)_0\,(x_\sigma)_0\right),\\ &\left(\eta_{TNB_2}\right)_3(\nu,\mu_0,\sigma_0) = -\kappa_1\left((x_\mu)_0\,(m_\sigma)_0 - (m_\mu)_0\,(x_\sigma)_0\right)\\ &\left(\eta_{TNB_2}\right)_4(\nu,\mu_0,\sigma_0) = \kappa_1\left((x_\mu)_0\,(z_\sigma)_0 - (z_\mu)_0\,(x_\sigma)_0\right)\\ &\qquad\qquad + (\kappa_3-\kappa_2)\left((x_\mu)_0\,(y_\sigma)_0 - (y_\mu)_0\,(x_\sigma)_0\right).\end{aligned}$$

For the curve $\gamma(\nu)$ to be a geodesic where TNB_2-Smarandache curve r_{TNB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4, the following conditions must hold:

$$\left\{\begin{array}{l} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0,\\ \kappa_1 = 0,\ \kappa_2 \neq \kappa_3,\\ (x_\mu)_0\,(y_\sigma)_0 = (y_\mu)_0\,(x_\sigma)_0,\ (x_\mu)_0\,(m_\sigma)_0 \neq (m_\mu)_0\,(x_\sigma)_0 \end{array}\right., \tag{37}$$

where $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$. Since $\kappa_1(v)\neq 0$, we have

Theorem 9. *$\gamma(\nu)$ is not a geodesic curve where TNB_2-Smarandache curve r_{TNB_2} of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_2}(\nu,\mu,\sigma)$ in G_4.*

CASE 9.

Here, by taking the TB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the TB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{TB_1B_2}(\nu,\mu,\sigma) = r_{TB_1B_2}(\nu) + \begin{bmatrix} x(\nu,\mu,\sigma)T(\nu) + y(\nu,\mu,\sigma)N(\nu) \\ +z(\nu,\mu,\sigma)B_1(\nu) + m(\nu,\mu,\sigma)B_2(\nu) \end{bmatrix}.$$

If TB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$\eta_{TB_1B_2}(\nu,\mu_0,\sigma_0) = \left(\eta_{TB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{TB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) + \left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),$$

where

$$\begin{aligned}
\left(\eta_{TB_1B_2}\right)_1(\nu,\mu_0,\sigma_0) &= 0,\\
\left(\eta_{TB_1B_2}\right)_2(\nu,\mu_0,\sigma_0) &= \kappa_3\left((x_\sigma)_0((m_\mu)_0 + (z_\mu)_0) - (x_\mu)_0((m_\sigma)_0 + (z_\sigma)_0)\right),\\
\left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0) &= (\kappa_2-\kappa_1)\left((x_\mu)_0(m_\sigma)_0 - (m_\mu)_0(x_\sigma)_0\right)\\
&\quad + \kappa_3\left((x_\mu)_0(y_\sigma)_0 - (y_\mu)_0(x_\sigma)_0\right),\\
\left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0) &= (\kappa_1-\kappa_2)\left((x_\mu)_0(z_\sigma)_0 - (z_\mu)_0(x_\sigma)_0\right)\\
&\quad + \kappa_3\left((x_\mu)_0(y_\sigma)_0 - (y_\mu)_0(x_\sigma)_0\right).
\end{aligned}$$

Thus,

Theorem 10. *$\gamma(\nu)$ is a geodesic curve where TB_1B_2-Smarandache curve $r_{TB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TB_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu,\mu_0,\sigma_0) = y(\nu,\mu_0,\sigma_0) = z(\nu,\mu_0,\sigma_0) = m(\nu,\mu_0,\sigma_0) = 0,\\ \kappa_3 \neq 0,\\ \left(\eta_{TB_1B_2}\right)_3(\nu,\mu_0,\sigma_0) = \left(\eta_{TB_1B_2}\right)_4(\nu,\mu_0,\sigma_0) = 0,\\ (x_\mu)_0((m_\sigma)_0 + (z_\sigma)_0) \neq (x_\sigma)_0((m_\mu)_0 + (z_\mu)_0) \end{cases} \tag{38}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

CASE 10.

Here, by taking the NB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the NB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$\varphi_{NB_1B_2}(\nu,\mu,\sigma) = r_{NB_1B_2}(\nu) + \begin{bmatrix} x(\nu,\mu,\sigma)T(\nu) + y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu) + m(\nu,\mu,\sigma)B_2(\nu) \end{bmatrix}.$$

If NB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu = \mu_0$ and $\sigma = \sigma_0$, then from (8), the normal of this hypersurface is

$$\eta_{NB_1B_2}(\nu,\mu_0,\sigma_0) = \left(\eta_{NB_1B_2}\right)_1(\nu,\mu_0,\sigma_0)T(\nu) + \left(\eta_{NB_1B_2}\right)_2(\nu,\mu_0,\sigma_0)N(\nu) + \left(\eta_{NB_1B_2}\right)_3(\nu,\mu_0,\sigma_0)B_1(\nu) + \left(\eta_{NB_1B_2}\right)_4(\nu,\mu_0,\sigma_0)B_2(\nu),$$

where

$$\begin{aligned}
\left(\eta_{NB_1B_2}\right)_1(\nu,\mu_0,\sigma_0) &= 0,\\
\left(\eta_{NB_1B_2}\right)_2(\nu,\mu_0,\sigma_0) &= \left(\frac{\kappa_2-\kappa_3}{\sqrt{3}}\right)\left((x_\mu)_0(m_\sigma)_0 - (m_\mu)_0(x_\sigma)_0\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_3}{\sqrt{3}}\left((z_\mu)_0 (x_\sigma)_0-(x_\mu)_0 (z_\sigma)_0\right),\\
(\eta_{NB_1B_2})_3(\nu,\mu_0,\sigma_0)=&\frac{\kappa_2}{\sqrt{3}}\left((x_\mu)_0 (m_\sigma)_0-(m_\mu)_0 (x_\sigma)_0\right)\\
&+\frac{\kappa_3}{\sqrt{3}}\left((x_\mu)_0 (y_\sigma)_0-(y_\mu)_0 (x_\sigma)_0\right),\\
(\eta_{NB_1B_2})_4(\nu,\mu_0,\sigma_0)=&\frac{\kappa_2}{\sqrt{3}}\left((z_\mu)_0 (x_\sigma)_0-(x_\mu)_0 (z_\sigma)_0\right)\\
&+(\frac{\kappa_3-\kappa_2}{\sqrt{3}})\left((x_\mu)_0 (y_\sigma)_0-(y_\mu)_0 (x_\sigma)_0\right).
\end{aligned}
$$

So,

Theorem 11. *$\gamma(\nu)$ is a geodesic curve where NB_1B_2-Smarandache curve $r_{NB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{NB_1B_2}(\nu,\mu,\sigma)$ in G_4 if the conditions*

$$
\left\{
\begin{array}{l}
x(\nu,\mu_0,\sigma_0)=y(\nu,\mu_0,\sigma_0)=z(\nu,\mu_0,\sigma_0)=m(\nu,\mu_0,\sigma_0)=0,\\
(\eta_{NB_1B_2})_2(\nu,\mu_0,\sigma_0)\neq 0,\ \ (\eta_{NB_1B_2})_3(\nu,\mu_0,\sigma_0)=(\eta_{NB_1B_2})_4(\nu,\mu_0,\sigma_0)=0
\end{array}
\right. \tag{39}
$$

are satisfied. Here, $\nu\in[L_1,L_2]$, $\mu_0\in[T_1,T_2]$, $\sigma_0\in[M_1,M_2]$.

CASE 11.

Here, by taking the TNB_1B_2-Smarandache curve of $\gamma(\nu)$ instead of the curve $\gamma(\nu)$ in (9), let us define a parametric hypersurface $\varphi_{TNB_1B_2}(\nu,\mu,\sigma)$ which is given with the aid of the TNB_1B_2-Smarandache curve of $\gamma(\nu)$ and the Frenet vectors of the curve $\gamma(\nu)$ as follows

$$
\varphi_{TNB_1B_2}(\nu,\mu,\sigma)=r_{TNB_1B_2}(\nu)+\left[\begin{array}{c} x(\nu,\mu,\sigma)T(\nu)+y(\nu,\mu,\sigma)N(\nu)\\ +z(\nu,\mu,\sigma)B_1(\nu)+m(\nu,\mu,\sigma)B_2(\nu)\end{array}\right].
$$

If TNB_1B_2-Smarandache curve of the curve $\gamma(\nu)$ is a isoparametric curve on a hypersurface $\varphi_{TNB_1B_2}(\nu,\mu,\sigma)$ in G_4 for $\mu=\mu_0$ and $\sigma=\sigma_0$, then from (8), the normal of this hypersurface is

$$
\begin{aligned}
\eta_{TNB_1B_2}(\nu,\mu_0,\sigma_0)=&(\eta_{TNB_1B_2})_1(\nu,\mu_0,\sigma_0)T(\nu)+(\eta_{TNB_1B_2})_2(\nu,\mu_0,\sigma_0)N(\nu)\\
&+(\eta_{TNB_1B_2})_3(\nu,\mu_0,\sigma_0)B_1(\nu)+(\eta_{TNB_1B_2})_4(\nu,\mu_0,\sigma_0)B_2(\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
&(\eta_{TNB_1B_2})_1(\nu,\mu_0,\sigma_0)=0,\\
&(\eta_{TNB_1B_2})_2(\nu,\mu_0,\sigma_0)=\kappa_2\left((m_\sigma)_0 (x_\mu)_0-(x_\sigma)_0 (m_\mu)_0\right)\\
&\qquad+\kappa_3\left(\left((x_\sigma)_0 (m_\mu)_0-(m_\sigma)_0 (x_\mu)_0\right)-\left((x_\mu)_0 (z_\sigma)_0-(z_\mu)_0 (x_\sigma)_0\right)\right),\\
&(\eta_{TNB_1B_2})_3(\nu,\mu_0,\sigma_0)=(\kappa_2-\kappa_1)\left((x_\mu)_0 (m_\sigma)_0-(m_\mu)_0 (x_\sigma)_0\right)\\
&\qquad+\kappa_3\left((x_\mu)_0 (y_\sigma)_0-(y_\mu)_0 (x_\sigma)_0\right),\\
&(\eta_{TNB_1B_2})_4(\nu,\mu_0,\sigma_0)=(\kappa_1-\kappa_2)\left((x_\mu)_0 (z_\sigma)_0-(z_\mu)_0 (x_\sigma)_0\right)
\end{aligned}
$$

$$+ (\kappa_3 - \kappa_2) \left((x_\mu)_0 (y_\sigma)_0 - (y_\mu)_0 (x_\sigma)_0 \right).$$

Finally, we get

Theorem 12. *$\gamma(\nu)$ is a geodesic curve where TNB_1B_2-Smarandache curve $r_{TNB_1B_2}$ of the curve $\gamma(\nu)$ is isoparametric on the hypersurface $\varphi_{TNB_1B_2}(\nu, \mu, \sigma)$ in G_4 if the conditions*

$$\begin{cases} x(\nu, \mu_0, \sigma_0) = y(\nu, \mu_0, \sigma_0) = z(\nu, \mu_0, \sigma_0) = m(\nu, \mu_0, \sigma_0) = 0, \\ \left(\eta_{TNB_1B_2}\right)_2 (\nu, \mu_0, \sigma_0) \neq 0, \ \ \left(\eta_{TNB_1B_2}\right)_3 (\nu, \mu_0, \sigma_0) = \left(\eta_{TNB_1B_2}\right)_4 (\nu, \mu_0, \sigma_0) = 0 \end{cases} \tag{40}$$

are satisfied. Here, $\nu \in [L_1, L_2]$, $\mu_0 \in [T_1, T_2]$, $\sigma_0 \in [M_1, M_2]$.

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