Mathematical Sciences and Applications E-Notes

https://doi.org/10.36753/mathenot.794789
9 (2) 74-80 (2021) - Research Article
ISSN: 2147-6268

Regularization of p-Adic Distributions Associated to Functions on p-Adic Fields With Moderate Variation

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Abstract
The p-adic distributions attached to ordinary functions defined on p-adic fields with moderate variation are studied. We first give a sufficient growth condition on ordinary functions to construct p-adic distributions. Then a moderate variation condition on functions for regularization of these p-adic distributions is imposed which provides a general method to construct p-adic measures. The p-adic integrals against these measures are also explicitly transformed to integrals against Bernoulli measures.

Keywords: *p-Adic analysis; p-Adic measures; p-Adic integration, Bernoulli measures, p-Adic gamma measures*

AMS Subject Classification (2020): *Primary: 11S80*

1. Introduction

The theory p-adic integration has been an important tool in number theory for many years. The construction of the p-adic zeta function by p-adic interpolation of the complex Riemann zeta function $\zeta(s)$ at negative integers was achieved by Leopoldt and Kubota by using the (p-adic) limits

$$\lim_{M\to\infty} \frac{1}{p^M} \sum_{j=0}^{p^M-1} f(x+j)$$

for a locally analytic function f on $\mathbb{Z}_p$ [1]. This idea has been conceptualized by expressing $\zeta(-r)$ for $r \in \mathbb{Z}_{\geq 1}$ as the p-adic integral of x^r against the p-adic measure $\mu_{\alpha,1}$ called as Mazur's Bernoulli measure [9]. Indeed more generally one can define a p-adic L-function as the p-adic integral of a Dirichlet character against a p-adic measure. Today it is well known that the special values of complex L-functions are also expressed in terms of p-adic integrals and this relation is also reflected to the p-adic properties of modular forms and moduli spaces [7, 8].

In a simple setting, a p-adic distribution denoted by μ on a compact subset X of $\mathbb{Q}_p$ can be defined as a finitely additive map from the compact open subsets of X to $\mathbb{C}_p$. Explicitly if $U \subset X$ is equal to the disjoint union of compact open subsets $U_1, ..., U_n$ then $\mu(U) = \mu(U_1) + ... + \mu(U_1)$. Indeed it is enough to check the additivity condition on the

Received : *14-09-2020,* ***Accepted :*** *22-12-2020*

compact open subsets of the form $a + (p^N)$ contained in X, i.e. μ extends to a p-adic distribution on X if and only if

$$\mu(a + (p^N)) = \sum_{b=0}^{p-1} \mu(a + bp^N + (p^{N+1})) \tag{1.1}$$

for any $a + (p^N)$ contained in X. We will call (1.1) as the *distribution property*. If additionally μ is bounded then we call μ as a p-adic measure on X. The reader is referred to [3, 6, 9] for the properties as well as equivalent characterizations of p-adic distributions and measures. We will mainly take X to be $\mathbb{Z}_p$ or $\mathbb{Z}_p^*$.

The p-adic integral of a continuous function $f(x)$ on X against a p-adic measure μ denoted by $\int_X f\,\mu$ is defined as

$$\int_X f\,\mu = \lim_{N\to\infty} \sum_{U \subset X} f(\tilde{a})\mu(U)$$

where U ranges over the compact open subsets of X of the form $a + (p^N)$ and $\tilde{a} \in U$ is arbitrary. Note that this limit exists and is independent of the choice of $\tilde{a} \in U$ [9, Theorem 6 of §II].

The standard examples of p-adic measures are Bernoulli measures, p-adic Gamma measures and the measure μ_z defined as

$$\mu_z(a + (p^N) = z^a/(1 - z^{p^N})$$

where $z \in \mathbb{C}_p$ with $|z-1|_p \geq 1$ (See [9], [5] and [10] respectively). Bernoulli measures is core for construction of p-adic zeta functions. The p-adic Gamma measures give a p-adic analogue of Euler constant [4]. Also the measure μ_z has an important role in the theory of p-adic polylogarithms [2].

We briefly recall the Bernoulli measures referring to [9] for the details. The k-th Bernoulli distribution $\mu_{B,k}$ is defined as

$$\mu_{B,k}(a + (p^N)) = p^{N(k-1)} B_k(a/p^N)$$

where $B_k(x)$ is the k-the Bernoulli polynomial. Note that $\mu_{B,k}$ is unbounded, so is not a measure. The k-th Bernoulli measure is obtained by *regularization* of $\mu_{B,k}$ as follows. Let $\alpha \in \mathbb{Z}_{\geq 1}$ for which $(p, \alpha) = 1$. Then it is clear that for any distribution μ on $\mathbb{Z}_p$, the map $U \mapsto \mu(\alpha U)$ on the compact open subsets of $\mathbb{Z}_p$ is also a distribution and so is

$$U \mapsto \mu_{k,\alpha}(U) := \mu_{k,\alpha}(U) - \alpha^{-k}\mu_{k,\alpha}(\alpha U).$$

It turns out that $\mu_{k,\alpha}$ is bounded, indeed $|\mu_{k,\alpha}(U)|_p \leq 1$ and so is a p-adic measure on $\mathbb{Z}_p$. The p-adic Gamma measures are also constructed in a similar way [4].

In this paper we will generalize this method of construction of p-adic distributions and measures on $\mathbb{Z}_p^*$ and show that we can associate p-adic measures to ordinary functions satisfying some growth and variation conditions. In particular we will see that the Bernoulli measures and p-adic Gamma measures correspond to polynomials and $x\log_p(x) - x$ respectively where $\log_p$ is Iwasawa's p-adic logarithm, but the method provided here is also valid for a larger class of functions. We will also give an expression for p-adic integrals against these measures in terms of Mazur's Bernoulli measure.

We fix the following notation for the rest of the paper. We will denote the ring of p-adic integers and its field of fractions by $\mathbb{Z}_p$ and $\mathbb{Q}_p$ respectively. Also $\mathbb{C}_p$ denotes the completion of a fixed algebraic closure of $\mathbb{Q}_p$ endowed with the normalized p-adic norm as $|p|_p = 1/p$. The group of units of $\mathbb{Z}_p$ is denoted by $\mathbb{Z}_p^*$, i.e. $\mathbb{Z}_p^* = \{\alpha \in \mathbb{Z}_p \ : \ |\alpha| = 1\}$.

2. Admissible functions with moderate variation

We give a basic definition for the functions that we will work on. Let $R \geq 0$ be a real number and $D_R := \{x \in \mathbb{Q}_p \,|\, |x|_p \geq R,\ x \notin \mathbb{Z}_{\leq 0}\}$.

Definition 2.1. We say that a function f defined on D_R is admissible if there exists $k \in \mathbb{Z}_{\geq 1}$ for which the following are satisfied;

i) The pointwise limit $F(x) := \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} f\left(\dfrac{x+j}{p^M}\right)$ exists for any $x \in \mathbb{Q}_p$ with $x \notin \mathbb{Z}_{\leq 0}$

ii) For some $\alpha \in \mathbb{Z}_{\geq 2}$ with $(p, \alpha) = 1$ the pointwise limit $F_\alpha(x) := \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} f\left(\dfrac{x+j}{\alpha p^M}\right)$ exists for any $x \in \mathbb{Q}_p$ with $x \notin \mathbb{Z}_{\leq 0}$.

For example $f(x) = x^k$ and $f(x) = x^k \log_p(x)$ for $k \in \mathbb{Z}_{\geq 0}$ are admissible functions, and $\alpha \geq 2$ can be any integer with $(p, \alpha) = 1$. In the sequel we always consider an admissible function together with a fixed α satisfying condition ii) of Definition 2.1 without any further reference. Now we show that admissible functions induce p-adic distributions.

Proposition 2.1. *Let f be an admissible function on some D_R. Then the maps μ_f and $\mu_{f,\alpha}$ defined as*

$$\mu_f(a + (p^N)) = p^{N(k-1)} F(a/p^N)$$
$$\mu_{f,\alpha}(a + (p^N)) = p^{N(k-1)} F_\alpha(a/p^N)$$

on the compact open subsets of $\mathbb{Z}_p^*$ extend to p-adic distributions on $\mathbb{Z}_p^*$.

Proof. We need to check the distribution property (1.1). Now we have that

$$\sum_{b=0}^{p-1} \mu_{f,\alpha}(a + bp^N + (p^{N+1})) = \sum_{b=0}^{p-1} p^{(N+1)(k-1)} F_\alpha\left(\frac{a + bp^N}{p^{N+1}}\right)$$
$$= p^{(N+1)(k-1)} \lim_{M\to\infty} p^{M(k-1)} \sum_{b=0}^{p-1} \sum_{j=0}^{p^M-1} f\left(\frac{(a+bp^N)/p^{N+1} + j}{\alpha p^M}\right)$$
$$= p^{N(k-1)} \lim_{M\to\infty} p^{(M+1)(k-1)} \sum_{b=0}^{p-1} \sum_{j=0}^{p^M-1} f\left(\frac{(a/p^N) + (b + jp)}{\alpha p^{M+1}}\right)$$
$$= p^{N(k-1)} \lim_{M'\to\infty} p^{M'(k-1)} \sum_{j'=0}^{p^{M'}-1} f\left(\frac{(a/p^N) + j'}{\alpha p^{M'}}\right)$$

where $M' = M + 1$ and $j' = b + jp$. But by definition the last sum is equal to

$$p^{N(k-1)} F_\alpha(a/p^N) = \mu_{f,\alpha}(a + (p^N))$$

which completes the proof (Note that the above arguments are also valid for $\alpha = 1$.) □

Remark 2.1. The condition $x \notin \mathbb{Z}_{\leq 0}$ is redundant in some cases. For example if $f(x)$ is a polynomial then f is well defined on $\mathbb{Q}_p$ and so μ_f and $\mu_{f,\alpha}$ extend to p-adic distributions also on $\mathbb{Z}_p$. But for example if $f(x) = \log_p(x)$ which is not defined at $x = 0$, then the condition $x \notin \mathbb{Z}_{\leq 0}$ is necessary.

Now we conceptualize the notion of regularization of p-adic distributions obtained in Proposition 2.1. We adopt the notation of Definition 2.1 and Proposition 2.1. Recall that since $\mu_{f,\alpha}$ is a p-adic distribution, the map

$$U \mapsto \mu_{f,\alpha}(\alpha U)$$

also extends to a p-adic distribution. Explicitly we have that

$$\mu_{f,\alpha}(\alpha U) = p^{N(k-1)} F_\alpha(\{\alpha a\ /p^N\})$$

where $\{x\}$ denotes the fractional part of x.

Definition 2.2. Let f be an admissible function and $\mu_f^{(\alpha)}$ be the p-adic distribution defined as

$$\mu_f^{(\alpha)}(U) = \mu_f(U) - \mu_{f,\alpha}(\alpha U) = p^{N(k-1)} F(a/p^N) - p^{N(k-1)} F_\alpha(\{\alpha a\ /p^N\})$$

for any compact open subset U of $\mathbb{Z}_p^*$. If $\mu_f^{(\alpha)}$ is bounded, and so extends to a p-adic measure then we say that $\mu_f^{(\alpha)}$ is a regularization of $\mu_{f,1}$.

Now we will give two main results. The first theorem below presents a sufficient condition on f which guarantees the boundedness of $\mu_f^{(\alpha)}$. The importance of it is that a *moderate variation* for an admissible function f is enough for the boundedness of $\mu_f^{(\alpha)}$. The second theorem will be useful to express the p-adic integrals against $\mu_f^{(\alpha)}$ in terms of p-adic integrals against Mazur's Bernoulli measure.

Theorem 2.1. *Let f be an admissible function, and $k \in \mathbb{Z}_{\geq 1}$. Suppose that*

$$\sup_{u \in \mathbb{Z}_p} \{|f(y+u) - f(y)|_p\} = O(|y^{k-1}|_p) \text{ as } |y|_p \to \infty \tag{2.1}$$

Then $|\mu_f^{(\alpha)}|_p$ is bounded, and so is a regularization of $\mu_{f,1}$.

Proof. By definition we have

$$\mu_f^{(\alpha)}(a + (p^N)) = p^{N(k-1)} \lim_{M \to \infty} p^{M(k-1)} \sum_{j=0}^{p^M - 1} \left[f\left(\frac{(a/p^N) + j}{p^M}\right) - f\left(\frac{\{\alpha a/p^N\} + j}{\alpha p^M}\right) \right].$$

Put $b = a/p^N$. So

$$\frac{\{\alpha b\} + j}{\alpha p^M} = \frac{\alpha b - [\alpha b] + j}{\alpha p^M} = \frac{b + \left(\dfrac{j - [\alpha b]}{\alpha}\right)}{p^M}$$

Since $p \nmid \alpha$, for a fixed M and any $j \in \{0, 1, ..., p^M - 1\}$, there exists a unique $j' \in \{0, 1, ..., p^M - 1\}$ such that $j' \equiv \alpha j + [\alpha b] \pmod{p^M}$, so the mapping $j \mapsto j'$ on the set $\{0, 1, 2, ..., p^M - 1\}$ is bijective. But then

$$\frac{b + j}{p^M} - \frac{\{\alpha b\} + j'}{\alpha p^M} = \frac{b + j}{p^M} - \frac{b + \left(\dfrac{j' - [\alpha b]}{\alpha}\right)}{p^M} = \frac{j - \left(\dfrac{j' - [\alpha b]}{\alpha}\right)}{p^M} \in \mathbb{Z}_p.$$

So by hypothesis for large enough M we have

$$\left| f\left(\frac{b + j}{p^M}\right) - f\left(\frac{\{\alpha b\} + j'}{\alpha p^M}\right) \right|_p \leq K \left| \frac{b + j}{p^M} \right|_p^{k-1} = K.p^{M(k-1)} |b + j|_p^{k-1}$$

for some $K > 0$. Now we distinguish two cases. First suppose that $N = 0$, so $b = a \in \mathbb{Z}_p$. But then since $b + j \in \mathbb{Z}_p$ we have that

$$K.p^{M(k-1)} |b + j|_p^{k-1} \leq K.p^{M(k-1)} \implies |\mu(\mathbb{Z}_p^*)|_p \leq \lim_{M \to \infty} |p^{M(k-1)}|_p K.p^{M(k-1)} = K.$$

Now suppose that $N \geq 1$. Since $(a, p) = 1$, we have $|b|_p > 1$. But in this case $|b + j|_p = |b|_p$ and so

$$\left| f\left(\frac{b + j}{p^M}\right) - f\left(\frac{\{\alpha b\} + j'}{\alpha p^M}\right) \right|_p \leq K p^{M(k-1)} |b|_p^{k-1} = K p^{M(k-1)} p^{N(k-1)}$$

which implies that $|\mu_f^{(\alpha)}(a + (p^N))|_p \leq K$ as desired. □

Theorem 2.2. *Let f be a function satisfying hypothesis of Theorem 2.1, and $M \in \mathbb{Z}_{\geq 1}$. Suppose that there exist $k \in \mathbb{Z}$ and functions $h(y)$ and $T(y, u, v)$ depending on f such that for all $y, u, v \in \mathbb{Q}_p$ with $|y|_p \geq p^M$, $|u - y|_p < p^M$, $|v - y|_p < p^M$ and $u - v \in \mathbb{Z}_p$ the following are satisfied;*

i) $f(y + u) - f(y + v) = (u - v)h(y) + T(y, u, v)$,

ii) The pointwise limit $H(a) := \lim_{r \to \infty} p^{r(k-1)} h(a/p^r)$ *exists and is bounded on* $\mathbb{Z}_p^*$ *and*

iii) $|T(y, v, u)|_p = O\left(\left|\dfrac{y^{k-2}}{p^M}\right|_p\right)$ *as* $|y|_p \to \infty$.

Then there exists $K > 0$ such that for all a and N,

$$|\mu_f^{(\alpha)}(a + (p^N)) - H(a)\mu_{\alpha,1}(a + (p^N))|_p \leq K/p^N$$

.

Proof. We adopt the same notation of the proof of Theorem 2.1. We set $y = (a/p^N)/p^M$, $u = j/p^M$ and $v = (j' - [\alpha b])/(\alpha p^M)$. So the hypothesis on y, u and v are clearly satisfied. Then

$$\begin{aligned}
\mu_f^{(\alpha)}(a+(p^N)) &= p^{N(k-1)} \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} \left[f\left(\frac{(a/p^N)+j}{p^M}\right) - f\left(\frac{\{\alpha a/p^N\}+j}{\alpha p^M}\right)\right] \\
&= \lim_{M\to\infty} p^{(M+N)(k-1)} h(a/p^{M+N}) \sum_{j=0}^{p^M-1} \frac{j-(j'-[\alpha a/p^N])/\alpha}{p^M} + p^{N(k-1)} \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} T(y,u,v) \\
&= H(a)\mu_{\alpha,1}(a+(p^N)) + p^{N(k-1)} \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} T(y,u,v).
\end{aligned}$$

Now we use the hypothesis iii) and have that

$$\begin{aligned}
|\mu_f^{(\alpha)}(a+(p^N)) - H(a)\mu_{\alpha,1}(a+(p^N))|_p &= \left| p^{N(k-1)} \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} T(y,u,v)\right|_p \\
&\leq \frac{1}{p^{N(k-1)}} \lim_{M\to\infty} \frac{1}{p^{M(k-1)}} p^M K |a^{k-2}|_p p^{(M+N)(k-2)} \\
&= K/p^N
\end{aligned}$$

as desired. ∎

An immediate consequence of Theorem 2.2 is the relation between integrals against $\mu_f^{(\alpha)}$ and Mazur's measure $\mu_{\alpha,1}$.

Corollary 2.1. *Let f be given as in Theorem 2.2 and suppose that the corresponding function $H(x)$ defined in Theorem 2.2 is integrable against $\mu_{\alpha,1}$. Let $g(x)$ be a continuous function on $\mathbb{Z}_p^*$. Then*

$$\int_{\mathbb{Z}_p^*} g(x)\ d\mu_f^{(\alpha)} = \int_{\mathbb{Z}_p^*} g(x)H(x)\ d\mu_{\alpha,1}$$

.

Proof. By definition

$$\int_{\mathbb{Z}_p^*} g(x)\ d\mu_f^{(\alpha)} = \lim_{N\to\infty} S_N, \text{ where } S_N = \sum_{a=0}^{p^N-1} g(a)\mu_f^{(\alpha)}(a+(p^N))$$

$$\int_{\mathbb{Z}_p^*} g(x)H(x)\ d\mu_{\alpha,1} = T_N, \text{ where } T_N = \sum_{a=0}^{p^N-1} g(a)H(a)\mu_{\alpha,1}(a+(p^N))$$

Now we see that

$$\begin{aligned}
|S_N - T_N|_p &= \left| \sum_{a=0}^{p^N-1} g(a)\left[\mu_f^{(\alpha)}(a+(p^N)) - H(a)\mu_{\alpha,1}(a+(p^N))\right]\right|_p \\
&\leq \max_{0\leq a\leq p^N-1} \left| g(a)\left[\mu_f^{(\alpha)}(a+(p^N)) - H(a)\mu_{\alpha,1}(a+(p^N))\right]\right|_p
\end{aligned}$$

But since $g(x)$ is continuous on $\mathbb{Z}_p$ it is also bounded, say $g(x) \leq M$. So by Theorem 2.2 there exists a constant K such that

$$|S_N - T_N|_p \leq \frac{KM}{p^N} \implies |S_N - T_N|_p \to 0 \text{ as } N \to \infty$$

∎

3. Examples

We devote this section to examples. We may omit some calculations for which many of them follow by considering the power series expansion of $\log_p(x)$.

Example 3.1. Let $f(x) = x^k$ (or any monic polynomial of degree k). Then

$$\begin{aligned}
F_\alpha(x) &= \lim_{M\to\infty} p^{M(k-1)} \sum_{j=0}^{p^M-1} \left(\frac{x+j}{\alpha p^M}\right)^k = \alpha^{-k} \lim_{M\to\infty} \frac{1}{p^M} \sum_{j=0}^{p^M-1} (x+j)^k \\
&= \alpha^{-k} \lim_{M\to\infty} \frac{1}{p^M} \sum_{j=0}^{p^M-1} \sum_{l=0}^{k} \binom{k}{l} x^{k-l} j^l = \alpha^{-k} \lim_{M\to\infty} \frac{1}{p^M} \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \frac{B_{l+1}(p^M) - B_{l+1}(0)}{l+1} \\
&= \alpha^{-k} \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \frac{1}{l+1} \lim_{M\to\infty} \frac{B_{l+1}(p^M) - B_{l+1}(0)}{p^M} = \alpha^{-k} \sum_{l=0}^{k} \binom{k}{l} x^{k-l} \frac{1}{l+1} \frac{dB_{l+1}(x)}{dx}\Big|_{x=0} \\
&= \alpha^{-k} \sum_{l=0}^{k} \binom{k}{l} x^{k-l} B_l(0) = \alpha^{-k} B_k(x)
\end{aligned}$$

So we have that

$$\mu_f^{(\alpha)}(a + (p^N)) = p^{N(k-1)} \left[B_k(a/p^N) - \alpha^{-k} B_k(\{\alpha a/p^N\})\right].$$

Note that by Proposition 2.1, $\mu_f^{(\alpha)}$ is a p-adic distribution. We also see that

$$f(y+u) - f(y) = kuy^{k-1} + O(y^{k-2}).$$

So for any $u \in \mathbb{Z}_p$ and $|y|_p > 1$, we obtain

$$|f(y+u) - f(y)|_p \le |y^{k-1}|_p.$$

Hence $f(x)$ satisfies the hypothesis of Theorem 2.1 and so $\mu_f^{(\alpha)}$ is a p-adic measure. Indeed $\mu_f^{(\alpha)}$ is the k-th Bernoulli measure. We may go further and use Theorem 2.2. We compute that

$$h(y) = ky^{k-1}, \ \ T(y,u,k) = y^{k-2} + \text{ (terms with lower degree in } y)$$

so that $f(x)$ satisfies the hypothesis of Theorem 2.2. Also $H(x) = kx^{k-1}$, so by Corollary 2.1 we have that

$$\int_{\mathbb{Z}_p^*} g(x)\, d\mu_{\alpha,k} = \int_{\mathbb{Z}_p^*} kg(x)x^{k-1}\, d\mu_{\alpha,1}.$$

Example 3.2. Let $f(x) = \log_p(x)$. Recall that $\log_p(p) = 0$. It is again easy to check that $f(x) = \log_p(x)$ satisfies the hypothesis of Theorem 2.1 with $k = 0$;

$$\begin{aligned}
f(y+u) - f(y) = \log_p(y+u) - \log_p(y) &= \log_p\left(1 + \frac{u}{y}\right) \\
\implies |f(y+u) - f(y)|_p &= |u/y|_p \le |y^{-1}|_p
\end{aligned}$$

Now for the corresponding p-adic distribution $\mu_{f,\alpha}$ we have that

$$\begin{aligned}
\mu_{f,\alpha}(\alpha U) &= \frac{1}{p^N} \lim_{M\to\infty} \frac{1}{p^M} \sum_{j=0}^{p^M-1} \log_p\left(\frac{\{\alpha a/p^N\} + j}{\alpha p^M}\right) \\
&= \frac{1}{p^N} \lim_{M\to\infty} \sum_{j=0}^{p^M-1} \frac{1}{p^M} \left(\log_p\left(\frac{\{\alpha a/p^N\} + j}{p^M}\right) - \log_p(\alpha)\right) \\
&= \mu_{f,1}(\alpha U) - \frac{\log_p(\alpha)}{p^N}
\end{aligned}$$

So we obtain that

$$\begin{aligned}\mu_f^{(\alpha)}(U) &= \mu_{f,1}(U) - \mu_{f,\alpha}(\alpha U)\\ &= \mu_{f,1}(U) - \mu_{f,1}(\alpha U) + \frac{\log_p(\alpha)}{p^N}.\end{aligned}$$

But this is exactly the same measure denoted by $\nu_{1,\alpha}$ which is the regularization of $\nu_{G,1}$ in the notation of [4]. Also we have that $|\mu_f^{(\alpha)}(U)|_p \leq 1$ which follows as $K = 1$ in the notation of Theorem 2.1. Also in the notation of Theorem 2.2 we see that $H(x) = 1/x$.

Example 3.3. Now let $f(x) = x\log_p x - x$. Then

$$\begin{aligned}f(y+u) - f(y) &= (y+u)\log_p(y+u) - (y+u) - y\log_p(y) + y\\ &= y\log_p(1+u/y) + u(\log_p(y+u) - 1)\end{aligned}$$

The first term is bounded by 1. Now let $y = 1/(vp^r)$ where $|v|_p = 1$ and $r \in \mathbb{Z}_{\geq 1}$. Then $\log_p(y+u) = \log_p(1+uvp^r)$, and so the second term is also bounded by 1. Hence we have $k = 1$ and $K = 1$ in Theorem 2.1. Then it follows that

$$\begin{aligned}\mu_f^{(\alpha)}(U) &= \mu_{f,1}(U) - \mu_{f,\alpha}(\alpha U)\\ &= \mu_{f,1}(U) - \frac{1}{\alpha}\mu_{f,1}(\alpha U) + \frac{\log_p(\alpha)}{\alpha}\left(\left\{\frac{\alpha a}{p^N}\right\} - \frac{1}{2}\right).\end{aligned}$$

In this case we obtain the p-adic Gamma measure $v_{0,\alpha}$ of J. Diamond [4]. Also if we carry on the calculations of Theorem 2.2 with $k = 1$ we see that $h(x) = H(x) = \log_p(x)$.

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