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ON THE STABILITY OF A THIRD ORDER DIFFERENCE EQUATION

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Abstract

In this paper, we are investigated the equilibrium points of difference equation $x_{n+1} = x_{n-1}x_{n-2} + A$, where A is a positive real number and the initial conditions are positive. We are also studied the local asymptotic stability of related difference equation. Particularly, we are examined the convergence of solutions of related equation.

Keywords: Difference equations, stability, convergence

ÜÇÜNCÜ DERECEDEN BİR FARK DENKLEMİNİN KARARLILIĞI ÜZERİNE

Öz

Bu çalışmada $x_{n+1} = x_{n-1}x_{n-2} + A$ fark denkleminin A pozitif bir reel sayı ve başlangıç koşulları pozitif iken denge noktaları incelendi. Ayrıca ilgili fark denkleminin lokal asimptotik kararlılığı çalışıldı. Özellikle ilgili denklemin çözümlerinin yakınsaklığı incelendi.

Anahtar Kelimeler: Fark denklemleri, kararlılık, yakınsaklık

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1. Introduction and Preliminaries

On account of the fact that many mathematical models need to discrete variables, the difference equations known to be composed of discrete variables have been studied frequently by many mathematicians in recent years. Particularly, stability analysis and convergence of solutions of difference equations has been huge interest between researchers.

In [6], Kent et al investigated the boundedness of solutions, periodicity of solutions, and existence of unbounded solutions of following difference equation

$$x_{n+1} = x_n x_{n-1} - 1. \tag{1}$$

Additionally, in [10] Liu et al and in [19] Wang et al studied some properties of solutions of related difference equation.

In [2], Amleh et al handled the stability of solutions and existence of bounded solutions of difference equations

$$x_{n+1} = x_n x_{n-1} + \alpha. \tag{2}$$

Moreover, in [15], Taşdemir et al studied existence of periodic solutions of Eq.(2).

In [7], Kent et al investigated the periodicity of solutions, existence of bounded or unbounded of solutions and stability of solutions of difference equation

$$x_{n+1} = x_{n-1} x_{n-2} - 1.$$

In this paper we investigate the convergence of solutions of following difference equations

$$x_{n+1} = x_{n-1} x_{n-2} + A, \tag{3}$$

where A is a positive real number and the initial conditions are positive. We also investigate the local asymptotic stability of difference equation (3). Particularly, we examine the convergence of solutions of related equation.

Definition 1 A difference equation of order third is an equation of the form

$$x_{n+1} = f\left(x_n, x_{n-1}, x_{n-2}\right),\ n = 0,1,\ldots \tag{4}$$

where f is a function that maps some set I^3 into I. The set I is usually an interval of real numbers, or a union of intervals, or a discrete set such as the set of integers $\mathbb{Z} = \{\ldots,-1,0,1,\ldots\}$.

A solution of Eq.(4) is a sequence $\{x_n\}_{n=-2}^{\infty}$ that satisfies Eq.(4) for all $n \geq 0$.

A solution of Eq.(4) that is constant for all $n \geq -2$ is called an *equilibrium solution* of Eq.(4). If

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$$x_n = \bar{x}, \text{ for all } n \geq -2$$

is an equilibrium solution of Eq.(4), then $\bar{x}$ is called an *equilibrium point*, or simply an *equilibrium* of Eq.(4).

Definition 2 (Linearized Equation) Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}\left(\bar{x}, \bar{x}, \ldots, \bar{x}\right), \text{ for } i = 0,1,\ldots,k$$

denote the partial derivative of $f\left(u_0, u_1, \ldots, u_k\right)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(4).

The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \ldots + q_k z_{n-k},\ k = 0,1,\ldots, \tag{5}$$

is called the linearized equation of Eq.(4) about the equilibrium point $\bar{x}$.

Definition 3 (Characteristic Equation) The equation

$$\lambda^{k+1} - q_0 \lambda^k - q_1 \lambda^{k-1} - \ldots - q_{k-1}\lambda - q_k = 0 \tag{6}$$

is called the characteristic equation of Eq.(5) about $\bar{x}$.

Definition 4 (Stability) Let $\bar{x}$ an equilibrium point of Eq.(4).

An equilibrium point $\bar{x}$ of Eq.(4) is called *locally stable* if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-2}^{\infty}$ is a solution of Eq.(4) with

$$\left|x_{-2} - \bar{x}\right| + \left|x_{-1} - \bar{x}\right| + \left|x_0 - \bar{x}\right| < \delta,$$

then

$$\left|x_n - \bar{x}\right| < \varepsilon, \text{ for all } n \geq -2.$$

An equilibrium point $\bar{x}$ of Eq.(4) is called *locally asymptotically stable* if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-2}^{\infty}$ is a solution of Eq.(4) with

$$\left|x_{-2} - \bar{x}\right| + \left|x_{-1} - \bar{x}\right| + \left|x_0 - \bar{x}\right| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

An equilibrium point $\bar{x}$ of Eq.(4) is called a *global attractor* if, for every solution $\{x_n\}_{n=-2}^{\infty}$ of Eq.(4), we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

An equilibrium point $\bar{x}$ of Eq.(4) is called *globally asymptotically stable* if it is locally stable, and a global attractor.

An equilibrium point $\bar{x}$ of Eq.(4) is called *unstable* if it is not locally stable.

Theorem 5 (Clark Theorem see ([3], p.6)) Assume that $q_0, q_1, \ldots, q_k$ are real numbers such that

$$|q_0| + |q_1| + \ldots + |q_k| < 1.$$

Then all roots of Eq.(6) lie inside the unit disk.

2. Main Results

Firstly, we find out the equilibrium points of Eq.(3). The next lemma gives the equilibrium points of related difference equation.

Lemma 6 Eq.(3) has two equilibrium points such that

$$\bar{x}_1 = \frac{1+\sqrt{1-4A}}{2}, \tag{7}$$

$$\bar{x}_2 = \frac{1-\sqrt{1-4A}}{2}. \tag{8}$$

Lemma 7 The equilibrium points of Eq.(3) have three conditions as follows:

i. If $A < \frac{1}{4}$ then the equilibrium points $\bar{x}_1$ and $\bar{x}_2$ are both different and real numbers.

ii. If $A = \frac{1}{4}$ then the equilibrium points $\bar{x}_1$ and $\bar{x}_2$ are equal and $\bar{x}_{1,2} = \frac{1}{2}$.

iii. If $A > \frac{1}{4}$ then the equilibrium points $\bar{x}_1$ and $\bar{x}_2$ are complex numbers.

Now, we investigate the local stability analysis of real equilbrium points of Eq.(3). Firstly, we examine the linearized equation of Eq.(3) about equilibrium point $\bar{x}$.

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Theorem 8 The linearized equation of Eq.(3) about equilibrium point $\bar{x}$ is

$$z_{n+1} - \bar{x}z_{n-1} - \bar{x}z_{n-2} = 0. \tag{9}$$

Proof. We consider Eq.(4). Thus we have the following partial derivatives at the equilibrium point $\bar{x}$ of Eq.(4):

$$q_0 = \frac{\partial f}{\partial x_n}\left(\bar{x},\bar{x},\bar{x}\right) = 0,$$

$$q_1 = \frac{\partial f}{\partial x_{n-1}}\left(\bar{x},\bar{x},\bar{x}\right) = \bar{x},$$

$$q_2 = \frac{\partial f}{\partial x_{n-2}}\left(\bar{x},\bar{x},\bar{x}\right) = \bar{x}.$$

Therefore we obtain the linearized equation of Eq.(3) about equilibrium point $\bar{x}$ is $z_{n+1} - \bar{x}z_{n-1} - \bar{x}z_{n-2} = 0$ as desired.

Lemma 9 The characteristic equation of Eq.(9) about equilibrium point $\bar{x}$ is

$$\lambda^3 - \bar{x}\lambda - \bar{x} = 0. \tag{10}$$

Note that the next theorem shows that the equilibrium point $\bar{x}_1$ of Eq.(3) is unstable.

Theorem 10 If $A < \frac{1}{4}$ then the equilibrium point $\bar{x}_1 = \frac{1+\sqrt{1-4A}}{2}$ of Eq.(3) is unstable.

Proof. The characteristic equation of Eq.(9) about equilibrium point $\bar{x}_1 = \frac{1+\sqrt{1-4A}}{2}$ is

$$\lambda^3 - \frac{1+\sqrt{1-4A}}{2}\lambda - \frac{1+\sqrt{1-4A}}{2} = 0.$$

Hence we have

$$2\lambda^3 - \left(1+\sqrt{1-4A}\right)\lambda - \left(1+\sqrt{1-4A}\right) = 0.$$

We consider a polynomial such that

$$P(\lambda) = 2\lambda^3 - \left(1+\sqrt{1-4A}\right)\lambda - \left(1+\sqrt{1-4A}\right).$$

Thus we obtain the followings:

$$P(1) = -2\sqrt{1-4A} < 0,$$

$$P(1+\sqrt{1-4A}) > 0.$$

Therefore $P(1) < 0 < P(1+\sqrt{1-4A})$ and we have $P(\lambda) = 0$ such that $1 < \lambda < 1+\sqrt{1-4A}$. So, absolute value of at least one root of characteristic equation is greater than 1. Hence, equilibrium point $\bar{x}_1 = \frac{1+\sqrt{1-4A}}{2}$ of Eq.(3) are unstable.

The following theorem shows that the equilibrium point $\bar{x}_2$ of Eq.(3) is locally asymptotically stable.

Theorem 11 Assume that $A<\frac{1}{4}$, then the equilibrium point $\bar{x}_2=\frac{1-\sqrt{1-4A}}{2}$ of Eq.(3) is locally asymptotically stable.

Proof. The characteristic equation of Eq.(9) about equilibrium point $\bar{x}_2=\frac{1-\sqrt{1-4A}}{2}$ is

$$\lambda^3-\frac{1-\sqrt{1-4A}}{2}\lambda-\frac{1-\sqrt{1-4A}}{2}=0.$$

Now we consider the Clark Theorem (see Theorem 5). According to this Theorem, if $|q_0|+|q_1|+...+|q_k|<1$ then absolute values of all roots of the characteristic equation of Eq.(9) about equilibrium point $\bar{x}$ is less than 1. Let $A<\frac{1}{4}$. Hence

$$|q_0|+|q_1|+|q_2|=|0|+\left|\frac{1-\sqrt{1-4A}}{2}\right|+\left|\frac{1-\sqrt{1-4A}}{2}\right|=\left|1-\sqrt{1-4A}\right|.$$

If $A<\frac{1}{4}$, then we obtain $0<1-\sqrt{1-4A}<1.$ So

$$|q_0|+|q_1|+|q_2|=\left|1-\sqrt{1-4A}\right|<1.$$

According to Clark Theorem, the equilibrium point $\bar{x}_2=\frac{1-\sqrt{1-4A}}{2}$ of Eq.(3) is locally asymptotically stable.

Lemma 12 Let A is a positive real number and the initial conditions are positive. Then $x_n\in(0,\infty)$ for all $n\geq -2.$

Lemma 13 Let $f\in C\left((0,\infty)^2\to(0,\infty)\right)$ is a function such that $f(x,y)=xy+A.$ If $\bar{x}_2<x<\bar{x}_1$ then $f(x,x)<x.$

The following theorem shows that every solutions of Eq.(11) with $x_{-2},x_{-1},x_0\in\left(\bar{x}_2,\bar{x}_1\right)$ converges to equilibrium point $\bar{x}_2$.

Theorem 14 Assume that $A<\frac{1}{4}$ and $f\in C\left((0,\infty)^2\to(0,\infty)\right)$ increases in both variables and that the difference equation

$$x_{n+1}=f\left(x_{n-1},x_{n-2}\right),\ n=0,1,... \tag{11}$$

has two consecutive equilibrium points $\bar{x}_1$ and $\bar{x}_2$ with $\bar{x}_2<\bar{x}_1$. If the initial conditions in $\left(\bar{x}_2,\bar{x}_1\right)$ then every solutions $\{x_n\}_{n=-2}^{\infty}$ of Eq.(11) with $x_{-2},x_{-1},x_0\in\left(\bar{x}_2,\bar{x}_1\right)$ converges to equilibrium point $\bar{x}_2$.

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Proof. Firstly we know from Lemma 13, $f(x,x) < x$, for $\bar{x}_2 < x < \bar{x}_1$. Then we choose a number t_0 such that $t_0 = \max\{x_{-2}, x_{-1}, x_0\}$ and let $\{t_n\}_{n=0}^{\infty}$ be the unique solution of the difference equation

$$t_{n+1} = f(t_n, t_n) = t_n^2 + A, n = 0,1,... \tag{12}$$

with initial condition t_0. Hence, we have from Eq.(11) and (12),

$$x_1 = f(x_{-1}, x_{-2}) \leq f(t_0, t_0) = t_1 < t_0,$$
$$x_2 = f(x_0, x_{-1}) \leq f(t_0, t_0) = t_1 < t_0,$$
$$x_3 = f(x_1, x_0) \leq f(t_0, t_0) = t_1 < t_0,$$
$$x_4 = f(x_2, x_1) \leq f(t_1, t_1) = t_2 < t_1 < t_0,$$
$$\vdots$$

Therefore we obtain that

$$\overline{\lim_{n\to\infty}} x_n \leq \lim_{n\to\infty} t_n.$$

Now we assume that $\lim_{n\to\infty} t_n = t$, then we have from (12), $t = \bar{x}_1$ or $t = \bar{x}_2$. Additionally we know that $\bar{x}_2 < \bar{x}_1$ and $\{t_n\}_{n=0}^{\infty}$ is a decreasing sequence. So, t_n converges to $\bar{x}_2$ as desired.

Theorem 15 If $A = \frac{1}{4}$ then the equilibrium point $\bar{x} = \frac{1}{2}$ of Eq.(3) is unstable.

Proof. Now we consider the characteristic equation of Eq.(9) about equilibrium point $\bar{x} = \frac{1}{2}$. Thus we have

$$\lambda^3 - \frac{1}{2}\lambda - \frac{1}{2} = 0. \tag{13}$$

Therefore we obtain the three roots of characteristic equation (13) as follows:

$$\lambda_1 = 1, \lambda_{2,3} = \frac{1+i}{2}.$$

Hence, we get $|\lambda_{2,3}| < 1 = |\lambda_1|$. So, equilibrium point $\bar{x} = \frac{1}{2}$ of Eq.(3) is unstable.

Example 16 Let $A = 0.2$ and the initial values $x_{-2} = 0.71, x_{-1} = 0.65$ and $x_0 = 0.68$. Then, every solutions of Eq.(3) converges to equilibrium point $\bar{x}_2 = 0.27639$. The following figure shows the first 100 terms of Eq.(3) and verifies to results of Theorem 14.

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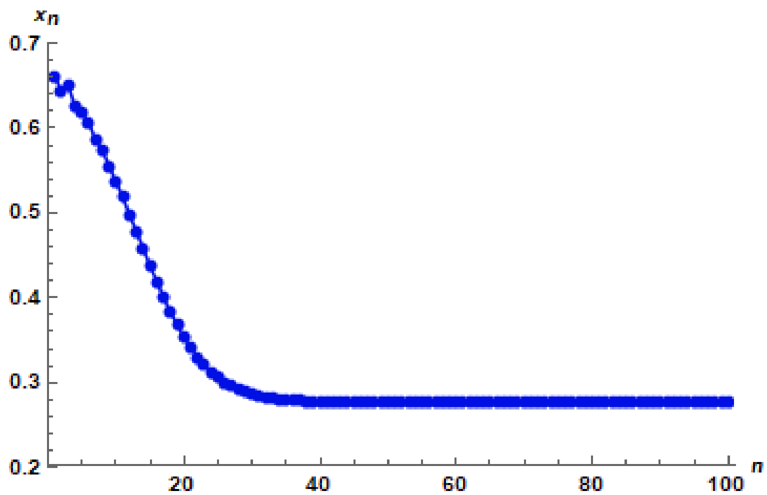

Figure 1 Plot of Eq.(3).

3. Conclusion

In this paper, we examine the equilibrium points of Eq.(3). We also find out that if $A < \frac{1}{4}$, then the equilibrium point $\overline{x}_1$ of Eq.(3) is unstable. Moreover, we reveal that if $A < \frac{1}{4}$, then the equilibrium point $\overline{x}_2$ of Eq.(3) is locally asymptotically stable and every solutions of Eq.(11) with $x_{-2}, x_{-1}, x_0 \in (\overline{x}_2, \overline{x}_1)$ converges to equilibrium point $\overline{x}_2$. Additionally, we obtain that if $A = \frac{1}{4}$ then the equilibrium point $\overline{x} = \frac{1}{2}$ of Eq.(3) is unstable. Finally we give an example for verifies to our results.

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