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On A Class of Fractional Differential Equations with Arbitrary Singularities

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Abstract

In this paper, we consider a class of singular fractional differential equations such that its right hand side has an arbitrary singularity on certain interval of the real axis. We obtain new results on the existence and uniqueness of solutions using some classical fixed point theorems.

Keywords: *Caputo derivative, differential equation, existence, fixed point, singularity, uniqueness.*
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1. Introduction

The fractional differential equations theory has attracted great attention during the recent years. This is because this new theory arises in many engineering and scientific disciplines such as mechanics, chemistry, biology, economics, control theory and signal processing, see [4, 5, 9, 10, 11]. Many authors investigated the existence and uniqueness of solutions for nonlinear fractional differential equations. We refer the reader to [1, 2, 3, 6, 7, 8, 12] for further information and applications.
On the other hand, the singular fractional differential problems are quite significant in realistic problems [11, 13, 14]. In [1], Z. Dahmani and M.Z. Sarikaya considered a Lane-Emden coupled fractional differential equations. They discussed the existence and uniqueness of solutions and some Ulam stabilities for the following problem:

$$\left\{\begin{array}{l} {}^cD^{\beta_1}({}^cD^{\alpha_1}+b_1g_1(t))x_1(t)+f_1(t,x_1(t),x_2(t))=h_1(t),t\in[0,1] \\ {}^cD^{\beta_2}({}^cD^{\alpha_2}+b_2g_2(t))x_2(t)+f_2(t,x_1(t),x_2(t))=h_2(t),t\in[0,1] \end{array}\right. \tag{1.1}$$
$$x_k(0) = 0,\ {}^cD^{\alpha_k}x_k(1)+b_kg_k(1)x_k(1)=0.\ k=1,2$$

where $0<\alpha_k<1, 0<\beta_k<1,\ k=1,2$, the derivatives D^{β_k} and D^{α_k} are in the sense of Caputo.
In this paper, we are concerned with a class of singular fractional differential equations with an arbitrary singularity $T_0\in J$. So, let us consider:

$$\left\{\begin{array}{l} {}^cD^{\beta_1}({}^cD^{\alpha_1}+b_1g_1(t))x_1(t)+f_1(t,x_1(t),x_2(t),{}^cD^{\gamma_1}x_1(t))=h_1(t),t\in J \\ {}^cD^{\beta_2}({}^cD^{\alpha_2}+b_2g_2(t))x_2(t)+f_2(t,x_1(t),x_2(t),{}^cD^{\gamma_2}x_2(t))=h_2(t),t\in J \end{array}\right. \tag{1.2}$$

(1.3)

$$x_k(0) = 0,\ {}^cD^{\alpha_k}x_k(T)+b_kg_k(T)x_k(T)=\int_0^T G_k(\tau)x_k(\tau)d\tau,\ k=1,2 \tag{1.4}$$

where $0<\gamma_k<\alpha_k<1, 0<\beta_k<1,\ k=1,2$ and $t\in J:=[0,T],T>0$, the derivatives D^{β_k}, $D^{\alpha_k}$and D^{γ_k} are in the sense of Caputo. The functions $g_k: J-\{T_0\}\to\mathbb{R}$ are supposed singular at $T_0\in J$ and the functions $f_k: J\times\mathbb{R}^3\to\mathbb{R}$ and $h_k: J\to\mathbb{R}$ will be specified later.

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2. Fractional Calculus Preliminaries

In this section, we present some preliminaries that we need to prove the main results [10, 14].

Definition 2.1. *The Riemann-Liouville fractional integral operator of order* $\alpha > 0$, *for a continuous function* f *on* $[0,\infty[$ *is defined as:*

$$J^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, & \alpha > 0, \\ f(t), & \alpha = 0 \end{cases} \tag{2.1}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx$.

Definition 2.2. *The Caputo derivative of order* α *for a function* $x : [0,\infty) \to \mathbb{R}$, *which is at least* $l-$*times differentiable can be defined as:*

$$D^{\alpha} x(t) = \frac{1}{\Gamma(l-\alpha)} \int_0^t (t-s)^{l-\alpha-1} x^{(l)}(s)\,ds = J^{l-\alpha} x^{(l)}(t), \tag{2.2}$$

for $l-1 < \alpha < l$, $l \in \mathbb{N}^*$.

Lemma 2.3. *For* $\alpha > 0$, *the general solution of the fractional differential equation* $D^{\alpha} x(t) = 0$ *is given by*

$$x(t) = \sum_{j=0}^{l-1} c_j t^j, \tag{2.3}$$

where $c_j \in \mathbb{R}$, $j = 0, ..., l-1$, $l = [\alpha] + 1$.

Lemma 2.4. *Let* $\alpha > 0$. *Then*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + \sum_{j=0}^{l-1} c_j t^j, \tag{2.4}$$

where $c_j \in \mathbb{R}$, $j = 0, 1, ..., l-1$, $l = [\alpha] + 1$.

Lemma 2.5. *Let* $q > p > 0$ *and* $v \in L^1([a,b])$. *Then*

$$D^p J^q v(t) = J^{q-p} v(t), t \in [a,b]. \tag{2.5}$$

Let us now prove the following lemma:

Lemma 2.6. *Suppose that* $(h_k)_{k=1}^2 \in C(J,\mathbb{R}), (f_k)_{k=1}^2 \in C(J \times \mathbb{R}^3, \mathbb{R})$ *and consider the problem:*

$$\begin{cases} {}^cD^{\beta_1}({}^cD^{\alpha_1} + b_1 g_1(t)) x_1(t) + f_1(t, x_1(t), x_2(t), {}^cD^{\gamma_1} x_1(t)) = h_1(t), & t \in J \\ {}^cD^{\beta_2}({}^cD^{\alpha_2} + b_2 g_2(t)) x_2(t) + f_2(t, x_1(t), x_2(t), {}^cD^{\gamma_2} x_2(t)) = h_2(t), & t \in J \end{cases}$$

$$0 < \gamma_k < \alpha_k < 1, 0 < \beta_k < 1, \quad k = 1,2$$

associated with the conditions:

$$\begin{aligned} x_k(0) &= 0\,, \; k = 1,2 \\ {}^cD^{\alpha_k} x_k(T) + b_k g_k(T) x_k(T) &= \int_0^T G_k(\tau) x_k(\tau)\,d\tau, \; k = 1,2. \end{aligned}$$

Then, 1.2 - 1.4 has a unique solution $(x_1, x_2)(t)$, *such that:*

$$\begin{aligned} x_k(t) = & \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \int_0^{\tau} \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} [h_k(s) - f_k(s,, x_1(s), x_2(s), {}^cD^{\gamma_k} x_k(s))]\,ds\,d\tau \\ & - \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} b_k g_k(\tau) x_k(\tau) d\tau + \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} G_k(\tau) x_k(\tau)\,d\tau - \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \left(\int_0^T \frac{(T-\tau)^{\beta_k-1}}{\Gamma(\beta_k)} [h_k(\tau) - f_k(\tau,, x_1(\tau), x_2(\tau), {}^cD^{\gamma_k} x_k(\tau))] \right), \; k = 1,2. \end{aligned} \tag{2.6}$$

Proof. We begin by applying Lemma 2.4 to 1.2. We can write:

$$(D^{\alpha_k} + b_k g_k(\tau)) x_k(\tau) = \int_0^{\tau} \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} (h_k(s)\,ds - f_k(s,, x_1(s), x_2(s), {}^cD^{\gamma_k} x_k(s)))\,ds - c^k,$$

where $k = 1,2$, $c^k \in \mathbb{R}$, and $0 < \beta_k < 1$.

By the same lemma, it yields that

$$x_k(t)=\int_0^t\frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)}\int_0^\tau\frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)}(h_k(s)ds-f_k(s,,x_1(s),x_2(s),{}^cD^{\gamma_k}x_k(s)))dsd\tau$$
$$-\int_0^t\frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)}b_kg_k(\tau)x_k(\tau)d\tau-\frac{c^kt^{\alpha_k}}{\Gamma(\alpha_k+1)}-c'^k, \tag{2.7}$$

where $k=1,2$ and $c'^k\in\mathbb{R}$.
Thanks to Lemma 2.5 and using the conditions 1.4 we obtain the values of c^k and c'^k. Substituting the last assertion in 2.7 we obtain 2.6. The proof of Lemma 2.6 is thus completed. □

Now, let us introduce the Banach space $(X\times X,\|(x_1,x_2)\|_{X\times X})$, with:

$$X:=\{x:x\in C(J,\mathbb{R}),{}^cD^\gamma x\in C(J,\mathbb{R})\},$$

endowed with the norm:

$$\|(x_1,x_2)\|_{X\times X}=\max(\|x_1\|_\infty,\|{}^cD^{\gamma_1}x_1\|_\infty,\|x_2\|_\infty,\|{}^cD^{\gamma_2}x_2\|_\infty);\ \|x\|_\infty=\sup_{t\in J}|x(t)|.$$

3. Formulation of Hypotheses

We begin by introducing the following hypotheses:
(H_1) : There exist nonnegative constants $(\mu_k)_j$, $j=1,2,3;k=1,2$, such that for all $t\in J$ and all (x_1,x_2,x_3), $(y_1,y_2,y_3)\in\mathbb{R}^3$, we have:

$$\begin{aligned}|f_k(t,x_1,x_2,x_3)-f_k(t,y_1,y_2,y_3)| &\le \sum_{j=1}^3(\mu_k)_j|x_j-y_j|\\ &\le L_k\sum_{j=1}^3|x_j-y_j|, L_k=\max_{j=1\dots3}\{(\mu_k)_j\}.\end{aligned} \tag{3.1}$$

(H_2) : For each $k=1,2$, the functions $f_k:J\times\mathbb{R}^3\longrightarrow\mathbb{R}$ and $h_k:J\to\mathbb{R}$ are continuous.
(H_3) : The functions $G_k:J\to\mathbb{R}$ are continuous and $\|G_k\|_\infty=\sup_{t\in J}|G_k(t)|=G_k, k=1,2$.
(H_4) : The functions $g_k:J-\{T_0\}\longrightarrow\mathbb{R}^+$ are continuous on $J-\{T_0\}$ and singular at T_0, i.e.: $\lim_{t\to T_0}g_k(t)=+\infty, k=1,2$.
(H_5) : For each $k=1,2$, there exists $\lambda_k, 0<\lambda_k<\alpha_k<1$, $w_k(t)=(t-T_0)^{\lambda_k}g_k(t)$ is continuous on J and $\|w_k\|_\infty=Sup_{t\in J}|w_k(t)|=Sup_{t\in J}\left|(t-T_0)^{\lambda_k}g_k(t)\right|=M_k$.
(H_6) : The functions $f_k:J\times\mathbb{R}^3\longrightarrow\mathbb{R}$ and $h_k:J\to\mathbb{R}$ are bounded respectively by S_k and R_k, i.e.:
For $k=1,2$, there exist $S_k>0,\forall t\in J,\forall u\in\mathbb{R}^3,|f_k(t,u)|\le S_k$ and there exist $R_k>0,\forall t\in J,|h_k(t)|\le R_k$.

Also, we set the following quantities:

$$\begin{aligned}W'_k\ &:\ =3L_k\left(\frac{T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}+\frac{T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}\right)+|b_k|M_k\frac{\Gamma(1-\lambda_k)}{\Gamma(\alpha_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\lambda_k}+\frac{G_k}{\Gamma(\alpha_k+1)}T^{\alpha_k+1},\\ W''_k\ &:\ =3L_k\left(\frac{T^{\alpha_k-\gamma_k+\beta_k}}{\Gamma(\alpha_k-\gamma_k+\beta_k+1)}+\frac{T^{\alpha_k-\gamma_k+\beta_k}}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k+1)}\right)+|b_k|M_k\frac{\Gamma(1-\lambda_k)}{\Gamma(\alpha_k-\gamma_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\gamma_k-\lambda_k}\\ &\quad+\frac{G_k}{\Gamma(\alpha_k-\gamma_k+1)}T^{\alpha_k-\gamma_k+1}\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}C_k\ :\ &=\frac{(S_k+R_k)(t_2^{\alpha_k+\beta_k}-t_1^{\alpha_k+\beta_k})}{\Gamma(\alpha_k+\beta_k+1)}+r|b_k|M_k\frac{\Gamma(1-\lambda_k)}{\Gamma(\alpha_k-\lambda_k+1)}\left((t_2-T_0)^{\alpha_k-\lambda_k}-(t_1-T_0)^{\alpha_k-\lambda_k}\right)\\ &+r\frac{G_k}{\Gamma(\alpha_k+1)}\left(t_2^{\alpha_k+1}-t_1^{\alpha_k+1}\right)+\frac{(S_k+R_k)(t_2^{\alpha_k+\beta_k}-t_1^{\alpha_k+\beta_k})}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}.\end{aligned} \tag{3.3}$$

3.1. Existence and Uniqueness of Solutions

The following main result is based on Banach contraction principle. We prove the following theorem:

Theorem 3.1. *Assume that $(H_i)_{i=1,2,...,5}$ are satisfied. Then, the system 1.2 - 1.4 has a unique solution on J provided that*

$$W_k \quad = \quad :\max\{W_k', W_k''\} < 1,\ k = 1,2. \tag{3.4}$$

Proof. We will proceed in two steps:
Step1: Define the nonlinear operator $T : X \times X \to X \times X$ by

$$T(x_1,x_2)(t) := (T_1(x_1,x_2)(t), T_2(x_1,x_2)(t)),\ t \quad \in \quad J, \tag{3.5}$$

where, for all $k = 1,2$,

$$\begin{aligned} T_k(x_1,x_2)(t) = \\ \int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} \int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)} [h_k(s) - f_k(s,,x_1(s),x_2(s),{}^cD^{\gamma_k}x_k(s))]dsd\tau \\ -\int_0^t \frac{(t-\tau)^{\alpha_k-1}}{\Gamma(\alpha_k)} b_k g_k(\tau)x_k(\tau)d\tau + \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \int_0^T G_k(\tau)x_k(\tau)d\tau \\ -\frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \int_0^T \frac{(T-\tau)^{\beta_k-1}}{\Gamma(\beta_k)} [h_k(\tau) - f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau))]d\tau. \end{aligned} \tag{3.6}$$

We need to prove that T is contractive. For all $t \in J, (x_1,x_2),(y_1,y_2) \in X \times X$, we can write:

$$\begin{aligned} T_k(x_1,x_2)(t) - T_k(y_1,y_2)(t) \\ = \frac{1}{\Gamma(\alpha_k+\beta_k)} \int_0^t (t-\tau)^{\alpha_k+\beta_k-1} [f_k(\tau,,y_1(\tau),y_2(\tau),{}^cD^{\gamma_k}y_k(\tau)) \\ -f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau))]d\tau \\ -\frac{b_k}{\Gamma(\alpha_k)} \int_0^t (t-\tau)^{\alpha_k-1} g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau + \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau \\ -\frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k)} \int_0^T (T-\tau)^{\beta_k-1} [f_k(\tau,,y_1(\tau),y_2(\tau),{}^cD^{\gamma_k}y_k(\tau)) \\ -f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau))]d\tau, k = 1,2. \end{aligned}$$

Hence,

$$\begin{aligned} |T_k(x_1,x_2)(t) - T_k(y_1,y_2)(t)| \\ \leq \Big| \frac{1}{\Gamma(\alpha_k+\beta_k)} \int_0^t (t-\tau)^{\alpha_k+\beta_k-1} [f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau)) \\ -f_k(\tau,,y_1(\tau),y_2(\tau),{}^cD^{\gamma_k}y_k(\tau))]d\tau| \\ + \left| \frac{b_k}{\Gamma(\alpha_k)} \int_0^t (t-\tau)^{\alpha_k-1} g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau \right| + \left| \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau \right| \\ + \Big| \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k)} \int_0^T (T-\tau)^{\beta_k-1} [f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau)) \\ -f_k(\tau,,y_1(\tau),y_2(\tau),{}^cD^{\gamma_k}y_k(\tau))]d\tau| \end{aligned}$$

Thanks to (H_1), we can write

$$\begin{aligned} |T_k(x_1,x_2)(t) - T_k(y_1,y_2)(t)| \\ \leq \left| \frac{L_k}{\Gamma(\alpha_k+\beta_k)} \int_0^t (t-\tau)^{\alpha_k+\beta_k-1} (|x_1-y_1| + |x_2-y_2| + |{}^cD^{\gamma_k}x_k - {}^cD^{\gamma_k}y_k|)(\tau)d\tau \right| \\ + \left| \frac{b_k}{\Gamma(\alpha_k)} \int_0^t (t-\tau)^{\alpha_k-1} g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau \right| + \left| \frac{t^{\alpha_k}}{\Gamma(\alpha_k+1)} \int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau \right| \\ + \left| \frac{L_k t^{\alpha_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k)} \int_0^T (T-\tau)^{\beta_k-1} (|x_1-y_1| + |x_2-y_2| + |{}^cD^{\gamma_k}x_k - {}^cD^{\gamma_k}y_k|)(\tau)d\tau \right|. \end{aligned}$$

Therefore,

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_\infty$$

$$\leq \frac{L_k}{\Gamma(\alpha_k+\beta_k)} \sup_{t\in[0,T]} \int_0^t (t-\tau)^{\alpha_k+\beta_k-1}\left(|x_1-y_1|+|x_2-y_2|+|{}^cD^{\gamma_k}x_k-{}^cD^{\gamma_k}y_k|\right)(\tau)d\tau$$

$$+\frac{|b_k|}{\Gamma(\alpha_k)} \sup_{t\in[0,T]} \int_0^t (t-\tau)^{\alpha_k-1}(\tau-T_0)^{-\lambda_k}\left|(\tau-T_0)^{\lambda_k}g_k(\tau)\right||x_k(\tau)-y_k(\tau)|\,d\tau$$

$$+\frac{1}{\Gamma(\alpha_k+1)} \sup_{t\in[0,T]} t^{\alpha_k}\int_0^T |G_k(\tau)|\,|x_k(\tau)-y_k(\tau)|\,d\tau$$

$$+\frac{L_k}{\Gamma(\alpha_k+1)\Gamma(\beta_k)} \sup_{t\in[0,T]} t^{\alpha_k}\int_0^T (T-\tau)^{\beta_k-1}\left(|x_1-y_1|+|x_2-y_2|+|{}^cD^{\gamma_k}x_k-{}^cD^{\gamma_k}y_k|\right)(\tau)d\tau.$$

By (H_3) and (H_5), we obtain

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_\infty$$

$$\leq \frac{3L_k}{\Gamma(\alpha_k+\beta_k)}\|(x_1-y_1),(x_2-y_2)\|_{X\times X} \sup_{t\in[0,T]} \int_0^t (t-\tau)^{\alpha_k+\beta_k-1}\,d\tau$$

$$+\frac{|b_k|}{\Gamma(\alpha_k)}M_k\|x_k-y_k\|_\infty \sup_{t\in[0,T]} \int_0^t (t-\tau)^{\alpha_k-1}(\tau-T_0)^{-\lambda_k}\,d\tau+\frac{T^{\alpha_k}}{\Gamma(\alpha_k+1)}TG_k\|x_k-y_k\|_\infty$$

$$+\frac{3L_kT^{\alpha_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k)}\|(x_1-y_1),(x_2-y_2)\|_{X\times X} \sup_{t\in[0,T]} \int_0^T (T-\tau)^{\beta_k-1}d\tau.$$

Hence,

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_\infty$$

$$\leq \left(\frac{3L_k}{\Gamma(\alpha_k+\beta_k+1)}T^{\alpha_k+\beta_k}+\frac{3L_k}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}T^{\alpha_k+\beta_k}\right)\|(x_1-y_1),(x_2-y_2)\|_{X\times X}$$

$$+\left(\frac{\Gamma(1-\lambda_k)|b_k|M_k}{\Gamma(\alpha_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\lambda_k}+\frac{G_k}{\Gamma(\alpha_k+1)}T^{\alpha_k+1}\right)\|x_K-y_K\|_\infty.$$

And then,

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_\infty$$

$$\leq \left[\left(\frac{3L_k}{\Gamma(\alpha_k+\beta_k+1)}+\frac{3L_k}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}\right)T^{\alpha_k+\beta_k}+\frac{\Gamma(1-\lambda_k)|b_k|M_k}{\Gamma(\alpha_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\lambda_k}+\frac{G_k}{\Gamma(\alpha_k+1)}T^{\alpha_k+1}\right]$$

$$\|(x_1-y_1),(x_2-y_2)\|_{X\times X}$$

Consequently,by 3.2 , for $k=1,2$

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_\infty \leq W_k'\|(x_1-y_1),(x_2-y_2)\|_{X\times X} \tag{3.7}$$

Step 2: For the derivative ${}^cD^{\gamma}T(x_1,x_2), k=1,2$, we can see that:

$$\begin{gathered}{}^cD^{\gamma}T(x_1,x_2)(t):=\\ \left({}^cD^{\gamma_1}T_1(x_1,x_2)(t),{}^cD^{\gamma_2}T_2(x_1,x_2)(t)\right),t\in J,k=1,2\end{gathered} \tag{3.8}$$

For $k=1,2$, we can write

$$\begin{gathered}{}^cD^{\gamma_k}T_k(x_1,x_2)(t)=\\ \int_0^t \frac{(t-\tau)^{\alpha_k-\gamma_k-1}}{\Gamma(\alpha_k-\gamma_k)}\int_0^\tau \frac{(\tau-s)^{\beta_k-1}}{\Gamma(\beta_k)}[h_k(s)-f_k(s,,x_1(s),x_2(s),{}^cD^{\gamma_k}x_k(s))]dsd\tau\\ -\int_0^t \frac{(t-\tau)^{\alpha_k-\gamma_k-1}}{\Gamma(\alpha_k-\gamma_k)}b_kg_k(\tau)x_k(\tau)d\tau+\frac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}\int_0^T G_k(\tau)x_k(\tau)d\tau\\ -\frac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}\int_0^T \frac{(T-\tau)^{\beta_k-1}}{\Gamma(\beta_k)}[h_k(\tau)-f_k(\tau,,x_1(\tau),x_2(\tau),{}^cD^{\gamma_k}x_k(\tau))]d\tau.\end{gathered} \tag{3.9}$$

We need to prove that ${}^{c}D^{\gamma}T$ is contractive. For all $t\in J,(x_1,x_2),(y_1,y_2)\in X\times X$ and each $k=1,2$ we have:

$$
\begin{aligned}
&{}^{c}D^{\gamma_k}T_k(x_1,x_2)(t)-{}^{c}D^{\gamma_k}T_k(y_1,y_2)(t)\\
&=\tfrac{1}{\Gamma(\alpha_k-\gamma_k+\beta_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k+\beta_k-1}\left(f_k(\tau,,y_1(\tau),y_2(\tau),{}^{c}D^{\gamma_k}y_k(\tau))\right.\\
&\left.-f_k(\tau,,x_1(\tau),x_2(\tau),{}^{c}D^{\gamma_k}x_k(\tau))\right)d\tau\\
&-\tfrac{b_k}{\Gamma(\alpha_k-\gamma_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k-1}g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau+\tfrac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}\int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau\\
&-\tfrac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k)}\int_0^T(T-\tau)^{\beta_k-1}\left(f_k(\tau,,y_1(\tau),y_2(\tau),{}^{c}D^{\gamma_k}y_k(\tau))\right.\\
&\left.-f_k(\tau,,x_1(\tau),x_2(\tau),{}^{c}D^{\gamma_k}x_k(\tau))\right)d\tau.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|{}^{c}D^{\gamma_k}T_k(x_1,x_2)(t)-{}^{c}D^{\gamma_k}T_k(y_1,y_2)(t)|\\
&\leq\left|\tfrac{1}{\Gamma(\alpha_k-\gamma_k+\beta_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k+\beta_k-1}[f_k(\tau,,x_1(\tau),x_2(\tau),{}^{c}D^{\gamma_k}x_k(\tau))\right.\\
&\left.-f_k(\tau,,y_1(\tau),y_2(\tau),{}^{c}D^{\gamma_k}y_k(\tau))]d\tau\right|\\
&+\left|\tfrac{b_k}{\Gamma(\alpha_k-\gamma_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k-1}g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau\right|+\left|\tfrac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}\int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau\right|\\
&+\left|\tfrac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k)}\int_0^T(T-\tau)^{\beta_k-1}[f_k(\tau,,x_1(\tau),x_2(\tau),{}^{c}D^{\gamma_k}x_k(\tau))\right.\\
&\left.-f_k(\tau,,y_1(\tau),y_2(\tau),{}^{c}D^{\gamma_k}y_k(\tau))]d\tau\right|
\end{aligned}
$$

The hypothesis (H_1) allow us to write

$$
\begin{aligned}
&|{}^{c}D^{\gamma_k}T_k(x_1,x_2)(t)-{}^{c}D^{\gamma_k}T_k(y_1,y_2)(t)|\\
&\leq\tfrac{L_k}{\Gamma(\alpha_k-\gamma_k+\beta_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k+\beta_k-1}(|x_1-y_1|+|x_2-y_2|+|{}^{c}D^{\gamma_k}x_k-{}^{c}D^{\gamma_k}y_k|)(\tau)d\tau\\
&+\left|\tfrac{b_k}{\Gamma(\alpha_k-\gamma_k)}\int_0^t(t-\tau)^{\alpha_k-\gamma_k-1}g_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau\right|+\left|\tfrac{t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}\int_0^T G_k(\tau)[x_k(\tau)-y_k(\tau)]d\tau\right|\\
&+\left|\tfrac{L_k t^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k)}\int_0^T(T-\tau)^{\beta_k-1}(|x_1-y_1|+|x_2-y_2|+|{}^{c}D^{\gamma_k}x_k-{}^{c}D^{\gamma_k}y_k|)(\tau)d\tau\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|{}^{c}D^{\gamma_k}T_k(x_1,x_2)-{}^{c}D^{\gamma_k}T_k(y_1,y_2)\|_\infty\\
&\leq\tfrac{L_k}{\Gamma(\alpha_k-\gamma_k+\beta_k)}\sup_{t\in[0,T]}\int_0^t(t-\tau)^{\alpha_k-\gamma_k+\beta_k-1}(|x_1-y_1|+|x_2-y_2|+|{}^{c}D^{\gamma_k}x_k-{}^{c}D^{\gamma_k}y_k|)(\tau)d\tau\\
&+\tfrac{|b_k|}{\Gamma(\alpha_k-\gamma_k)}\sup_{t\in[0,T]}\int_0^t(t-\tau)^{\alpha_k-\gamma_k-1}(\tau-T_0)^{-\lambda_k}\left|(\tau-T_0)^{\lambda_k}g_k(\tau)\right||x_k(\tau)-y_k(\tau)|d\tau\\
&+\tfrac{1}{\Gamma(\alpha_k-\gamma_k+1)}\sup_{t\in[0,T]}t^{\alpha_k-\gamma_k}\int_0^T|G_k(\tau)||x_k(\tau)-y_k(\tau)|d\tau\\
&+\tfrac{L_k}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k)}\sup_{t\in[0,T]}t^{\alpha_k-\gamma_k}\int_0^T(T-\tau)^{\beta_k-1}(|x_1-y_1|+|x_2-y_2|+|{}^{c}D^{\gamma_k}x_k-{}^{c}D^{\gamma_k}y_k|)(\tau)d\tau.
\end{aligned}
$$

Using (H_3) and (H_5), we have

$$\|{}^cD^{\gamma_k}T_k(x_1,x_2) - {}^cD^{\gamma_k}T_k(y_1,y_2)\|_\infty$$

$$\leq \frac{3L_k}{\Gamma(\alpha_k-\gamma_k+\beta_k)}\|(x_1-y_1),(x_2-y_2)\|_{X\times X}\sup_{t\in[0,T]}\int_0^t(t-\tau)^{\alpha_k-\gamma_k+\beta_k-1}\,d\tau$$

$$+\frac{|b_k|}{\Gamma(\alpha_k-\gamma_k)}M_k\|x_k-y_k\|_\infty\sup_{t\in[0,T]}\int_0^t(t-\tau)^{\alpha_k-\gamma_k-1}(\tau-T_0)^{-\lambda_k}\,d\tau+\frac{T^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)}TG_k\|x_k-y_k\|_\infty$$

$$+\frac{3L_kT^{\alpha_k-\gamma_k}}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k)}\|(x_1-y_1),(x_2-y_2)\|_{X\times X}\sup_{t\in[0,T]}\int_0^T(T-\tau)^{\beta_k-1}d\tau.$$

Hence,

$$\|{}^cD^{\gamma_k}T_k(x_1,x_2) - {}^cD^{\gamma_k}T_k(y_1,y_2)\|_\infty$$

$$\leq\left(\frac{3L_k}{\Gamma(\alpha_k-\gamma_k+\beta_k+1)}T^{\alpha_k-\gamma_k+\beta_k}+\frac{3L_k}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k+1)}T^{\alpha_k-\gamma_k+\beta_k}\right)\|(x_1-y_1),(x_2-y_2)\|_{X\times X}$$

$$+\left(\frac{\Gamma(1-\lambda_k)|b_k|M_k}{\Gamma(\alpha_k-\gamma_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\gamma_k-\lambda_k}+\frac{G_k}{\Gamma(\alpha_k-\gamma_k+1)}T^{\alpha_k-\gamma_k+1}\right)\|x_k-y_k\|_\infty.$$

Thus,

$$\|{}^cD^{\gamma_k}T_k(x_1,x_2) - {}^cD^{\gamma_k}T_k(y_1,y_2)\|_\infty$$

$$\leq\left[\left(\frac{3L_k}{\Gamma(\alpha_k-\gamma_k+\beta_k+1)}+\frac{3L_k}{\Gamma(\alpha_k-\gamma_k+1)\Gamma(\beta_k+1)}\right)T^{\alpha_k-\gamma_k+\beta_k}\right.$$

$$\left.+\frac{\Gamma(1-\lambda_k)|b_k|M_k}{\Gamma(\alpha_k-\gamma_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\gamma_k-\lambda_k}+\frac{G_k}{\Gamma(\alpha_k-\gamma_k+1)}T^{\alpha_k-\gamma_k+1}\right]$$

$$\|(x_1-y_1),(x_2-y_2)\|_{X\times X}$$

Consequently,by 3.2, for $k=1,2$, it yields that

$$\|{}^cD^{\gamma_k}T_k(x_1,x_2) - {}^cD^{\gamma_k}T_k(y_1,y_2)\|_\infty\leq W_k^{"}\|(x_1-y_1),(x_2-y_2)\|_{X\times X} \tag{3.10}$$

which implies that

$$\|T_k(x_1,x_2)-T_k(y_1,y_2)\|_{X\times X}\leq W_k\|(x_1-y_1,x_2-y_2)\|_{X\times X},$$

Finally, we obtain

$$\|T(x_1,x_2)-T(y_1,y_2)\|_{X\times X}\leq\max_{k=1,2}W_k\|(x_1-y_1,x_2-y_2)\|_{X\times X}.$$

Then by 3.4 , T is contractive. □

3.2. Existence of Solutions

This result is based on Schaefer fixed point theorem . We have:

Theorem 3.2. *Assume that the hypotheses $(H_i)_{i=2,\dots,6}$ are satisfied. Then the problem 1.2 - 1.4 has at least one solution on J.*

Proof. We will prove the theorem through the following steps:

Step1: The continuity of the functions $f_k,h_k,w_k,k=1,2$ implies that T is continuous on $X\times X$.

Step2: We define the set:

$$\Omega_r:=\{(x_1,x_2)\in X\times X,\|(x_1,x_2)\|_{X\times X}\leq r\},\text{ where } r>0.$$

For $(x_1,x_2)\in\Omega_r, k=1,2$, we obtain:

$$\|T_k(x_1,x_2)\|_X\leq\frac{(S_k+R_k)T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}+r|b_k|M_k\frac{\Gamma(1-\lambda_k)}{\Gamma(\alpha_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\lambda_k}+r\frac{G_k}{\Gamma(\alpha_k+1)}T^{\alpha_k+1}+\frac{(S_k+R_k)T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}. \tag{3.11}$$

This is to say that

$$\|T(x_1,x_2)\|_{X\times X}\leq\sum_{k=1}^{2}\left(\frac{(S_k+R_k)T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+\beta_k+1)}+r|b_k|M_k\frac{\Gamma(1-\lambda_k)}{\Gamma(\alpha_k-\lambda_k+1)}(T-T_0)^{\alpha_k-\lambda_k}+r\frac{G_k}{\Gamma(\alpha_k+1)}T^{\alpha_k+1}+\frac{(S_k+R_k)T^{\alpha_k+\beta_k}}{\Gamma(\alpha_k+1)\Gamma(\beta_k+1)}\right) \tag{3.12}$$

Hence, the operator T maps bounded sets into bounded sets in $X\times X$.

Step3: Equi-continuity of $T(\Omega_r)$:
For $t_1,t_2\in J, t_1<t_2, (x_1,x_2)\in\Omega_r$ and $k=1,2$, we have:

$$\|T_k(x_1,x_2)(t_2)-T_k(x_1,x_2)(t_1)\|_X\leq C_k \tag{3.13}$$

In 3.13, the right hand sides C_k (given in 3.3) are independent of x_1,x_2. and tend to zero as t_1 tends to t_2. Then, as a consequence of Steps 1,2,3 and by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Step4: We show that the set defined by:

$$\Omega:=\{(x_1,x_2)\in X\times X,(x_1,x_2)=\mu T(x_1,x_2),0<\mu<1\},$$

is bounded:

Let $(x_1,x_2)\in\Omega$, then $(x_1,x_2)=\mu T(x_1,x_2)$, for some $0<\mu<1$. Hence, for $t\in J$, we have:

$$x_1(t)=\mu T_1(x_1,x_2)(t), x_2(t)=\mu T_2(x_1,x_2)(t). \tag{3.14}$$

Thus,

$$\|(x_1,x_2)\|_{X\times X}=\mu\|T(x_1,x_2)\|_{X\times X}, 0<\mu<1. \tag{3.15}$$

Since the functions f_k and h_k are bounded, then by 3.13 , we obtain:

$$\|(x_1,x_2)\|_{X\times X}\leq\mu\max_{k=1,2}C_k, 0<\mu<1. \tag{3.16}$$

Consequently, Ω is bounded.

As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of 1.2 - 1.4. □

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