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Research Article

A weighted Topp-Leone G family of distributions: properties, applications for modelling reliability data and different method of estimation

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Abstract

Based on the Topp-Leone distribution, we propose a new family of continuous distributions with one shape parameter called the weighted Topp-Leone family. We study some basic properties including quantile function, asymptotic, mixture for cdf and pdf, various entropies and order statistics.Then we study Lindley case as special case with more details. The maximum likelihood estimates of parameters are compared with various methods of estimations by conducting a simulation study. Finally, three real data sets are illustration the purposes.

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1. Introduction

Topp and Leone [35] introduced a family of of J-shaped density. The cumulative distribution function (cdf) of Topp-Leone distribution (TL) is given by

$$\Pi(x) = [x(2-x)]^{\alpha} = [1-(1-x)^2]^{\alpha}, \tag{1.1}$$

where $0 < x < 1$ and $0 < \alpha < 1$. The probability density function (PDF) and hazard rate function (HRF) of TL distribution are given by

$$\pi(x) = 2\alpha(1-x)[x(2-x)]^{\alpha} = [1-(1-x)^2]^{\alpha-1}, \tag{1.2}$$

and

$$\tau(x) = \frac{2\alpha(1-x)[x(2-x)]^{\alpha}}{1-[x(2-x)]^{\alpha}} = \frac{[1-(1-x)^2]^{\alpha-1}}{1-[(1-x)^2]^{\alpha}}. \tag{1.3}$$

TL distribution is bathtub hazard rate for any $0 < \alpha < 1$. Topp-Leone family of distribution was introduced by [2]. The cdf and pdf of Topp-Leone family are

$$F_{TL-G}(x;\alpha,\boldsymbol{\xi}) = \left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha}, \tag{1.4}$$

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$$f_{TL-G}(x;\alpha,\boldsymbol{\xi}) = 2\alpha\, g(x;\boldsymbol{\xi}\bar{G}(x;\boldsymbol{\xi}\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha-1} \quad (1.5)$$

where $\alpha > 0$ is a shape parameter, $G(.;\boldsymbol{\xi})$ is the baseline cdf, $g(x;\boldsymbol{\xi}) = \frac{\partial G(x)}{\partial x}$ is the baseline pdf for $G(.)$ and $\boldsymbol{\xi}$ is the vector of parameters for baseline $G(.)$.

The various extensions of the Topp-Leone distribution have been studied by several authors. For example: Topp-Leone family of distributions by [2], Topp-Leone generator of distributions [32], Topp-Leone generated family of distributions by [31], Topp-Leone Nadarajah-Haghighi distribution by [37], Transmuted Topp-Leone G family of distributions by [36], ToppLeone odd log-logistic family of distributions by [8], odd log-logistic Topp-Leone G family of distributions by [1], type II Topp-Leone generalized family by [13], Topp-Leone odd Lindley-G family of distributions by [30], type II generalized Toppleone family by [20], Topp-Leone generalized Odd log-logistic family by [22], new power Topp-Leone generalized family by [6] and Frechet Topp-Leone family by [29].

In this paper we introduced a weighted Topp-Leone family of distribution. The CDF of new family is given by

$$F(x;\alpha,\boldsymbol{\xi}) = \Pi(G(x;\boldsymbol{\xi})) = \frac{2\,F_{TL}(x;\boldsymbol{\xi})}{1+F_{TL}(x;\boldsymbol{\xi})} = \frac{2\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha}}{1+\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha}}, \quad (1.6)$$

We denote it by $-G(\alpha,\boldsymbol{\xi})$

The related pdf and HRF of $WTL-G(\alpha,\boldsymbol{\xi})$ family are given by

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{\partial F(x;\alpha,\boldsymbol{\xi})}{\partial x} = \frac{4\alpha\, g(x;\boldsymbol{\xi})\bar{G}(x,\boldsymbol{\xi})\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha-1}}{\left\{1+\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha}\right\}^2} \quad (1.7)$$

$$h(x;\alpha,\boldsymbol{\xi}) = \frac{4\alpha\, g(x;\boldsymbol{\xi})\bar{G}(x;\boldsymbol{\xi})\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha-1}}{1-\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{2\alpha}}. \quad (1.8)$$

It is easy to show that $f(x;\alpha,\boldsymbol{\xi}) = \frac{2\,f_{TL}(x;\boldsymbol{\xi})}{[1+F_{TL}(x;\boldsymbol{\xi})]^2}$ which verifies that the pdf of $WTL-G(\alpha,\boldsymbol{\xi})$ is a weighted Topp-Leone family with weight $w(x) = \frac{2}{[1+F_{TL}(x;\boldsymbol{\xi})]^2}$

Theorem 1.1. *Let $G(x;\boldsymbol{\xi})$ be an identifiable distribution, then he cdf (1.6) is identifiable.*

Proof. First note that $F(x;\alpha_1,\boldsymbol{\xi}) = F(x;\alpha_2,\boldsymbol{\xi})$ implies that

$$\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha_1} = \left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]^{\alpha_2}$$

Using generalized binomial expansion,

$$\sum_{i=0}^{\infty}\binom{\alpha_1}{i}\bar{G}(x;\boldsymbol{\xi})^{2\,i} = \sum_{i=0}^{\infty}\binom{\alpha_1}{i}\bar{G}(x;\boldsymbol{\xi})^{2\,i}$$

for any $x \in \mathbf{R}$. Taking $\bar{G}(x;\boldsymbol{\xi}) = 1$, implies $\binom{\alpha_1}{i} = \binom{\alpha_2}{i}$, then $\alpha_1 = \alpha_2$. □

The key goal of this research is to introduce an extra parameter to a family of lifetime distribution functions to bring more flexibility to the given family. We call this new method as a weighted Topp-Leone family of continuous distributions, called the WTL-G family, and to study some of its statistical properties. Furthermore, the key motivations for using WTL-G family in the practice are the followings:

1. A very simple and convenient method of adding additional parameters to modify the existing distributions.
2. To improve the characteristics and flexibility of the existing distributions.
3. To introduce the extended version of the baseline distribution whose cdf, and hrf, have closed form.
4. To provide better fits than the competing modified models having higher number of parameters than the proposed model.

The rest of this paper is organized as follows: In the above, new family of distributions was proposed. Various properties of the proposed distribution are explored in Section 2. These properties include quantile function, asymptotic, mixture for cdf and pdf, various entropies and order statistics. In section 3, we consider Lindley as special case and studied it with more details. The maximum likelihood estimation of parameters are compared with various methods of estimations by conducting simulation study in section 4,5. Real data sets are analysed to show the performance of the new family in Section 6. In Section 7, some concluding remarks are considered.

2. Main properties

In this section, we study some properties of WTL-G including quantile function, asymptotic, mixture for cdf and pdf, various entropies and order statistics.

2.1. Quantile function

Let $U \sim U(0,1)$, if $Q_G(u) = G^{-1}(u)$, denote the quantile function of baseline G, then

$$X_u = Q_G \left\{ 1 - \left[1 - \left(\frac{u}{2-u}\right)^{\frac{1}{\alpha}} \right]^{0.5} \right\} \tag{2.1}$$

has cdf (1.6).

2.2. Asymptotic

Let $\delta = \inf\{x|G(x) > 0\}$. Then, the asymptotics of Equations (1.6), (1.7) and (1.8) as $x \to \delta$ are given by

$$F(x) \sim 2^{\alpha+1}\, G(x)^{\alpha},$$
$$f(x) \sim \alpha\, 2^{\alpha+1}\, g(x) G(x)^{\alpha-1},$$
$$h(x) \sim \alpha\, 2^{\alpha+1}\, g(x) G(x)^{\alpha-1}.$$

The asymptotics of Equations (1.6), (1.7) and (1.8) as $x \to \infty$ are given by

$$1 - F(x) \sim \frac{\alpha}{2}\, \bar{G}(x)^2,$$
$$f(x) \sim \alpha\, g(x) \bar{G}(x),$$
$$h(x) \sim \frac{2\, g(x)}{\bar{G}(x)}.$$

2.3. Mixture for cdf and pdf

First, we define the exponentiated-G (“Exp-G”) distribution for an arbitrary parent distribution $G(x)$, say $W \sim \mathrm{Exp}^k G$, if W has cdf and pdf given by

$$H_k(x) = G(x)^k \qquad \text{and} \qquad h_k(x) = k\, g(x)\, G(x)^{k-1},$$

respectively. This transformed model is also called the Lehmann type I distribution, say $\mathrm{Exp}^k(G)$.

Using geometric expansion and generalized binomial expansion we can write

$$\begin{aligned}
F(x) = \frac{2\left[1-\bar{G}(x)^2\right]^{\alpha}}{1+\left[1-\bar{G}(x)^2\right]^{\alpha}} &= 2\sum_{i=0}^{\infty}(-1)^i\left[1-\bar{G}(x)^2\right]^{\alpha(i+1)} \\
&= 2\sum_{i,j=0}^{\infty}(-1)^{i+j}\binom{\alpha(i+1)}{j}\bar{G}(x)^{2j} \\
&= 2\sum_{i,j=0}^{\infty}\sum_{k=0}^{2j}(-1)^{i+j+k}\binom{\alpha(i+1)}{j}\binom{2j}{k}G(x)^k \qquad (2.2)\\
&= 2\sum_{i,k=0}^{\infty}\sum_{j=[\frac{k}{2}]}^{\infty}(-1)^{i+j+k}\binom{\alpha(i+1)}{j}\binom{2j}{k}G(x)^k \\
&= \sum_{k=0}^{\infty}a_k\,G(x)^k = \sum_{k=0}^{\infty}a_k\,H_k(x)
\end{aligned}$$

where

$$a_k = 2\sum_{i=0}^{\infty}\sum_{j=[\frac{k}{2}]}^{\infty}(-1)^{i+j+k}\binom{\alpha(i+1)}{j}\binom{2j}{k}. \qquad (2.3)$$

The pdf of X follows by differentiating (2.2) as

$$f(x) = \sum_{k=0}^{\infty}a_{k+1}\,h_{k+1}(x), \qquad (2.4)$$

Equation (2.4) show that the WTL-G density function is a linear combination of Exp-G densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the Exp-G distributions.

2.4. Entropies

2.4.1. Renyi and Shannon entropies. An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Renyi [28] and Shannon [33] entropies. The Renyi entropy of a random variable with pdf $f(x)$ is defined by

$$I_R(c) = \frac{1}{1-c}\log\left(\int_0^{\infty}f^c(x)dx\right),$$

for $c>0$ and $c\neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the special case of the Renyi entropy when $c\uparrow 1$. Direct calculation yields

$$\begin{aligned}
E\{-\log[f(X)]\} &= -\log(4\alpha) - E\{\log[g(X;\boldsymbol{\xi})]\} - E\left\{\log\left[\bar{G}(x;\boldsymbol{\xi})\right]\right\} \\
&+ (1-\alpha)E\left\{\log\left[1-\bar{G}(x;\boldsymbol{\xi})^2\right]\right\} + 2E\left\{\log\left[1+(1-\bar{G}(x;\boldsymbol{\xi})^2)^{\alpha}\right]\right\}.
\end{aligned}$$

First, we define and obtain

$$A(a_1,a_2,a_3;\alpha) = \int_0^1\frac{(1-u)^{a_1}(1-(1-u)^2)^{a_2}}{(1+(1-(1-u)^2)^{\alpha})^{a_3}}du.$$

After some simple manipulation we can obtain

$$A(a_1, a_2, a_3; \alpha) = \sum_{i,j=0}^{\infty} \frac{(-1)^j \binom{-a_3}{i} \binom{a_2 + \alpha\, i}{j}}{a_1 + 2\, j + 1}.$$

After some algebraic manipulations, we obtain

$$E\left\{\log\left[\bar{G}(x; \boldsymbol{\xi})\right]\right\} = 4\alpha \frac{\partial}{\partial t} A(1 + t, \alpha - 1, 2; \alpha)|_{t=0},$$

$$E\left\{\log\left[1 - \bar{G}(x; \boldsymbol{\xi})^2\right]\right\} = 4\alpha \frac{\partial}{\partial t} A(1, \alpha + t - 1, 2; \alpha)|_{t=0},$$

$$E\left\{\log\left[\bar{G}(x; \boldsymbol{\xi})\right]\right\} = 4\alpha \frac{\partial}{\partial t} A(1, \alpha - 1, 2 - t; \alpha)|_{t=0}.$$

The simplest formula for the entropy of X is given by

$$\begin{aligned} E\{-\log[f(X)]\} = &- \log(4\alpha) - E\{\log[g(X; \boldsymbol{\xi})]\} \\ &- 4\alpha \frac{\partial}{\partial t} A(1 + t, \alpha - 1, 2; \alpha)|_{t=0} \\ &+ 4\alpha(1 - \alpha) \frac{\partial}{\partial t} A(1, \alpha + t - 1, 2; \alpha)|_{t=0} \\ &+ 8\alpha \frac{\partial}{\partial t} A(1, \alpha - 1, 2 - t; \alpha)|_{t=0} \end{aligned}$$

After some algebraic developments, we obtain an alternative expression for $I_R(c)$

$$I_R(c) = \frac{c}{1 - c} \log(4\alpha) + \frac{1}{1 - c} \log\left[\sum_{i,j,k=0}^{\infty} w_{i,j,k} I(c, k))\right],$$

where $I(c, k) = \int_{-\infty}^{\infty} g(x; \boldsymbol{\xi})^c\, G(x; \boldsymbol{\xi})^k\, dx$ and

$$w_{i,j,k} = (-1)^{j+k} \binom{-2c}{i} \binom{\alpha(c - 1 + i)}{j} \binom{c + 2\, j}{k}$$

2.4.2. Residual entropy and cumulative residual entropy. The residual entropy of X is given by

$$\mathcal{E}(X) = -\int_0^{\infty} F(x) \log(F(x)) dx \tag{2.5}$$

and the cumulative residual entropy of X is given by

$$\mathcal{CE}(X) = -\int_0^{\infty} \bar{F}(x) \log(\bar{F}(x)) dx \tag{2.6}$$

After some simple algebra using geometric expansion and generalized binomial expansion, for WTL-G$(\alpha, \boldsymbol{\xi})$ we can obtain,

$$\mathcal{E}(X) = 2 \sum_{i,k,l=0}^{\infty} \sum_{j=0}^{i+1} \sum_{r=0}^{2\,l} \frac{(-1)^{j+l+r} \binom{i+1}{j} \binom{-i-2}{k} \binom{\alpha(j+k+1)}{l} \binom{2\,l}{r} I(0, r)}{i + 1} \tag{2.7}$$

and

$$\begin{aligned} \mathcal{CE}(X) = \; & 2 \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{2\,k} \frac{(-1)^{k+l} 2^{i+1} \binom{-i-1}{j} \binom{\alpha(i+j+1)}{k} \binom{2\,k}{l} I(0, k)}{i + 1} \\ & \times \left[\binom{\alpha(i+j+1)}{k} - \binom{\alpha(i+j+2)}{k}\right] \end{aligned} \tag{2.8}$$

2.5. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from the WTL-G family of distributions. We can write the density of the ith order statistic, say $X_{i:n}$, as

$$f_{i:n}(x) = K\, f(x)\, F^{i-1}(x)\, \{1 - F(x)\}^{n-i} = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1},$$

where $K = n!/[(i-1)!\,(n-i)!]$.

After some algebraic developments, we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, h_{r+k+1}(x),$$

where $h_{r+k}(x)$ denotes the exp-G density function with power parameter $r+k$,

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\,a_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\,j!},$$

and a_k is defined in Equation (2.3). Here, the quantities $f_{j+i-1,k}$ are obtained recursively by $f_{j+i-1,0} = a_0^{j+i-1}$ and (for $k \geq 1$)

$$f_{j+i-1,k} = (k\,a_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, a_m\, f_{j+i-1,k-m}. \tag{2.9}$$

Based on the expansion (2.9), we can obtain some structural properties (ordinary and incomplete moments, generating function, etc.) for the WTL-G order statistics from those exp-G properties.

3. Lindley case

In this section we study WTL-Lindley by taking $G(x) = 1 - (1 + \frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda \mathrm{x}}$ and $g(x) = \frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda \mathrm{x}}$ for $x > 0$ and $\lambda > 0$ as Lindley cdf and pdf in equations (1.6,1.7 and 1.8). We denote it by WTL-Li(λ, α). The cdf and pdf of WTL-Li(λ, α) are given by

$$F(x) = \frac{2\left[1 - (1 + \frac{\lambda x}{1+\lambda})^2\, \mathrm{e}^{-2\lambda \mathrm{x}}\right]^{\alpha}}{1 + \left[1 - (1 + \frac{\lambda x}{1+\lambda})^2\, \mathrm{e}^{-2\lambda \mathrm{x}}\right]^{\alpha}} \tag{3.1}$$

and

$$f(x) = \frac{4\alpha\,\lambda^2\,(1+x)(1+\lambda+\lambda\,x)\mathrm{e}^{-2\lambda \mathrm{x}}\left[1 - (1 + \frac{\lambda \mathrm{x}}{1+\lambda})^2\, \mathrm{e}^{-2\lambda \mathrm{x}}\right]^{\alpha-1}}{(1+\lambda)^2 \left\{1 + \left[1 - (1 + \frac{\lambda x}{1+\lambda})^2\, \mathrm{e}^{-2\lambda \mathrm{x}}\right]^{\alpha}\right\}^2} \tag{3.2}$$

Figures 1 and 2. provide the pdf and the HRF of WTL-Li(λ, α) for selected parameter values. These graphs show that the pdf of WTL-Li(λ, α) is unimodal, right skew or almost symmetric. The HRF of WTL-Li(λ, α) can be decreasing, increasing, bathtub, upside down and bathtub-upside down shape.

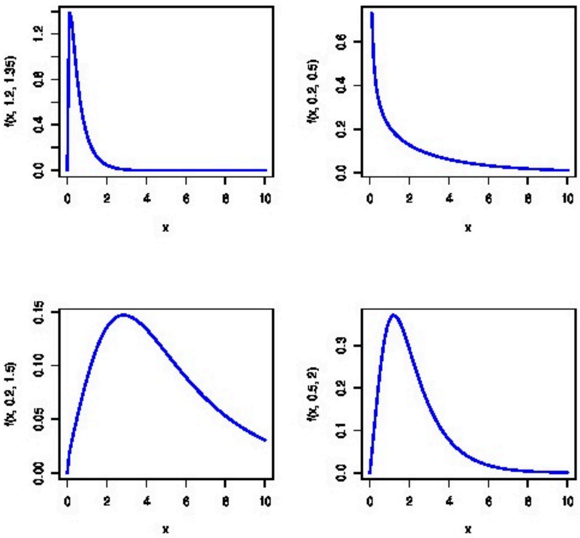

Figure 1. The sample curves of density function of WTL-Li(λ, α).

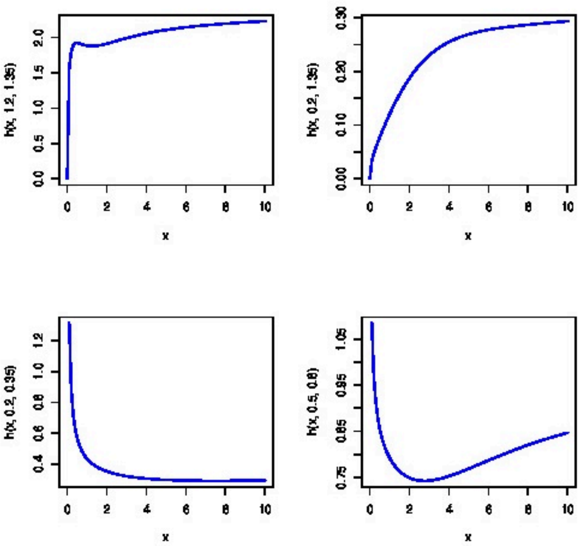

Figure 2. The sample curves of hazard rate function of WTL-Li(λ, α).

3.1. Asymptotic proporties

The asymptotic of cdf, pdf and hrf of the WTL-Li(λ, α) distribution as $x \to 0$ are, respectively, given by

$$F(x) \sim 2^{\alpha+1}\lambda^{\alpha}\, x^{\alpha},$$
$$f(x) \sim \alpha\, 2^{\alpha+1}\lambda^{\alpha}\, x^{\alpha-1},$$
$$h(x) \sim \alpha\, 2^{\alpha+1}\lambda^{\alpha}\, x^{\alpha-1}.$$

The asymptotic of cdf, pdf and hrf of the WTL-Li(λ, α) distribution as $x \to \infty$ are, respectively, as follows:

$$
\begin{aligned}
&1 - F(x) \sim \frac{\alpha\,\lambda^2}{2(1+\lambda)^2} x^2\, \mathrm{e}^{-2\lambda\,\mathrm{x}},\\
&f(x) \sim \frac{\alpha\,\lambda^3}{(1+\lambda)^2} x^2\, \mathrm{e}^{-2\lambda\,\mathrm{x}},,\\
&h(x) \sim 2\lambda.
\end{aligned}
$$

These equations show the effect of parameters on the tails of the WTL-Li distribution.

3.2. Extreme value

If $\bar{X} = (X_1 + ... + X_n)/n$ denotes the sample mean, then by the usual central limit theorem, $\sqrt{n}(\bar{X} - E(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. Let $\tau(x) = \frac{1}{\lambda}$, we obtain following equations for the cdf in of WTL-Li(λ, α) as

$$
\lim_{t\to 0} \frac{F(t\,x)}{F(t)} = \lim_{t\to 0} \frac{\left[1 - \left(1 + \frac{\lambda\,t\,x}{1+\lambda}\right)^2 \mathrm{e}^{-2\lambda\,t\,x}\right]^\alpha}{\left[1 - \left(1 + \frac{\lambda\,t}{1+\lambda}\right)^2 \mathrm{e}^{-2\lambda\,t}\right]^\alpha} = x^\alpha \tag{3.3}
$$

and

$$
\lim_{t\to\infty} \frac{1 - F(t + x\,\tau(t))}{1 - F(t)} = \lim_{t\to\infty} \frac{1 - G(t + x\,\tau(t))^\gamma}{1 - G(t)^\gamma} = \mathrm{e}^{-x}. \tag{3.4}
$$

Thus, from [23], there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$
Pr\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{e}^{-\mathrm{x}}},
$$

and

$$
Pr\left[c_n(m_n - d_n) \le x\right] \to 1 - \mathrm{e}^{-\mathrm{x}^\alpha},
$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in [23], one can see that $b_n = F^{-1}(1 - \frac{1}{n})$ and $a_n = \lambda$, where $F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$.

3.3. Quantile function

Quantile function is generally used to find representations in terms of lookup tables for key percentiles. Let X be an WTL-Li(λ, α) distributed random variable. The quantile function, $Q(p)$, defined by $F[Q(p)] = p$ is the root of the equation as

$$
[1 + \lambda + \lambda\,Q(p)]\,\mathrm{e}^{-\lambda\,Q(p)} = (1+\lambda)\left[1 - \left(\frac{p}{2-p}\right)^{\frac{1}{\alpha}}\right]^{0.5} \tag{3.5}
$$

for $0 < p < 1$. Substituting $Z(p) = -1 - \lambda - \lambda\,Q(p)$, one can write (3.5) as

$$
Z(p)\,\mathrm{e}^{Z(p)} = -(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1 - \left(\frac{\mathrm{p}}{2-\mathrm{p}}\right)^{\frac{1}{\alpha}}\right]^{0.5}. \tag{3.6}
$$

Hence, the solution $Z(p)$ is given by

$$
Z(p) = W_{-1}\left[-(1+\lambda)\mathrm{e}^{-1-\lambda}\left[1 - \left(\frac{\mathrm{p}}{2-\mathrm{p}}\right)^{\frac{1}{\alpha}}\right]^{0.5}\right], \tag{3.7}
$$

where $W_{-1}[.]$ is the negative branch of Lambert function [11]. Inserting (3.7), we obtain

$$Q(p) = -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1} \left[-(1+\lambda)\mathrm{e}^{-1-\lambda} \left[1 - (\frac{\mathrm{p}}{2-\mathrm{p}})^{\frac{1}{\alpha}} \right]^{0.5} \right]. \tag{3.8}$$

Now, we propose the following algorithm for generating random data from the WTL-Li(λ, α) distribution.

(a) The first algorithm is based on generating random data from the Lindley distribution mixture the exponential and gamma distributions.

Algorithm 1 (Mixture form of the Lindley distribution)

- Generate $U_i \sim$ Uniform$(0,1)$, $\quad i = 1, \ldots, n$;
- Generate $V_i \sim$ Exponential(λ), $\quad i = 1, \ldots, n$;
- Generate $W_i \sim$ Gamma$(2, \lambda)$, $\quad i = 1, \ldots, n$;
- If $1 - \left[1 - (\frac{U_i}{2-U_i})^{\frac{1}{\alpha}} \right]^{0.5} \leq \frac{\lambda}{1+\lambda}$ set $X_i = V_i$, otherwise, set $X_i = W_i$, $\quad i = 1, \ldots, n$.

(b) The second algorithm is based on generating random data from the inverse cdf of the WTL-Li(λ, α) distribution.
If $U \sim U(0,1)$, then

$$X_u = -1 - \frac{1}{\lambda} - \frac{1}{\lambda} W_{-1} \left[-(1+\lambda)\mathrm{e}^{-1-\lambda} \left[1 - (\frac{\mathrm{u}}{2-\mathrm{u}})^{\frac{1}{\alpha}} \right]^{0.5} \right]. \tag{3.9}$$

has cdf (3.1).

3.4. Moments, incomplete moments and moment generating functions

The formulae obtained in this paper can be simply used in some mathematical and statistical software such as Mathematica, Maple and R. These software be able to deal with complex analytic expressions. To determine mathematical properties of WTL-Li distribution, use of some algebraic expansions can be more efficient than computing these properties directly. In what follows, we derive the nth moment, kth central moment and moment generating function of WTL-Li(λ, α) distribution. In addition, we provide the nth incomplete moment, mean deviations, Bonferroni and Lorenz curves and present numerical values of skewness and kurtosis using the first four ordinary moments. First of all, assume that $X \sim WTL - Li(\lambda, \alpha)$. Using (2.4), we define

$$A(a_1, a_2, a_3, a_4; \lambda) = \int_0^\infty x^{a_1} (1+x)^{a_2} \mathrm{e}^{-a_3 x} \left[1 - (1 + \frac{\lambda}{1+\lambda} x) \mathrm{e}^{-\lambda x} \right]^{a_4} \mathrm{d}x.$$

By using generalized binomial expansion it can be shown that

$$A(a_1, a_2, a_3, a_4; \lambda) = \sum_{l,r=0}^{\infty} \sum_{k=0}^{l} (-1)^l \binom{a_4}{l} \binom{l}{k} \binom{a_2}{r} (\frac{\lambda}{1+\lambda})^l \times \frac{\Gamma(a_1 + 1 + k + r)}{(\lambda\, l + a_3)^{a_1+1+k+r}}. \tag{3.10}$$

So, the nth moment of WTL-Li distribution is given by

$$E\left[X^n\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, a_{k+1}\; A(n, 1, \lambda, k; \lambda). \tag{3.11}$$

The central moments $\mu_k = E(X - \mu)^k$ of WTL-Li(λ, α) distribution can be derived from (3.11) as

$$\mu_k = E(X - \mu)^k = \sum_{r=0}^{k} \binom{k}{r} \mu'_r (-\mu)^{k-r}. \tag{3.12}$$

where $\mu'_k = E(X^k)$, $\mu = \mu'_1 = E(X)$ and k is an integer value.

The mean and variance of X can be particularly obtained using Eq.s (3.11) and (3.12). Furthermore, these equations are used to derive the skewness as

$$S = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}}$$

and the kurtosis as

$$K = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}.$$

It is to highlight that the Eq. (3.11) can be easily computed numerically using the mathematical or statistical software. For this purpose, one can compute this equation for a large natural number, say N, instead of infinity in the sums. Therefore, several quantities of X such as moments, skewness and kurtosis can be computed numerically using Eq. (3.11). Table 1 shows numerical values of the first four ordinary moments, skewness and kurtosis of the WTL-Li(λ, α) distribution for different values of parameters (λ, α). Also, the skewness and kurtosis plots of the WTL-Li(λ, α) distribution for selected values of WTL-Li(λ, α) are drawn in Figure 3.

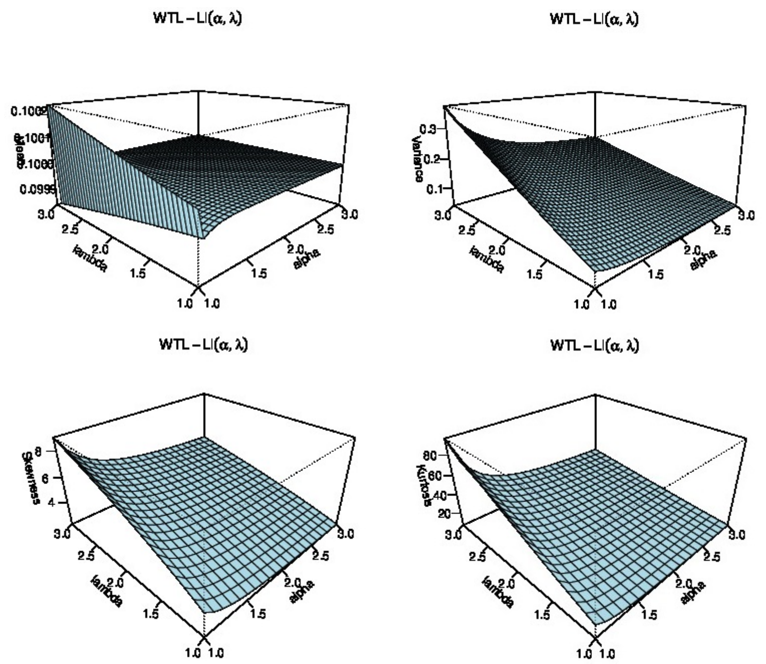

Figure 3. Plots of mean, variance, skewness and kurtosis for WTL-Li(λ, α) distribution.

Moreover, it is easy to verify that the moment generating function for WTL-Li(λ, α) distribution is given by

$$M_X(t) = E\left[\mathrm{e}^{\mathrm{t\,X}}\right] = \frac{\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)\, a_{k+1}\, A(0, 1, \lambda - t, k; \lambda).$$

In order to obtain the nth incomplete moment of the WTL-Li(λ,α) distribution let us define

$$\begin{aligned} B(a_1,a_2,a_3,a_4;y,\lambda) &= \int_0^y x^{a_1}\,(1+x)^{a_2}\,\mathrm{e}^{-a_3\,x}\left[1-(1+\frac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_4}\mathrm{d}x \\ &= \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}(\frac{\lambda}{1+\lambda})^l\times\frac{\gamma(a_1+1+k+r,\frac{y}{\lambda\,l+a_3})}{(\lambda\,l+a_3)^{a_1+1+k+r}}, \end{aligned} \tag{3.13}$$

where $\gamma(\lambda,z)=\int_0^z t^{\lambda-1}\,\mathrm{e}^{-t}\,dt$ stands for the incomplete gamma function. Note that the second equality of (3.13) is obtained by generalized binomial expansion. Hence, using (3.13) the nth incomplete moment of the WTL-Li(λ,α) distribution is derived by

$$m_n(y) = E\left[X^n\,|X<y\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,a_{k+1}\;B(n,1,\lambda,k,y;\lambda). \tag{3.14}$$

In what follows, we provide two measures of deviation, i.e. mean deviation about the mean (δ_1) and the mean deviation about the median (δ_2). By definition of these measures, it is easy to show that

$$\delta_1\,(X) = 2\mu\,F(\mu) - 2\int_0^{\mu} x\,f(x)\,\mathrm{d}x$$

and

$$\delta_2\,(X) = \mu - 2\int_0^{M} x\,f(x)\,\mathrm{d}x,$$

where M denotes the median of X. Therefore, it can be verified that measures $\delta_1\,(X)$ and $\delta_2\,(X)$ are given by

$$\delta_1\,(X) = 2\mu\,F(\mu) - \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)a_{k+1}\,A(1,1,\lambda,k;\lambda)$$

and

$$\delta_2\,(X) = \mu - \frac{2\,\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+\alpha)a_{k+1}\,B(1,1,\lambda,k;M,\lambda).$$

Theorem 3.1. *Let $X\sim WTL-Li(\lambda,\alpha)$, then $E(X^n)<\infty$ for $n=1,2,\cdots$.*

Proof. Note that $1<1+(1+\frac{\lambda\,x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}<2$ and $1-(1+\frac{\lambda\,x}{1+\lambda})^2\,\mathrm{e}^{-2\lambda\,\mathrm{x}}<1-\mathrm{e}^{-2\lambda\,\mathrm{x}}$, then

$$\begin{aligned} E(X^n) &= n\int_0^{\infty} x^{n-1}\times\frac{1-\left[1-(1+\frac{\lambda\,x}{1+\lambda})^2\,\mathrm{e}^{-2\lambda\,\mathrm{x}}\right]^{\alpha}}{1+\left[1-(1+\frac{\lambda\,x}{1+\lambda})^2\,\mathrm{e}^{-2\lambda\,\mathrm{x}}\right]^{\alpha}} \\ &< n\int_0^{\infty} x^{n-1}\times\left\{1-\left[1-(1+\frac{\lambda\,x}{1+\lambda})^2\,\mathrm{e}^{-2\lambda\,\mathrm{x}}\right]^{\alpha}\right\} \\ &< n\int_0^{\infty} x^{n-1}\times\left\{1-\left[1-\mathrm{e}^{-2\lambda\,\mathrm{x}}\right]^{\alpha}\right\}<\infty \end{aligned} \tag{3.15}$$

□

It means that all series in this section are convergent.

4. Estimation

4.1. Maximum-likelihood estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the WTL-Li distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the WTL-Li(λ, α) distribution. The log-likelihood function for the vector of parameters $\theta = (\lambda, \alpha)^T$ can be written as

$$\begin{aligned} l(\theta) &= n \log(\frac{4\alpha\lambda^2}{(1+\lambda)^2}) + \sum_{i=1}^{n} \log(1+x_i) + \sum_{i=1}^{n} \log(1+\lambda+\lambda\, x_i) - 2\lambda \sum_{i=1}^{n} x_i \\ &+ (\alpha-1)\sum_{i=1}^{n} \log(q_i) - 2\sum_{i=1}^{n} \log(1+q_i^{\alpha}), \end{aligned} \tag{4.1}$$

where $q_i = 1 - (1 + \frac{\lambda x}{1+\lambda})^2 \mathrm{e}^{-2\lambda\, \mathrm{x_i}}$ is a transformed observation.

The log-likelihood can be maximized either directly by using the SAS (Procedure NLMixed) or by solving the nonlinear likelihood equations obtained by differentiating Eq. (4.1). The components of the score vector $U(\theta)$ are given by

$$\begin{aligned} U_\lambda(\theta) &= \frac{2n}{\lambda(1+\lambda)} + \sum_{i=1}^{n} \frac{1+x_i}{1+\lambda+\lambda\, x_i} - 2\sum_{i=1}^{n} x_i \\ &+ \sum_{i=1}^{n} \frac{q_i^{(\lambda)}}{q_i} - 2\alpha \sum_{i=1}^{n} \frac{q_i^{(\lambda)} q_i^{\alpha-1}}{1+q_i^{\alpha}} \end{aligned}$$

and

$$U_\alpha(\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(q_i) - 2\sum_{i=1}^{n} \frac{q_i^{\alpha} \log(q_i)}{1+q_i^{\alpha}},$$

where $q_i^{(\lambda)} = \frac{2[x_i+\lambda(x_i^2+x_i-1)]}{1+\lambda}(1 + \frac{\lambda x}{1+\lambda})\mathrm{e}^{-2\lambda\, \mathrm{x_i}}$.

4.2. The other estimation methods

There are several approaches to estimate the parameters of distributions that each of them has its characteristic features and benefits. In this subsection five of those methods are briefly introduced and will be numerically investigated in the simulation study. A useful summary of these methods can be seen in [12]. Here $\{t_i; i = 1, 2, ..., n\}$ and $\{t_{i:n}; i = 1, 2, ..., n\}$ is the random sample and associated order statistics and F is the distribution function of WTL-Li distribution.

• **Least squares and weighted least squares estimators:** Least squares and weighted least squares estimators The Least Squares (LSE) and weighted Least Squares Estimators (WLSE) are introduced by [34]. The LSE's and WLSE's are obtained by minimizing the following functions:

$$S_{\mathrm{LSE}}(\lambda, \alpha) = \sum_{i=1}^{n} \left(F(t_{i:n}; \lambda, \alpha) - \frac{i}{n+1} \right)^2$$

and

$$S_{\mathrm{WLSE}}(\lambda, \alpha) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left(F(t_{i:n}; \lambda, \alpha) - \frac{i}{n+1} \right)^2.$$

• **Cramér–von–Mises estimator:** Cramér–von–Mises Estimator (CME) is introduced by [9]. The CMEs is obtained by minimizing the following function:

$$S_{\mathrm{CME}}(\lambda,\alpha)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(t_{i:n};\lambda,\alpha)-\frac{2i-1}{2n}\right)^2.$$

• **Anderson–Darling and right-tailed Anderson–Darling:** The Anderson Darling (ADE) and Right-Tailed Anderson Darling Estimators (RTADE) are introduced by [4]. The ADE's and RTADE's are obtained by minimizing the following functions:

$$S_{\mathrm{ADE}}(\lambda,\alpha)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F(t_i;\lambda,\alpha)+\log\overline{F}(t_{n+1-i};\lambda,\alpha)\}$$

and

$$S_{\mathrm{RTADE}}(\lambda,\alpha)=\frac{n}{2}-2\sum_{i=1}^{n}F(t_i;\lambda,\alpha)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\overline{F}(t_{n+1-i};\lambda,\alpha),$$

where $\overline{F}(\cdot)=1-F(\cdot)$.

4.3. Bayesian estimation

In this subsection, we derive the Bayes estimates of the unknown parameters of the model in (1.7). There are various ways to choose the priors. Here, we consider piecewise independent gamma priors on all parameters α and λ parameters are, respectively, given by

$$g_1(\alpha)=\frac{b_1^{a_1}\alpha^{a_1-1}\mathrm{e}^{-\mathrm{b}_1\alpha}}{\Gamma(a_1)},\quad \alpha>0,a_1,b_1>0,$$
$$g_2(\lambda)=\frac{b_2^{a_2}\alpha^{a_2-1}\mathrm{e}^{-\mathrm{b}_2\lambda}}{\Gamma(a_2)},\quad \lambda>0,a_2,b_2>0,$$

respectively. Thus, the joint prior distribution of α and λ can be written as

$$g(\alpha,\lambda)\propto\alpha^{a_1-1}\mathrm{e}^{-\mathrm{b}_1\alpha}\times\lambda^{\mathrm{a}_2-1}\mathrm{e}^{-\mathrm{b}_2\lambda}. \tag{4.2}$$

The joint posterior distribution of α and λ is given by

$$\begin{aligned}\pi(\alpha,\lambda|\vec{x}) &= \frac{L(\vec{x}|\alpha,\lambda)g(\alpha,\lambda)}{\int_0^\infty\int_0^\infty L(\vec{x}|\alpha,\lambda)g(\alpha,\lambda)d\alpha d\lambda}\\ &= \alpha^{n+a_1-1}e^{-\alpha\, b_1)}\frac{\lambda^{2n+b_2-1}}{(1+\lambda)^{2\,n}}e^{-\lambda(b_2+2\sum_{i=1}^n x_i)}\times\\ &\quad\times\prod_{i=1}^{n}(1+\lambda+\lambda\, x_i)\times\prod_{i=1}^{n}q_i^{\alpha-1}\times\prod_{i=1}^{n}[1+q_i^{\alpha}]^{-2},\end{aligned} \tag{4.3}$$

Therefore, the Bayes estimator of any function of α and λ, say, $\phi(\alpha,\lambda)$ under SELF is the posterior expectation of $\phi(\alpha,\lambda)$ and is given by

$$E[\phi(\alpha,\lambda)|\vec{x}] = \frac{\int_0^\infty\int_0^\infty\phi(\alpha,\lambda)L(\vec{x}|\alpha,\lambda)g(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty\int_0^\infty L(\vec{x}|\alpha,\lambda)g(\alpha,\lambda)d\alpha d\lambda}. \tag{4.4}$$

From the above equation we observe that the Bayes estimator is in the form of ratio of two integrals for which closed form solution is not available. The above ratio of integrals can be solved numerically.

5. Simulation study

In order to explore the estimators introduced above we consider the one model that have been used in this section, and investigate MSE of those estimators for different samples. For instance according to what has been mentioned above, for $(\lambda, \alpha) = (0.5, 0.6), (0.5, 1.5), (0.3, 0.8)$. The performance of each method of parameters estimations for the WTL-Li(λ, α) distribution with respect to sample size n is considered. To do this, a simulation study is done based on following steps:

Step 1. Generate one thousand samples of size n from (5). This work is done simply by quantile function and generated data from uniform distribution.

Step 2. Compute the estimates for the one thousand samples, say $\left(\hat{\alpha}_i, \hat{\beta}_i\right)$ for $i = 1, 2, ..., 10^4$.

Step 3. Compute the biases and mean squared errors by

$$Bias_{\varepsilon}(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\varepsilon}_i - \varepsilon)$$

and

$$MSE_{\varepsilon}(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\varepsilon}_i - \varepsilon)^2,$$

for $\varepsilon = \alpha, \lambda$. We repeated these steps for $n = 50, 100, \cdots, 1000$ with mentioned special case of parameters. So computing $bias_{\varepsilon}(n)$ and $MSE_{\varepsilon}(n)$ for $\varepsilon = \lambda, \alpha$ and $n = 50, 100, \cdots, 1000$. To obtain the value of the estimators, we have used the optim function and the Nelder-Mead method in the statistical package R version 3.4.4. The results are shown in Figures 4-6.

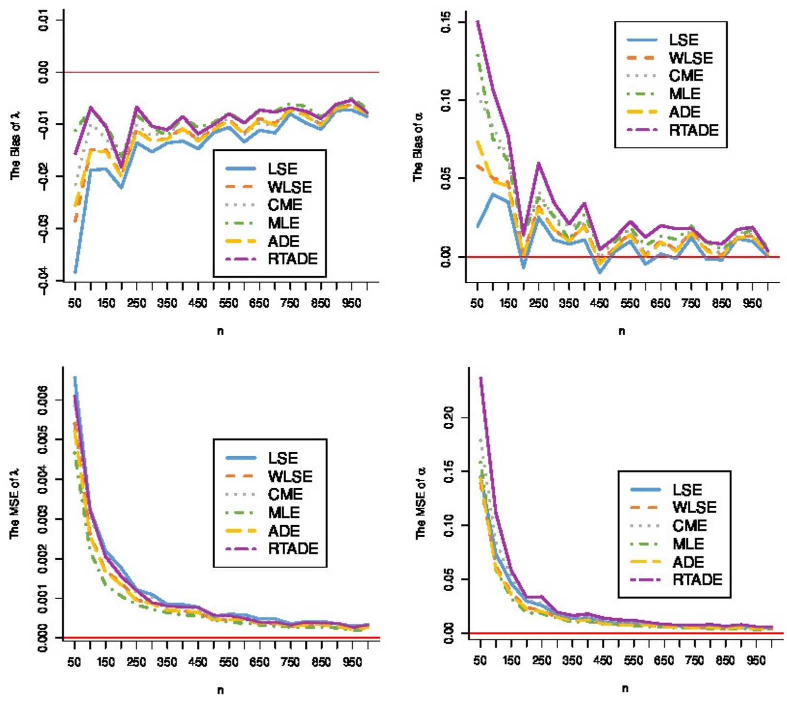

Figure 4. Bias of estimations for parameter values $(\lambda, \alpha) = (0.5, 0.6)$.

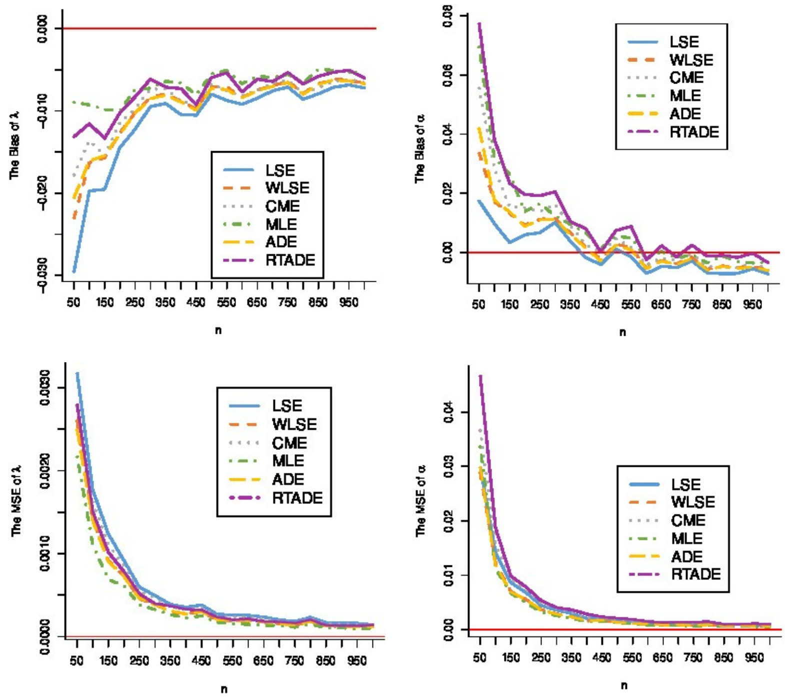

Figure 5. Bias of estimations for parameter values $(\lambda, \alpha) = (0.5, 1.5)$.

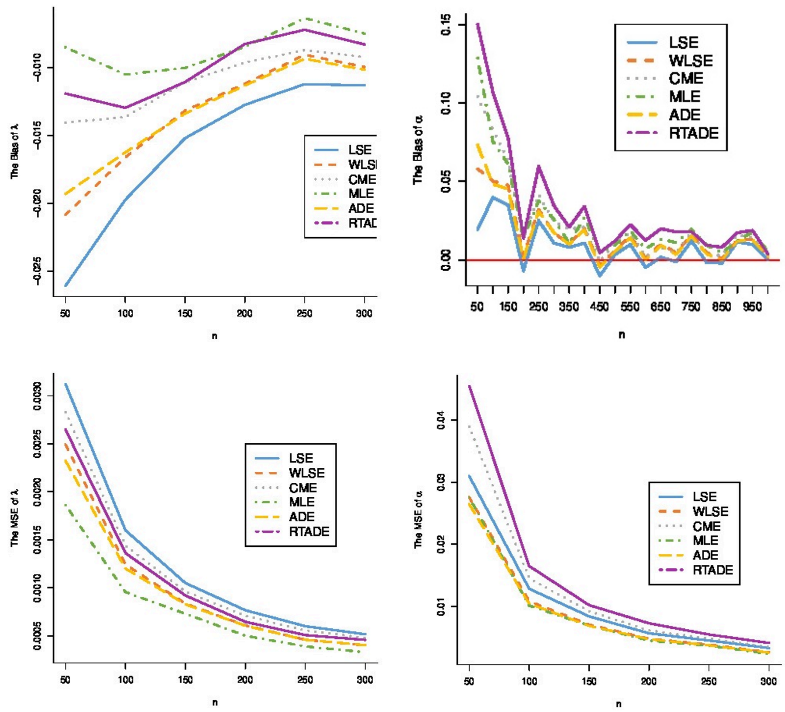

Figure 6. Bias of estimations for parameter values $(\lambda, \alpha) = (0.3, 0.8)$.

One can see that MSE plots for two parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the distribution parameters WTL-Li. Also

- for estimating α, LSE method has the minimum amount of bias, however for large sample size, all methods have same behaviour.
- for estimating λ, MLE method has the minimum amount of bias, however for large sample size, all methods have same behaviour.
- for estimating α, MLE method has the minimum amount of MSE, however for large sample size, all methods have same behaviour.
- for estimating λ, MLE method has the minimum amount of MSE, however for large sample size, all methods have same behaviour.

6. Applications

In this section, we present two applications by fitting the WTL-Li model and some famous models. The Cramér–von Mises (W*), Anderson-Darling (A*), Kolmogorow-Smirnow (K-S) and log-likelihood function (l) have been chosen for comparison the models for two examples.

The Weibull distribution (W), Lindley (Li) distribution [15], Generalized Lindley (GL) [25], Power Lindley (PL) distribution [14], Topp-Leone Lindley distribution (TL-Li) [2], Odd Log-Logistic Lindley (OLL-Li) [26], MarshallOlkin extended generalized Lindley distribution (MOGLi) [7] and Generalized Exponential (GE) [19] have been selected for comparison in two examples. The pdf of these models are given in Appendix. The parameters of models have been estimated by the MLE method.

6.1. Data set I

The first data set is related to failure time of 40 devices given by [24], p120. The data are: 0.602, 0.603, 0.603, 0.615, 0.652, 0.663, 0.688, 0.705, 0.761, 0.770, 0.868, 0.884, 0.898, 0.901, 0.911, 0.918, 0.935, 0.953, 0.983, 1.009, 1.040, 1.097, 1.097, 1.148, 1.296, 1.343, 1.422, 1.540, 1.555, 1.653, 1.752, 1.885, 2.015, 2.015, 2.030, 2.040, 2.123, 2.175, 2.443, 2.548. In the Table 1, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. As you see, the WTL-Li distribution is selected as the best model with more criteria. The histogram of the data set I and the plots of fitted PDF are displayed in Figure 9.

Table 1. Result for data set I.

model	estimatted parameters (se)			W^*	A^*	$K-S$	$-l$
WTL-Li (λ,α)	1.282	6.745		**0.125**	**0.813**	**0.111**	29.07
	(0.173)	(2.117)					
MOGLi(λ,α,β)	2.213	7.318	0.430	0.131	1.030	0.582	28.98
	(0.605)	(2.342)	(0.437)				
TL-Li(λ,α)	1.452	6.618		0.147	0.915	0.113	29.17
	(0.182)	(2.276)					
PL(λ,α)	0.804	2.043		0.240	1.395	0.157	32.02
	(0.123)	(0.227)					
Li(λ)	1.166			0.175	1.057	0.343	45.77
	(0.140)						
GL(λ,α)	2.615	7.290		0.142	0.893	0.117	24.22
	(0.348)	(2.534)					
OLL-Li(λ,α)	0.996	2.315		0.184	1.125	0.117	31.27
	(0.061)	(0.298)					
W(λ,α)	0.430	2.395		0.226	2.319	0.159	31.58
	(0.096)	(0.288)					
GE(λ,α)	2.248	8.705		0.135	0.858	0.113	29.17
	(0.336)	(3.006)					

6.2. Data set II

The second real data set is related to the marks of the slow pace students in Mathematics in the final examination were analysed by [18]. The data are: 29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23,11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31.

Similar to the previous application example, we have Tables 2. As it is clear, the WTL-Li is selected as the best model with more criteria. The histogram of the data set II and the plots of fitted PDF are displayed in Figure 10. Unimodality of profile likelihood functions of parameters for both data sets are given in Figures 7 and 8. P-P plots for both data sets are given in Figures 11 and 12.

Table 2. Result for data set II.

model	estimatted parameters (se)			W^*	A^*	$K-S$	$-l$
WTL-Li (λ,α)	0.042	1.310		**0.052**	**0.311**	**0.087**	196.75
	(0.006)	(0.252)					
MOGLi(λ,α,β)	0.061	1.539	0.299	0.189	1.340	0.705	196.46
	(0.023)	(0.314)	(0.319)				
TL-Li(λ,α)	0.049	1.123		0.084	0.482	0.113	197.61
	(0.006)	(0.243)					
PL(λ,α)	0.057	1.078		0.082	0.472	0.107	197.54
	(0.020)	(0.102)					
Li(λ)	0.074			0.077	0.443	0.092	197.84
	(0.007)						
GL(λ,α)	0.084	1.275		0.073	0.419	0.105	52.674
	(0.012)	(0.287)					
OLL-Li(λ,α)	0.072	1.126		0.075	0.435	0.102	197.40
	(0.007)	(0.139)					
W(λ,α)	0.006	1.486		0.108	0.609	0.112	198.28
	(0.003)	(0.122)					
GE(λ,α)	0.065	2.521		0.057	0.337	0.093	196.80
	(0.010)	(0.582)					

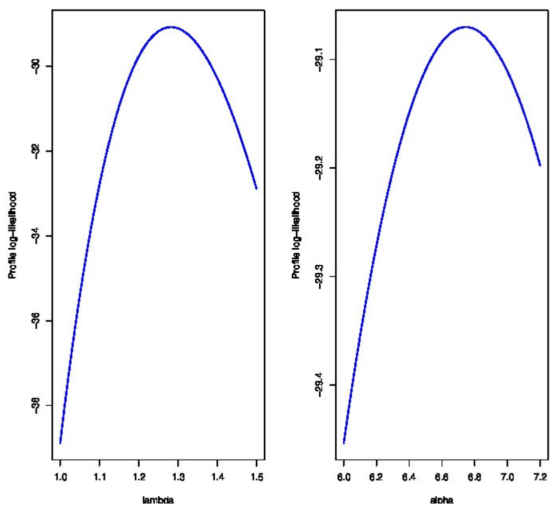

Figure 7. Unimodality of profile likelihood functions of parameters for data set I.

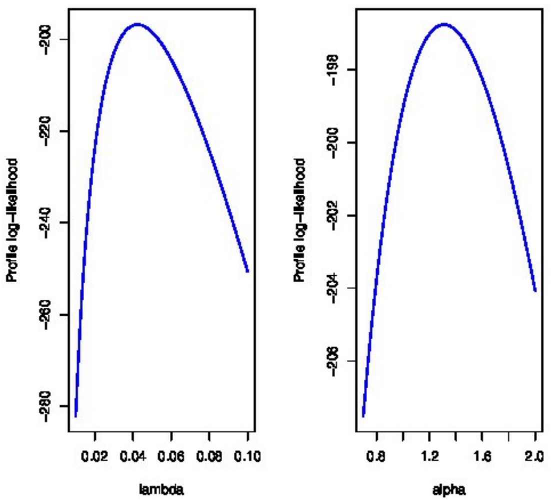

Figure 8. Unimodality of profile likelihood functions of parameters for data set II.

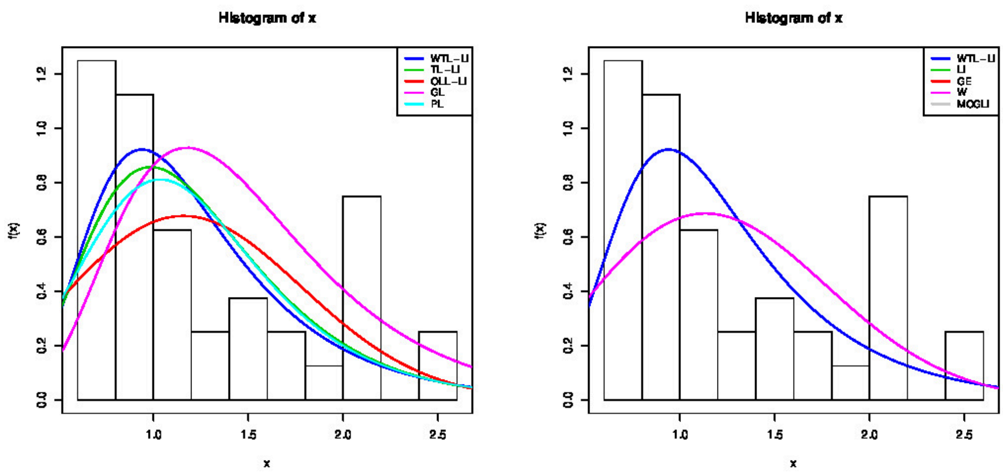

Figure 9. Histogram and fitted pdfs for data set I.

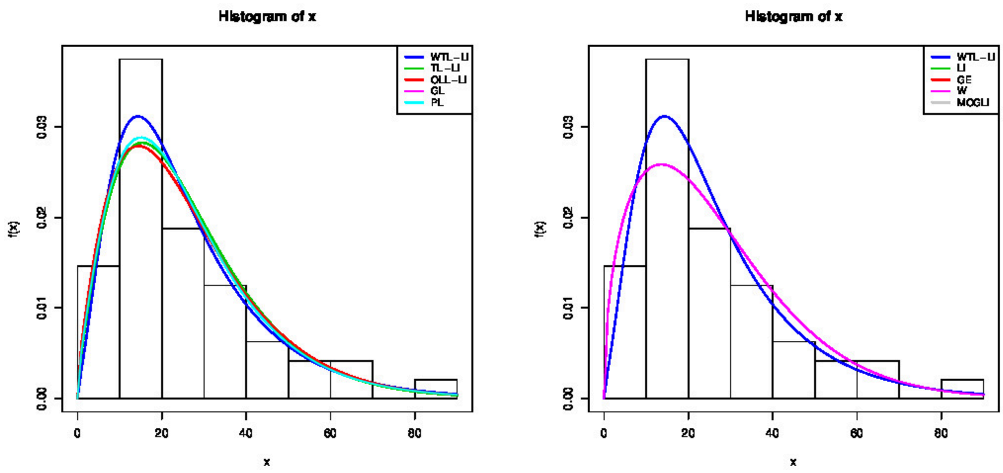

Figure 10. Histogram and fitted pdfs for data set II.

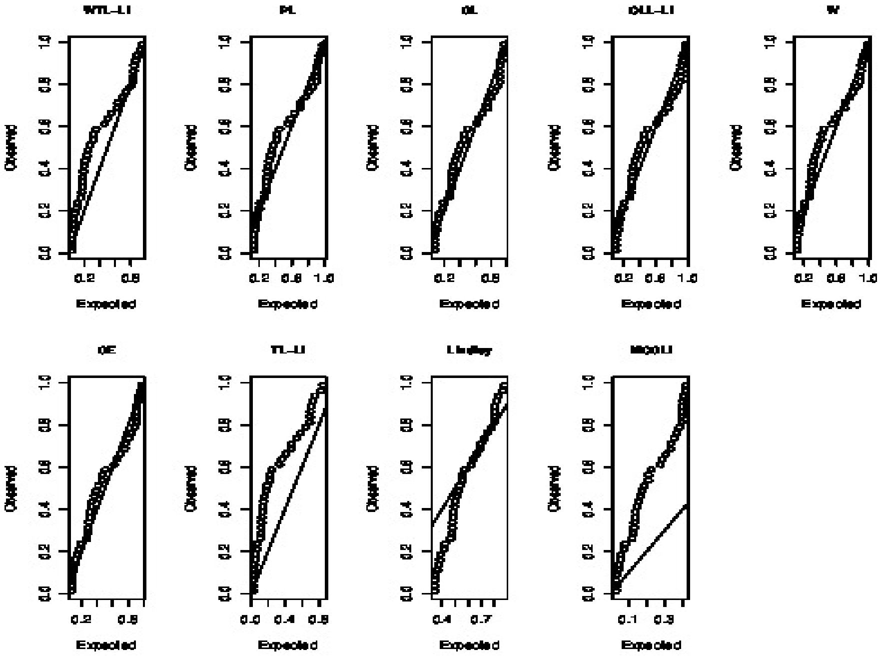

Figure 11. PP plots for data set I.

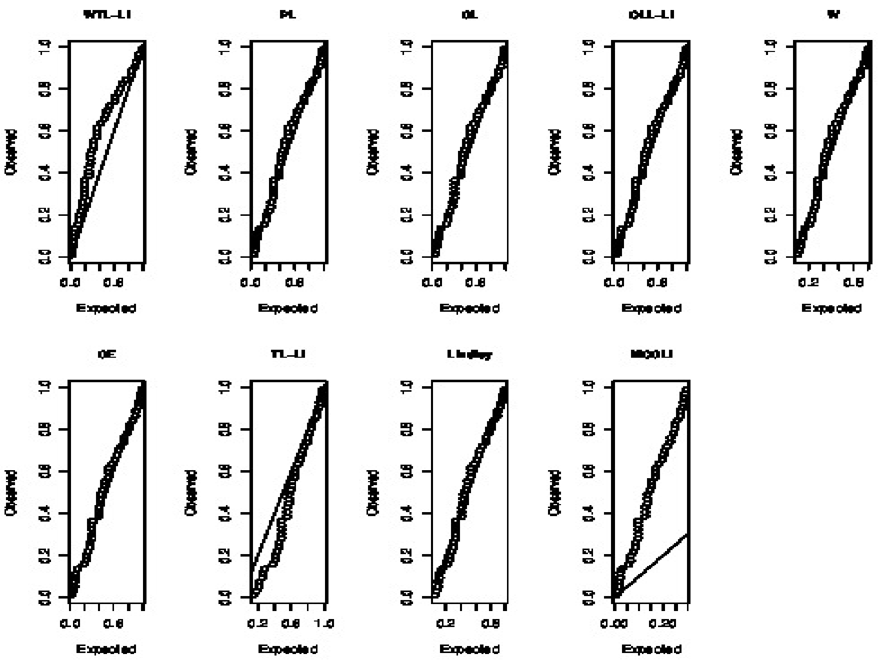

Figure 12. PP plots for data set II.

7. Conclusions

We introduce a new weighted extension of Topp-Leone family of distributions called the weighted Topp-Leone (WTL-G) family. Some properties of the new family, such as quantile function, asymptotic, mixture for cdf and PDF, various entropies and order statistics are obtained. Then we study Lindley case with more details. We estimate the parameters using maximum likelihood and other different methods. The Bias and MSE plots of parameters for all methods, will approach to zero with the increase in the volume of the sample which verifies the validity of the these estimation methods. The flexibility of this distribution is assessed by applying it to real data sets and comparing purpose distribution with others. The results of tables and figures illustrate the new models provide consistently better fits than other competitive models for these data sets. So Applications demonstrate the importance of the new family.

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Appendix: pdf of competitive models in application section

$$f_{Li}(x;\alpha) = \frac{\lambda^2}{1+\lambda}(1+x)\mathrm{e}^{-\lambda \mathrm{x}} \qquad \mathrm{x} > 0, \lambda > 0,$$

$$f_{TL-Li}(x;\alpha,\lambda) = \tfrac{2\alpha\,\lambda^2}{1+\lambda}(1+x)(1+\tfrac{\lambda\,x}{1+\lambda})\mathrm{e}^{-2\lambda\,\mathrm{x}}\left[1-(1+\tfrac{\lambda\,\mathrm{x}}{1+\lambda})^2\,\mathrm{e}^{-2\lambda\,\mathrm{x}}\right]^{\alpha-1} \qquad \mathrm{x} > 0, \lambda > 0, \alpha > 0,$$

$$f_{OLL-Li}(x;\alpha,\lambda)=\frac{\alpha\lambda^2\,(1+x)\mathrm{e}^{-\alpha\lambda\,\mathrm{x}}(1+\frac{\lambda\,\mathrm{x}}{1+\lambda})^{\alpha-1}\big[1-(1+\frac{\lambda\,\mathrm{x}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\big]^{\alpha-1}}{(1+\lambda)\big\{\big[1-(1+\frac{\lambda\,x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\big]^{\alpha}+\big[(1+\frac{\lambda\,x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\big]^{\alpha}\big\}^2}\qquad x>0,\alpha>0,\lambda>0,$$

$$f_{PL}(x;\alpha,\lambda)=\frac{\lambda^2\,\alpha}{1+\lambda}x^{\alpha-1}(1+x^{\alpha})\mathrm{e}^{-\lambda\,\mathrm{x}^{\alpha}}\qquad \mathrm{x}>0,\alpha>0,\lambda>0,$$

$$f_{GL}(x;\alpha,\lambda)=\frac{\alpha\,\lambda^2}{1+\lambda}\,(1+x)\mathrm{e}^{-\lambda\,\mathrm{x}}\left[1-(1+\frac{\lambda\,\mathrm{x}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha-1}\qquad \mathrm{x}>0,\alpha>0,\lambda>0,$$

$$f_{MOGLi}(x;\lambda,\alpha,\beta)=\frac{\beta\,\alpha\lambda^2\,(1+x)\mathrm{e}^{-\lambda\,\mathrm{x}}\left[1-(1+\frac{\lambda\mathrm{x}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha-1}}{(1+\lambda)\left\{\beta+(1-\beta)\left[1-(1+\frac{\lambda\,x}{1+\lambda})\mathrm{e}^{-\lambda,\mathrm{x}}\right]^{\alpha}\right\}^2}\quad x>0,\alpha,\beta,\lambda>0,$$

$$f_{GE}(x;\alpha,\lambda)=\alpha\,\lambda\,\mathrm{e}^{-\lambda\,\mathrm{x}}(1-\mathrm{e}^{-\lambda\,\mathrm{x}})^{\alpha-1}\qquad \mathrm{x}>0,\alpha>0,\lambda>0,$$

$$f_{W}(x;\alpha,\lambda)=\alpha\,\lambda\,(x^{\alpha-1}\mathrm{e}^{-\lambda\,\mathrm{x}^{\alpha}}\qquad \mathrm{x}>0,\alpha>0,\lambda>0.$$