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Research Article

A note on Terai's conjecture concerning primitive Pythagorean triples

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Abstract

Let f, g be positive integers such that $f > g$, $\gcd(f, g) = 1$ and $f \not\equiv g \pmod 2$. In 1993, N. Terai conjectured that the equation $x^2 + (f^2 - g^2)^y = (f^2 + g^2)^z$ has only one positive integer solution $(x, y, z) = (2fg, 2, 2)$. This is a problem that has not been solved yet. In this paper, using elementary number theory methods with some known results on higher Diophantine equations, we prove that if $f = 2^r s$ and $g = 1$, where r, s are positive integers satisfying $2 \nmid s$, $r \geq 2$ and $s < 2^{r-1}$, then Terai's conjecture is true.

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1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers, respectively. Let (a, b, c) be a primitive Pythagorean triple with $2 \mid a$. Then we have

$$\begin{aligned} &a = 2fg,\ b = f^2 - g^2,\ c = f^2 + g^2\ f, g \in \mathbb{N}, \\ &f > g,\ \gcd(f, g) = 1,\ f \not\equiv g \pmod 2. \end{aligned} \tag{1.1}$$

Many of the unsolved problems in number theory are related to the properties of such triples. In 1993, N. Terai [11] proposed the following conjecture about the generalized Ramanujan-Nagell equation with primitive Pythagorean triples.

Conjecture 1.1. *Let (a, b, c) be a primitive Pythagorean triple with $2 \mid a$. Then the equation*

$$x^2 + b^y = c^z \tag{1.2}$$

has only one solution $(x, y, z) = (a, 2, 2)$.

This is a problem that is far from resolved. So far it has only been proved for a few special cases, mainly where b or c is an odd prime power (see Section 2.3 of [7]). In practical terms, Conjecture 1.1 has been verified in the following cases:

(i) (N. Terai [11]) $b \equiv 1 \pmod 4$, $b^2 + 1 = 2c$, b and c are odd primes satisfying some conditions.

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(ii) (X.-G. Chen and M.-H. Le [2]) $b \not\equiv 1 \pmod{16}$, $b^2+1=2c$, b and c are odd primes.
(iii) (M.-H. Le [5]) $b \equiv \pm 3 \pmod 8$, $b > 8 \cdot 10^6$, c is an odd prime power.
(iv) (P.-Z. Yuan [12]; P.-Z. Yuan and J.-B. Wang [13]) $b \equiv \pm 3 \pmod 8$, c is an odd prime power.
(v) (M.-H. Le [6]; J.-Y. Hu and H. Zhang [4]) $b \equiv 7 \pmod 8$, b or c is an odd prime power.

Recently, a survey paper on the conjecture of Terai has been published by G. Soydan, M. Demirci, I.N. Cangül and A. Togbé (see [10] for the details about this conjecture).

By (1.1), (1.2) can be expressed as

$$x^2 + (f^2 - g^2)^y = (f^2 + g^2)^z, \ x, y, z \in \mathbb{N}. \tag{1.3}$$

In this paper, using elemantary number theory methods with some known results on higher Diophantine equations, we deal with (1.3) in a case where neither b or c is a prime power. We prove the following result:

Theorem 1.2. *If f, g satisfy*

$$f = 2^r s, \ g = 1, \ r, s \in \mathbb{N}, \ 2 \nmid s, \ r \geq 2, \ s < 2^{r-1}, \tag{1.4}$$

then (1.3) has only one solution $(x, y, z) = (2^{r+1}s, 2, 2)$.

Thus it can be seen that if f, g satisfy (1.4), then Conjecture 1.1 is true.

2. Preliminaries

Let n be a positive integer.

Lemma 2.1 (Theorems 1.75 and 1.76 of [8])**.** *For any complex numbers α and β, we have*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} (\alpha+\beta)^{n-2i} (\alpha\beta)^i.$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!} \in \mathbb{N}, \ i = 0, 1 \cdots, [n/2], \tag{2.1}$$

$[n/2]$ is the integer part of $n/2$.

Lemma 2.2 (The equality (2.35) of [3])**.** *If $2 \nmid n$, then*

$$\sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i} = (-1)^{(n^2-1)/8} 2^{(n-1)/2}.$$

Lemma 2.3 ([9])**.** *Every solution (X, Y, Z) of the equation*

$$X^2 + Y^2 = Z^n, \ X, Y, Z \in \mathbb{N}, \ \gcd(X, Y) = 1, \ 2 \mid Y \tag{2.2}$$

can be expressed as

$$Z = u^2 + v^2, \ u, v \in \mathbb{N} \ \gcd(u, v) = 1, \ 2 \mid v,$$
$$X + Y\sqrt{-1} = \lambda_1 (u + \lambda_2 v \sqrt{-1})^n \ \lambda_1, \lambda_2 \in \{1, -1\}.$$

Lemma 2.4 ([1])**.** *If $n \geq 4$, then the equation*

$$X^4 + Y^2 = Z^n, \ X, Y, Z \in \mathbb{N}, \ \gcd(X, Y) = 1 \tag{2.3}$$

has no solutions.

Lemma 2.5. *Let r, s be positive integers satisfying $2 \nmid s$ and $s < 2^{r-1}$. Then the equation*

$$2^{2r} s^2 + 1 = u^2 + v^2, \ u, v \in \mathbb{N}, \ \gcd(u, v) = 1, \ 2^r \mid v \tag{2.4}$$

has only the solution

$$(u, v) = (1, 2^r s). \tag{2.5}$$

Proof. We now assume that (u,v) is a solution of (2.4). Since $2^r \mid v$, we have $2 \nmid u$ and

$$v = 2^r t,\ t \in \mathbb{N}. \tag{2.6}$$

Substitute (2.6) into (2.4), we get

$$2^{2r}(s^2 - t^2) = u^2 - 1. \tag{2.7}$$

When $u = 1$, by (2.7), we have $t = s$. Hence, by (2.6), we get (2.5).

When $u > 1$, let $\zeta = (-1)^{(u-1)/2}$. Then $u - \zeta$ and $u + \zeta$ are positive integers satisfying

$$u - \zeta \equiv 0 \pmod 4,\ u + \zeta \equiv 2 \pmod 4. \tag{2.8}$$

Since $u^2 - 1 = (u - \zeta)(u + \zeta)$, by (2.7) and (2.8), we get $u - \zeta \equiv 0 \pmod{2^{2r-1}}$. So we have

$$u = 2^{2r-1}\ell + \zeta,\ \ell \in \mathbb{N}. \tag{2.9}$$

Substitute (2.9) into (2.7), we get

$$s^2 - t^2 = \ell(2^{2r-2}\ell + \zeta). \tag{2.10}$$

Therefore, by (2.10), we have

$$s^2 - 1 \geq s^2 - t^2 \geq 2^{2r-2} + \zeta \geq 2^{2r-2} - 1,$$

whence we get $s \geq 2^{r-1}$, a contradiction. Thus, (2.4) has only the solution (2.5). The lemma is proved. □

3. Proof of Theorem 1.2

We now assume that (x,y,z) is a solution of (1.3), and f, g satisfy (1.4). Then we have

$$x^2 + (2^{2r}s^2 - 1)^y = (2^{2r}s^2 + 1)^z,\ x, y, z \in \mathbb{N}. \tag{3.1}$$

Since $2^{2r}s^2 - 1$ and $2^{2r}s^2 + 1$ are both odd, by (3.1), we get

$$2 \mid x. \tag{3.2}$$

Further, since $2^{2r}s^2 - 1 \equiv -1 \pmod 4$, $2^{2r}s^2 + 1 \equiv 1 \pmod 4$ and $x^2 \equiv 0 \pmod 4$ by (3.2), we get from (3.1) that

$$2 \mid y. \tag{3.3}$$

Also, by (3.1) and (3.3), we have $(2^{2r}s^2 + 1)^z > (2^{2r}s^2 - 1)^y \geq (2^{2r}s^2 - 1)^2 > (2^{2r}s^2 + 1)$, whence we get

$$z \geq 2. \tag{3.4}$$

By (3.1), (3.3) and (3.4), we have

$$\begin{aligned} x^2 &= (2^{2r}s^2 + 1)^z - (2^{2r}s^2 - 1)^y \\ &= 2^{2r}s^2 \left((z + y) + 2^{2r}s^2 \left(\sum_{i=2}^{z} \binom{z}{i} (2^{2r}s^2)^{i-2} - \sum_{j=2}^{y} (-1)^j \binom{y}{j} (2^{2r}s^2)^{j-2} \right) \right). \end{aligned} \tag{3.5}$$

We see from (3.5) that $2^r s \mid x$ and

$$\left(\frac{x}{2^r s}\right)^2 \equiv z + y \pmod{2^{2r}s^2}.$$

So we have

$$x = 2^r s x_1,\ x_1 \in \mathbb{N} \tag{3.6}$$

and

$$x_1^2 \equiv z + y \pmod 4. \tag{3.7}$$

On the other hand, we see from (3.1), (3.2) and (3.3) that (2.2) has a solution

$$(X, Y, Z) = ((2^{2r}s^2 - 1)^{y/2}, x, 2^{2r}s^2 + 1) \tag{3.8}$$

for $n = z$. Applying Lemma 2.3 to (3.8), we have

$$2^{2r}s^2 + 1 = u^2 + v^2,\ u, v \in \mathbb{N},\ \gcd(u, v) = 1,\ 2 \mid v \tag{3.9}$$

and

$$(2^{2r}s^2 - 1)^{y/2} + x\sqrt{-1} = \lambda_1(u + \lambda_2 v\sqrt{-1})^z,\ \lambda_1, \lambda_2 \in \{1, -1\}. \tag{3.10}$$

When $2 \nmid z$, we get from (3.10) that

$$(2^{2r}s^2 - 1)^{y/2} = u\left|\sum_{i=0}^{(z-1)/2}(-1)^i\binom{z}{2i}u^{z-2i-1}v^{2i}\right| \tag{3.11}$$

and

$$x = v\left|\sum_{i=0}^{(z-1)/2}(-1)^i\binom{z}{2i+1}u^{z-2i-1}v^{2i}\right|. \tag{3.12}$$

Since $2 \mid v$ and $2 \nmid z$, we have $2 \nmid u$ and

$$2 \nmid \sum_{i=0}^{(z-1)/2}(-1)^i\binom{z}{2i+1}u^{z-2i-1}v^{2i}. \tag{3.13}$$

Hence, by (3.6), (3.12) and (3.13), we get

$$2^r \mid v \tag{3.14}$$

Therefore, by Lemma 2.5, we deduce from (3.9) and (3.14) that u and v satisfy (2.5). Substitute (2.5) into (3.11), we have

$$(2^{2r}s^2 - 1)^{y/2} = \left|\sum_{i=0}^{(z-1)/2}(-1)^i\binom{z}{2i}(2^{2r}s^2)^i\right|. \tag{3.15}$$

Since $y/2 \geq 1$ and $2^{2r}s^2 \equiv 1 \pmod{2^{2r}s^2 - 1}$, by (3.15), we get

$$\sum_{i=0}^{(z-1)/2}(-1)^i\binom{z}{2i} \equiv 0 \pmod{2^{2r}s^2 - 1}. \tag{3.16}$$

Further, by Lemma 2.2, we obtain from (3.16) that

$$2^{(z-1)/2} \equiv 0 \pmod{2^{2r}s^2 - 1}. \tag{3.17}$$

But, since $2^{2r}s^2 - 1$ is an odd integer with $2^{2r}s^2 - 1 > 1$, (3.17) is impossible. So, we have

$$2 \mid z. \tag{3.18}$$

We see from (3.1), (3.2),(3.3) and (3.18) that (2.2) has a solution

$$(X, Y, Z) = \left((2^{2r}s^2 - 1)^{y/2}, x, (2^{2r}s^2 + 1)^{z/2}\right) \tag{3.19}$$

for $n = 2$. Applying Lemma 2.3 to (3.19), we get

$$\begin{aligned}&(2^{2r}s^2 - 1)^{y/2} = U^2 - V^2,\ x = 2UV,\ (2^{2r}s^2 + 1)^{z/2} = U^2 + V^2,\\ &U, V \in \mathbb{N},\ U > V\ \ \gcd(U, V) = 1,\ U \not\equiv V \pmod 2.\end{aligned} \tag{3.20}$$

Further, by (3.20), we have

$$(2^{2r}s^2 - 1)^y = (U^2 - V^2)^2 \geq (U + V)^2 > U^2 + V^2 = (2^{2r}s^2 + 1)^{z/2} > (2^{2r}s^2 - 1)^{z/2},$$

whence we get

$$y > \frac{z}{2}. \tag{3.21}$$

By (3.3), (3.7) and (3.18), we have

$$2 \mid x_1. \tag{3.22}$$

Hence, by (3.7) and (3.22), we get

$$z+y\equiv 0 \pmod 4. \tag{3.23}$$

If $4 \mid z$, then from (3.23) we get $4 \mid y$. It follows from (3.1) that (2.3) has a solution $(X,Y,Z)=((2^{2r}s^2-1)^{y/4}, x, 2^{2r}s^2+1)$ for $n=z$. But, since $4 \mid z$ and $z\geq 4$, by Lemma 2.4, it is impossible. So we have

$$z\equiv y\equiv 2 \pmod 4. \tag{3.24}$$

Since $2 \mid z$, by (3.10), we have

$$(2^{2r}s^2-1)^{y/2}=\left|\sum_{i=0}^{z/2}(-1)^i\binom{z}{2i}u^{z-2i}v^{2i}\right| \tag{3.25}$$

and

$$x=uv\left|\sum_{i=0}^{z/2-1}(-1)^i\binom{z}{2i+1}u^{z-2i-2}v^{2i}\right|. \tag{3.26}$$

Since $2\nmid u$, $2 \mid v$ and $2 \,||\, z$ by (3.24), we have

$$2 \;||\; \sum_{i=0}^{z/2-1}(-1)^i\binom{z}{2i+1}u^{z-2i-2}v^{2i}. \tag{3.27}$$

Hence, by (3.6), (3.22), (3.26) and (3.27), we get (3.14). Therefore, by Lemma 2.5, we see from (3.9) and (3.14) that u and v satisfy (2.5).

Substitute (2.5) into (3.25), we have

$$(2^{2r}s^2-1)^{y/2}=\left|\sum_{i=0}^{z/2}(-1)^i\binom{z}{2i}(2^{2r}s^2)^i\right|. \tag{3.28}$$

Let

$$\theta=1+2^r s\sqrt{-1},\ \bar{\theta}=1-2^r s\sqrt{-1}. \tag{3.29}$$

By (3.28) and (3.29), we get

$$(2^{2r}s^2-1)^{y/2}=\frac{1}{2}\left|\theta^z+\bar{\theta}^z\right|. \tag{3.30}$$

Further let

$$\alpha=\theta^2,\ \beta=\bar{\theta}^2. \tag{3.31}$$

By (3.29) and (3.31), we have

$$\alpha+\beta=-2(2^{2r}s^2-1),\ \alpha\beta=(2^{2r}s^2+1)^2. \tag{3.32}$$

Since $z/2$ is odd by (3.24), applying Lemma 2.1 to (3.30), we get from (3.32) that

$$\begin{aligned}(2^{2r}s^2-1)^{y/2-1} &= \left|\frac{\alpha^{z/2}+\beta^{z/2}}{\alpha+\beta}\right| \\ &= \left|\sum_{j=0}^{(z/2-1)/2}(-1)^j\begin{bmatrix}z/2\\ j\end{bmatrix}(-2(2^{2r}s^2-1))^{z/2-2j-1}(2^{2r}s^2+1)^{2j}\right| \\ &= \left|\sum_{j=0}^{(z/2-1)/2}(-1)^j\begin{bmatrix}z/2\\ (z/2-1)/2-j\end{bmatrix}(4(2^{2r}s^2-1)^2)^j(2^{2r}s^2+1)^{z/2-2j-1}\right|. \end{aligned} \tag{3.33}$$

If $y>2$, then we have

$$y\geq 6 \tag{3.34}$$

by (3.24). Since

$$\begin{bmatrix}z/2\\ (z/2-1)/2\end{bmatrix}=\frac{z}{2} \tag{3.35}$$

by (2.1), we see from (3.33), (3.34) and (3.35) that

$$\frac{z}{2} \equiv 0 \pmod{2^{2r}s^2 - 1}. \tag{3.36}$$

Let p be an odd prime divisor of $2^{2r}s^2 - 1$, and let

$$p^{\gamma} \mid\mid \frac{z}{2},\ p^{\delta} \mid\mid 2^{2r}s^2 - 1,\ p^{\delta_j} \mid\mid 2j+1,\ j \geq 1. \tag{3.37}$$

Then we have

$$\delta_j \leq \frac{\log(2j+1)}{\log p} \leq j,\ j \geq 1. \tag{3.38}$$

By (2.1), if $2 \nmid n$, then

$$\begin{bmatrix} n \\ (n-1)/2 - j \end{bmatrix} = n \binom{(n-1)/2 + j}{2j} \frac{1}{2j+1},\ j = 0, \cdots, \frac{n-1}{2}. \tag{3.39}$$

Hence, by (3.35), (3.37),(3.38) and (3.39), we get

$$p^{\gamma} \ \mid\mid \ \begin{bmatrix} z/2 \\ (z/2-1)/2 \end{bmatrix} (2^{2r}s^2+1)^{z/2-1} \tag{3.40}$$

and

$$\begin{aligned} &\begin{bmatrix} z/2 \\ (z/2-1)/2 - j \end{bmatrix} (4(2^{2r}s^2+1)^2)^j (2^{2r}s^2+1)^{z/2-2j-1} \\ &\equiv \frac{z}{2}(2^{2r}s^2+1)^{z/2-2j-1} \binom{(z/2-1)/2+j}{2j} \frac{4^j(2^{2r}s^2-1)^{2j}}{2j+1} \\ &\equiv 0 \pmod{p^{\gamma+1}},\ j = 1, \cdots, \frac{z/2-1}{2}. \end{aligned} \tag{3.41}$$

Therefore, by (3.33), (3.37), (3.40) and (3.41), we obtain

$$(\frac{y}{2} - 1)\delta = \gamma. \tag{3.42}$$

Taking p through over all the distinct prime divisors of $2^{2r}s^2 - 1$, by (3.37) and (3.42), we have

$$\frac{z}{2} \equiv 0 \pmod{(2^{2r}s^2-1)^{y/2-1}},$$

whence we get

$$\frac{z}{2} \geq (2^{2r}s^2-1)^{y/2-1}. \tag{3.43}$$

The combination of (3.21) and (3.43) yields

$$y > (2^{2r}s^2-1)^{y/2-1}. \tag{3.44}$$

But, by (3.34), (3.44) is false. So we have $y = 2$.

Since $y = 2$, by (3.18) and (3.21), we get $z = 2$. Thus, by (3.1), $(x, y, z) = (2^{r+1}s, 2, 2)$ is the unique solution of (1.3). The theorem is proved.

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