



Some compact and non-compact embedding theorems for the function spaces defined by fractional Fourier transform

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Abstract

The fractional Fourier transform is a generalization of the classical Fourier transform through an angular parameter α . This transform uses in quantum optics and quantum wave field reconstruction, also its application provides solving some differential equations which arise in quantum mechanics. The aim of this work is to discuss compact and non-compact embeddings between the spaces $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ which are the set of functions in $L_w^1(\mathbb{R}^d)$ whose fractional Fourier transform are in $L_w^p(\mathbb{R}^d)$. Moreover, some relevant counterexamples are indicated.

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1. Introduction

Throughout this article, we study on \mathbb{R}^d . For any function $f : \mathbb{R}^d \rightarrow \mathbb{C}$, the translation and modulation operators are defined as $T_y f(t) = f(t - y)$ and $M_w f(t) = e^{iwt} f(t)$ for all $y, w \in \mathbb{R}^d$, respectively [18]. $C_C(\mathbb{R}^d)$ denotes the space of continuous complex function on \mathbb{R}^d whose support is compact, and also $C_0(\mathbb{R}^d)$ denotes the space of continuous complex functions on \mathbb{R}^d which vanish at infinity [17]. Besides we write the Lebesgue space $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, for $1 \leq p < \infty$. A weight (Beurling weight) function w on \mathbb{R}^d is a measurable and locally bounded function that satisfies $w(x) \geq 1$ and $w(x + y) \leq w(x)w(y)$ (submultiplicative, [11]) for all $x, y \in \mathbb{R}^d$. Throughout the article we consider Beurling weights. We define, for $1 \leq p < \infty$,

$$L_w^p(\mathbb{R}^d) = \{f \mid fw \in L^p(\mathbb{R}^d)\}.$$

It is well known that $L_w^p(\mathbb{R}^d)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$ [16].

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Let w_1 and w_2 be two weight functions. We say that $w_1 \prec w_2$ if there exists $c > 0$, such that $w_1(x) \leq cw_2(x)$ for all $x \in \mathbb{R}^d$, [9]. We define the Fourier transform \hat{f} (or $\mathcal{F}f$) of a function $f \in L^1(\mathbb{R})$ as

$$\hat{f}(w) = \mathcal{F}f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)e^{-iwt} dt.$$

The fractional Fourier transform was first studied by Wiener [20] in 1929. In [20], this new transformation has been viewed as a generalization of the Fourier transform. The importance of this transformation has been understood by Namias' work [14]. Namias used the fractional Fourier transform as a tool for solving ordinary and partial differential equations arising in quantum mechanics. McBride and Kerr [13], using the results of Namias, described this transformation in the Schwartz space and examined its properties. This transform has many applications in signal processing, optics, filtering and time-frequency analysis [1, 2, 6, 15]. The fractional Fourier transform is a generalization of the Fourier transform through an angle parameter α and can be considered as a rotation by an angle α in the time-frequency plane. The fractional Fourier transform with angle α of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} K_\alpha(u, t) f(t) dt,$$

where,

$$K_\alpha(u, t) = \begin{cases} \sqrt{\frac{1-i \cot \alpha}{2\pi}} e^{i\left(\frac{u^2+t^2}{2}\right) \cot \alpha - iut \operatorname{cosec} \alpha}, & \text{if } \alpha \text{ is not multiple of } \pi \\ \delta(t - u), & \text{if } \alpha = 2k\pi, k \in \mathbb{Z} \\ \delta(t + u), & \text{if } \alpha = (2k + 1)\pi, k \in \mathbb{Z} \end{cases}$$

and δ , Dirac delta function. The fractional Fourier transform with $\alpha = \frac{\pi}{2}$ corresponds to the Fourier transform, [2-4, 14, 15, 21].

The fractional Fourier transform can be extended for higher dimensions as [4]:

$$(\mathcal{F}_{\alpha_1, \dots, \alpha_n} f)(u_1, \dots, u_n) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_{\alpha_1, \dots, \alpha_n}(u_1, \dots, u_n; t_1, \dots, t_n) f(t_1, \dots, t_n) dt_1 \dots dt_n,$$

or shortly

$$\mathcal{F}_\alpha f(u) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} K_\alpha(u, t) f(t) dt,$$

where

$$K_\alpha(u, t) = K_{\alpha_1, \dots, \alpha_n}(u_1, \dots, u_n; t_1, \dots, t_n) = K_{\alpha_1}(u_1, t_1) K_{\alpha_2}(u_2, t_2) \dots K_{\alpha_n}(u_n, t_n).$$

Let w and ω be weight functions on \mathbb{R}^d and $1 \leq p < \infty$. The space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ consists of all $f \in L_w^1(\mathbb{R}^d)$ such that $\mathcal{F}_\alpha f \in L_\omega^p(\mathbb{R}^d)$. The linear space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is a Banach space under the norm:

$$\|f\|_{A_{\alpha,p}^{w,\omega}} = \|f\|_{1,w} + \|\mathcal{F}_\alpha f\|_{p,\omega}.$$

The space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is translation and modulation invariant space [19]. The family of all translation and modulation invariant spaces, studied recently in [5]. In this work we discuss compact and non-compact embeddings between the spaces $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. General compactness criteria for function spaces are given in [7] or more general in [8].

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i = 2k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Then by definition of fractional Fourier transform, $\mathcal{F}_\alpha f(u) = f(u)$ for all $u \in \mathbb{R}^d$. Thus we have

$$A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) = \left\{ f \in L_w^1(\mathbb{R}^d) \mid f \in L_\omega^p(\mathbb{R}^d) \right\} = L_w^1(\mathbb{R}^d) \cap L_\omega^p(\mathbb{R}^d). \tag{1.1}$$

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i = (2k + 1)\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Then by definition of fractional Fourier transform, $\mathcal{F}_\alpha f(u) = f(-u)$ for all $u \in \mathbb{R}^d$. We may assume without loss of generality that ω symmetric, i.e., satisfy $\omega(x) = \omega(-x)$ for all $x \in \mathbb{R}^d$. Again, we obtain (1.1). Throughout this study, unless otherwise indicated, we get $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. In this paper we extend the results of [12] to the fractional Fourier transform. An angle parameter α provides us a new aspect of results that have been established in [12]. Additionally, we give some examples of why a necessary condition cannot be provided.

2. Some compact and non-compact embedding theorems for the function spaces $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$

We begin with the following basic findings that will be used as a tool to show the inclusion properties and non-compact embeddings of the spaces $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$.

Proposition 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w and ω be weight functions on \mathbb{R}^d and $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Then*

$$\|T_y M_b g\|_{A_{\alpha,p}^{w,\omega}} \leq w(y) \|g\|_{1,w} + \|\mathcal{F}_\alpha g\|_{p,\omega} \tag{2.1}$$

and so $T_y M_b g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$, where $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$.

Proof. Let $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ and let $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Then, $g \in L_w^1(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L_\omega^p(\mathbb{R}^d)$. We may write by [10, Proposition 1.7]

$$\begin{aligned} \|T_y M_b g\|_{1,w} &= \int_{\mathbb{R}^d} |g(x - y)| \left| e^{i(x-y)b} \right| w(x) dx = \|T_y g\|_{1,w} \\ &\leq w(y) \|g\|_{1,w}. \end{aligned} \tag{2.2}$$

Also, we have

$$\begin{aligned} \|\mathcal{F}_\alpha(T_y M_b g)\|_{p,\omega}^p &= \int_{\mathbb{R}^d} |\mathcal{F}_\alpha(T_y M_b g)(u)|^p \omega^p(u) du \\ &= \left| \prod_{j=1}^d \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}} \right|^p \\ &\quad \times \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (T_y M_b g)(t) e^{j=1 \sum_{j=1}^d \left(\frac{i}{2}(u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j\right)} dt \right|^p \omega^p(u) du \\ &= \left| \prod_{j=1}^d \sqrt{\frac{1 - i \cot \alpha_j}{2\pi}} \right|^p \end{aligned}$$

$$\begin{aligned} & \times \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g(t-y) e^{\sum_{j=1}^d i(t_j-y_j)b_j} \right. \\ & \times \left. e^{\sum_{j=1}^d \left(\frac{i}{2}(u_j^2+t_j^2)\cot\alpha_j - iu_j t_j \operatorname{cosec}\alpha_j\right)} dt \right|^p \omega^p(u) du. \end{aligned}$$

By substitution $t - y = l$, we get

$$\begin{aligned} \|\mathcal{F}_\alpha(T_y M_b g)\|_{p,\omega}^p &= \left| \prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}} \right|^p \tag{2.3} \\ & \times \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g(l) e^{\sum_{j=1}^d \left(\frac{i}{2}(u_j^2+(l_j+y_j)^2)\cot\alpha_j - iu_j(l_j+y_j)\operatorname{cosec}\alpha_j\right)} \right. \\ & \times \left. e^{\sum_{j=1}^d -il_j y_j \cot\alpha_j} dl \right|^p \omega^p(u) du \\ &= \left| \prod_{j=1}^d \sqrt{\frac{1-i\cot\alpha_j}{2\pi}} \right|^p \left| \int_{\mathbb{R}^d} e^{\sum_{j=1}^d \left(\frac{i}{2}y_j^2 \cot\alpha_j - iu_j y_j \operatorname{cosec}\alpha_j\right)} \right|^p \\ & \times \left| \int_{\mathbb{R}^d} g(l) e^{\sum_{j=1}^d \left(\frac{i}{2}(u_j^2+l_j^2)\cot\alpha_j - iu_j l_j \operatorname{cosec}\alpha_j\right)} dl \right|^p \omega^p(u) du \\ &= \int_{\mathbb{R}^d} |\mathcal{F}_\alpha g(u)|^p \omega^p(u) du \\ &= \|\mathcal{F}_\alpha g\|_{p,\omega}^p. \end{aligned}$$

Therefore, we obtain inequality (2.1). This means that $T_y M_b g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. □

The following proposition will be needed to show the inclusion properties of the space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$.

Proposition 2.2. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w and ω be weight functions on \mathbb{R}^d . For every $0 \neq g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ there exists constants $c(g), c(\mathcal{F}_\alpha g) > 0$ such that*

$$c(g) w(y) \leq \|T_y M_b g\|_{A_{\alpha,p}^{w,\omega}} \leq w(y) \|g\|_{A_{\alpha,p}^{w,\omega}} \tag{2.4}$$

and

$$c(\mathcal{F}_\alpha g) \omega(a) \leq \|M_z g\|_{A_{\alpha,p}^{w,\omega}} \leq \omega(a) \|g\|_{A_{\alpha,p}^{w,\omega}} \tag{2.5}$$

where $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ and $a = (z_1 \sin \alpha_1, \dots, z_d \sin \alpha_d)$ for all $y = (y_1, \dots, y_d), z = (z_1, \dots, z_d) \in \mathbb{R}^d$, respectively.

Proof. Let us take $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ and $a = (z_1 \sin \alpha_1, \dots, z_d \sin \alpha_d)$ for all $y = (y_1, \dots, y_d), z = (z_1, \dots, z_d) \in \mathbb{R}^d$, respectively. Let $0 \neq g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Thus

$g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^p_\omega(\mathbb{R}^d)$. By [10, Proposition 1.7], there exists $c(g) > 0$ such that

$$c(g) w(y) \leq \|T_y g\|_{1,w} \leq w(y) \|g\|_{1,w}. \tag{2.6}$$

Using (2.2), (2.3) and (2.6), we obtain

$$\begin{aligned} c(g) w(y) &\leq \|T_y g\|_{1,w} \leq \|T_y M_b g\|_{1,w} + \|\mathcal{F}_\alpha(T_y M_b g)\|_{p,\omega} \\ &\leq w(y) \|g\|_{1,w} + w(y) \|\mathcal{F}_\alpha g\|_{p,\omega} = w(y) \|g\|_{A^{w,\omega}_{\alpha,p}}. \end{aligned}$$

This means inequality (2.4) holds.

By Proposition 3 (2) in [19], we have

$$\|\mathcal{F}_\alpha(M_z g)\|_{p,\omega} = \|T_a \mathcal{F}_\alpha g\|_{p,\omega}. \tag{2.7}$$

Again, by [10, Proposition 1.7], there exists $c(\mathcal{F}_\alpha g) > 0$ such that

$$c(\mathcal{F}_\alpha g) \omega(a) \leq \|T_a \mathcal{F}_\alpha g\|_{p,\omega} \leq \omega(a) \|\mathcal{F}_\alpha g\|_{p,\omega}. \tag{2.8}$$

Combining (2.7), (2.8) and the equality $\|M_z g\|_{1,w} = \|g\|_{1,w}$, we get

$$\begin{aligned} c(\mathcal{F}_\alpha g) \omega(a) &\leq \|T_a \mathcal{F}_\alpha g\|_{p,\omega} \leq \|M_z g\|_{1,w} + \|\mathcal{F}_\alpha(M_z g)\|_{p,\omega} \\ &\leq \omega(a) \|g\|_{1,w} + \omega(a) \|\mathcal{F}_\alpha g\|_{p,\omega} = \omega(a) \|g\|_{A^{w,\omega}_{\alpha,p}}. \end{aligned}$$

□

The proof of the following lemma is very similiar to the proof of Lemma 11 in [19], and therefore we omit the details.

Lemma 2.3. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w, ω and v be weight functions on \mathbb{R}^d . If $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d) \subset L^1_v(\mathbb{R}^d)$, then $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$ is a Banach space under the norm $\|g\| = \|g\|_{A^{w,\omega}_{\alpha,p}} + \|g\|_{1,v}$.*

Theorem 2.4. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w, ω and v be weight functions on \mathbb{R}^d . Then $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d) \subset L^1_v(\mathbb{R}^d)$ if and only if $v \prec w$.*

Proof. Assume that $v \prec w$. Hence, there exists $c_1 > 0$ such that $v(x) \leq c_1 w(x)$ for all $x \in \mathbb{R}^d$. Let $g \in A^{w,\omega}_{\alpha,p}(\mathbb{R}^d)$. Thus $g \in L^1_w(\mathbb{R}^d)$ and $\mathcal{F}_\alpha g \in L^p_\omega(\mathbb{R}^d)$. We write

$$\|gv\|_1 \leq c_1 \|gw\|_1$$

and then

$$\|g\|_{1,v} \leq c_1 \|g\|_{1,w}.$$

This implies

$$\|g\|_{1,v} \leq c_1 \|g\|_{1,w} + c_1 \|\mathcal{F}_\alpha g\|_{p,\omega} = c_1 \|g\|_{A^{w,\omega}_{\alpha,p}} < \infty.$$

Therefore, $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d) \subset L^1_v(\mathbb{R}^d)$.

Conversely, assume that $A^{w,\omega}_{\alpha,p}(\mathbb{R}^d) \subset L^1_v(\mathbb{R}^d)$. Let us take $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. By Proposition 2.2, there exists constants $c_2, c_3 > 0$ such that

$$c_2 w(y) \leq \|T_y M_b g\|_{A^{w,\omega}_{\alpha,p}} \leq c_3 w(y). \tag{2.9}$$

By [9, Lemma 2.2] and (2.2), there exists constants $c_4, c_5 > 0$ such that

$$c_4 v(y) \leq \|T_y g\|_{1,v} = \|T_y M_b g\|_{1,v} \leq c_5 v(y). \tag{2.10}$$

From Lemma 2.3 the space $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ is a Banach space under the norm $\|g\|$ for all $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Then the closed graph theorem implies the existence of a constant $c_6 > 0$ such that $\|g\|_{1,v} \leq c_6 \|g\|_{A_{\alpha,p}^{w,\omega}}$ for all $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Since $T_y M_b g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$, we get

$$\|T_y M_b g\|_{1,v} \leq c_6 \|T_y M_b g\|_{A_{\alpha,p}^{w,\omega}}. \tag{2.11}$$

Thus, combining (2.9), (2.10) and (2.11), we obtain

$$c_4 v(y) \leq \|T_y g\|_{1,v} = \|T_y M_b g\|_{1,v} \leq c_6 \|T_y M_b g\|_{A_{\alpha,p}^{w,\omega}} \leq c_6 c_3 w(y).$$

Let $c = \frac{c_6 c_3}{c_4}$. Then we have $v(y) \leq cw(y)$ for all $y \in \mathbb{R}^d$. □

The following theorem is the extension of Theorem 12 in [19].

Theorem 2.5. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w_1, w_2, ω_1 and ω_2 be weight functions on \mathbb{R}^d . Then $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ if and only if $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$.*

Proof. First of all, suppose that $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$. Then it is clear that $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ by [19, Proposition 13].

Now, assume that $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. By Proposition 2.2, there exist constants $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 w_1(y) \leq \|T_y M_b g\|_{A_{\alpha,p}^{w_1,\omega_1}} \leq c_2 w_1(y) \tag{2.12}$$

and

$$c_3 w_2(y) \leq \|T_y M_b g\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_4 w_2(y) \tag{2.13}$$

where $b = (-y_1 \cot \alpha_1, \dots, -y_d \cot \alpha_d)$ for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$. According to Lemma 11 in [19], the space $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ is a Banach space under the norm

$$\|g\| = \|g\|_{A_{\alpha,p}^{w_1,\omega_1}} + \|g\|_{A_{\alpha,p}^{w_2,\omega_2}} \tag{2.14}$$

for all $g \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Therefore, $w_2 \prec w_1$ follows from the closed graph theorem and applying the same technique in the proof of the Theorem 2.4. Again, by Proposition 2.2, there exist constants $c_5, c_6, c_7, c_8 > 0$ such that

$$c_5 \omega_1(a) \leq \|M_z g\|_{A_{\alpha,p}^{w_1,\omega_1}} \leq c_6 \omega_1(a) \tag{2.15}$$

and

$$c_7 \omega_2(a) \leq \|M_z g\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_8 \omega_2(a) \tag{2.16}$$

where $a = (z_1 \sin \alpha_1, \dots, z_d \sin \alpha_d)$ for all $z = (z_1, \dots, z_d) \in \mathbb{R}^d$. Then the closed graph theorem implies the existence of a constant $c_9 > 0$ such that $\|g\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_9 \|g\|_{A_{\alpha,p}^{w_1,\omega_1}}$ for all $g \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Since $M_z g \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$, we get

$$\|M_z g\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_9 \|M_z g\|_{A_{\alpha,p}^{w_1,\omega_1}}. \tag{2.17}$$

Thus, combining (2.15), (2.16) and (2.17), we obtain

$$c_7 \omega_2(a) \leq \|M_z g\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_9 \|M_z g\|_{A_{\alpha,p}^{w_1,\omega_1}} \leq c_9 c_6 \omega_1(a).$$

Let $c = \frac{c_9 c_6}{c_7}$. Then we have $\omega_2(a) \leq c\omega_1(a)$ for all $a \in \mathbb{R}^d$. □

Theorem 2.6. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let $1 \leq p < \infty$ and w, ω and v be weight functions on \mathbb{R}^d . If $v \prec w$ and the function ω is bounded then $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \hookrightarrow L_v^1(\mathbb{R}^d)$ is not compact.

Proof. Since $v \prec w$, then there exists $c_1 > 0$ such that $v(x) \leq c_1 w(x)$ for all $x \in \mathbb{R}^d$ and also $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \subset L_v^1(\mathbb{R}^d)$ by Theorem 2.4. Besides the unit map I from $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ into $L_v^1(\mathbb{R}^d)$ is bounded

$$\|I(f)\|_{1,v} = \|f\|_{1,v} \leq c_1 \|f\|_{1,w} \leq c_1 \|f\|_{A_{\alpha,p}^{w,\omega}}$$

for all $f \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$.

Let $g \in C_C(\mathbb{R}^d) \cap A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ be a fixed arbitrary function. We take sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = M_{t_n} g$. From the hypothesis, there exists $c_2 \geq 1$ such that $\omega(x) \leq c_2$ for all $x \in \mathbb{R}^d$. Then we write

$$\|g_n\|_{A_{\alpha,p}^{w,\omega}} = \|M_{t_n} g\|_{A_{\alpha,p}^{w,\omega}} \leq \|g\|_{1,w} + c_2 \|\mathcal{F}_\alpha g\|_{p,\omega} \leq c_2 \|g\|_{A_{\alpha,p}^{w,\omega}}. \tag{2.18}$$

Hence, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Now, we will show that there is no subsequence of $(g_n)_{n \in \mathbb{N}}$ that is convergent in $L_v^1(\mathbb{R}^d)$. Let $h \in C_C(\mathbb{R}^d)$ be given. We get

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \int_{\mathbb{R}^d} |M_{t_n} g(x)| |h(x)| dx \\ &\leq \|h\|_\infty \|g\|_1 \\ &= \|h\|_\infty \|T_{t_n} g\|_1. \end{aligned} \tag{2.19}$$

Since $g \in C_C(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, $T_{t_n} g \rightarrow 0$ as $n \rightarrow \infty$ and then there exists $N \in \mathbb{N}$ such that $\text{supp} g(x - t_n) = \emptyset$ for all $n \geq N$ and all $x \in \mathbb{R}^d$. On the other hand we get $\text{supp} g(x - t_j) = K_j$ where $j = 1, 2, \dots, N - 1$. Let $K = K_1 \cup K_2 \cup \dots \cup K_{N-1}$. Then, using (2.19) we write

$$\left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| \leq \|h\|_\infty \int_K |T_{t_n} g(x)| dx. \tag{2.20}$$

Also we have

$$|T_{t_n} g(x)| = |g(x - t_n)| \leq \sup_{x - t_n \in \mathbb{R}^d} |g(x - t_n)| = \|g\|_\infty \tag{2.21}$$

for all $n \in \mathbb{N}$. Then by bounded converge theorem and using (2.20), (2.21) we obtain

$$\int_{\mathbb{R}^d} g_n(x) h(x) dx \rightarrow 0 \tag{2.22}$$

as $n \rightarrow \infty$. Assume that $(g_n)_{n \in \mathbb{N}}$ converges to $f \neq 0$ in $L_v^1(\mathbb{R}^d)$. Thus we may write

$$\left| \int_{\mathbb{R}^d} (g_n - f)(x) h(x) dx \right| \leq \int_{\mathbb{R}^d} |(g_n - f)(x)| |h(x)| dx \leq \|h\|_\infty \|g_n - f\|_{1,v}.$$

Since $\|g_n - f\|_{1,v} \rightarrow 0$ as $n \rightarrow \infty$, then we have

$$\int_{\mathbb{R}^d} (g_n - f)(x) h(x) dx \rightarrow 0$$

as $n \rightarrow \infty$. Hence by using (2.22), we obtain

$$\int_{\mathbb{R}^d} f(x) h(x) dx = 0$$

for all $h \in C_C(\mathbb{R}^d)$. Then $f = 0$ almost everywhere. This is a contradiction. Therefore only possible limit of $(g_n)_{n \in \mathbb{N}}$ in $L_v^1(\mathbb{R}^d)$ is zero. It is known that there exist $c_3(g), c_4 > 0$ such that

$$c_3(g) v(x) \leq \|T_x g\|_{1,v} \leq c_4 v(x) \tag{2.23}$$

for all $g \in L_v^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, [9]. Therefore, we obtain

$$\|g_n\|_{1,v} = \|M_{t_n} g\|_{1,v} = \|g\|_{1,v} \geq \frac{\|T_{t_n} g\|_{1,v}}{v(t_n)} \geq \frac{c_3(g) v(t_n)}{v(t_n)} = c_3(g). \tag{2.24}$$

Then $(g_n)_{n \in \mathbb{N}}$ does not converge to zero and also it is impossible to find convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L_v^1(\mathbb{R}^d)$. In that case, the embedding I is not compact. \square

Let us give an example that the converse of the above theorem is not correct.

Example 2.7. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w, ω and v be weight functions on \mathbb{R}^d such that $v(x) = w(x) = (1 + \|x\|)^2$ and $\omega(x) = 1 + \|x\|$, for all $x \in \mathbb{R}^d$. It is easy to see that these functions are weight functions. Also let us take $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_1 = \alpha_2 = \dots = \alpha_d$. Then the embedding I from $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ into $L_v^1(\mathbb{R}^d)$ is not compact and the function ω is not bounded. Indeed, $v \prec w$ then $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \subset L_v^1(\mathbb{R}^d)$ and also the unit map I is continuous from $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ into $L_v^1(\mathbb{R}^d)$. Let us take any function $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ and fix it. Besides, we take sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{T_{t_n} g}{w(t_n)} = \frac{T_{t_n} g}{(1 + \|t_n\|)^2}$. Therefore, we have

$$\|g_n\|_{A_{\alpha,p}^{w,\omega}} = \left\| \frac{T_{t_n} g}{w(t_n)} \right\|_{A_{\alpha,p}^{w,\omega}} = \frac{1}{w(t_n)} \|T_{t_n} g\|_{A_{\alpha,p}^{w,\omega}}. \tag{2.25}$$

Also we write

$$\omega(x) = 1 + \|x\| \leq (1 + \|x\|)^2 = w(x). \tag{2.26}$$

Moreover we have the inequality

$$\|T_x g\|_{A_{\alpha,p}^{w,\omega}} \leq w(x) \|g\|_{1,w} + \omega(x \cos \alpha_1) \|\mathcal{F}_\alpha g\|_{p,\omega} \tag{2.27}$$

for all $x \in \mathbb{R}^d$, by Theorem 4 in [19]. Then, by using (2.26) we get

$$\begin{aligned} \|T_x g\|_{A_{\alpha,p}^{w,\omega}} &\leq w(x) \|g\|_{1,w} + \omega(x \cos \alpha_1) \|\mathcal{F}_\alpha g\|_{p,\omega} \\ &= w(x) \|g\|_{1,w} + (1 + |\cos \alpha_1| \|x\|) \|\mathcal{F}_\alpha g\|_{p,\omega} \\ &\leq w(x) \|g\|_{1,w} + (1 + \|x\|) \|\mathcal{F}_\alpha g\|_{p,\omega} \\ &\leq w(x) \|g\|_{A_{\alpha,p}^{w,\omega}}. \end{aligned}$$

Thus we have

$$\|g_n\|_{A_{\alpha,p}^{w,\omega}} = \left\| \frac{T_{t_n} g}{w(t_n)} \right\|_{A_{\alpha,p}^{w,\omega}} \leq \frac{w(t_n)}{w(t_n)} \|g\|_{A_{\alpha,p}^{w,\omega}} = \|g\|_{A_{\alpha,p}^{w,\omega}}.$$

Hence, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Let $h \in C_C(\mathbb{R}^d)$ be given. We write

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \frac{1}{w(t_n)} \int_{\mathbb{R}^d} |T_{t_n} g(x)| |h(x)| dx \\ &\leq \frac{1}{w(t_n)} \|h\|_{\infty} \|g\|_1. \end{aligned} \tag{2.28}$$

Since $w(x)$ tends to infinity as $x \rightarrow \infty$, $w(t_n)$ tends to infinity as $n \rightarrow \infty$. Then we obtain

$$\int_{\mathbb{R}^d} g_n(x) h(x) dx \rightarrow 0 \tag{2.29}$$

as $n \rightarrow \infty$. After that, by using the same technique in the proof of Theorem 2.6, we obtain that the embedding I is not compact. On the other hand, $\omega(x) \rightarrow \infty$ as $x \rightarrow \infty$, and then the function ω is not bounded.

Theorem 2.8. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let $1 \leq p < \infty$ and w, ω and v be weight functions on \mathbb{R}^d . If $v \prec w$ and function $\frac{v(x)}{w(x)}$ does not tend to zero for $x \rightarrow \infty$, then the embedding*

$$I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \hookrightarrow L_v^1(\mathbb{R}^d)$$

is never compact.

Proof. Since $v \prec w$, Theorem 2.4 and the proof of Theorem 2.6 implies that $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \subset L_v^1(\mathbb{R}^d)$ and continuity of the unit map I , respectively.

First of all, suppose that $w(x) \rightarrow \infty$ as $x \rightarrow \infty$. Let us take any fixed function $g \in A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Also we take sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we define a sequence $(b_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $b_{nj} = -t_{nj} \cot \alpha_j$. To show that the mapping $I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is never compact, define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{T_{t_n} M_{b_n} g}{w(t_n)}$. Therefore, we have

$$\|g_n\|_{A_{\alpha,p}^{w,\omega}} = \left\| \frac{T_{t_n} M_{b_n} g}{w(t_n)} \right\|_{A_{\alpha,p}^{w,\omega}} = \frac{1}{w(t_n)} \|T_{t_n} M_{b_n} g\|_{A_{\alpha,p}^{w,\omega}}.$$

By using (2.1) we write

$$\|g_n\|_{A_{\alpha,p}^{w,\omega}} \leq \frac{w(t_n)}{w(t_n)} \|g\|_{A_{\alpha,p}^{w,\omega}} = \|g\|_{A_{\alpha,p}^{w,\omega}}. \tag{2.30}$$

Hence, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Now, we will show that there wouldn't be a subsequence of $(g_n)_{n \in \mathbb{N}}$ which is convergent in $L_v^1(\mathbb{R}^d)$. We have

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \frac{1}{w(t_n)} \int_{\mathbb{R}^d} |T_{t_n} M_{b_n} g(x)| |h(x)| dx \\ &\leq \frac{1}{w(t_n)} \|h\|_{\infty} \|g\|_1 \end{aligned} \tag{2.31}$$

for all $h \in C_C(\mathbb{R}^d)$. Since $w(t_n) \rightarrow \infty$ as $n \rightarrow \infty$, then by using inequality (2.31) we obtain

$$\int_{\mathbb{R}^d} g_n(x) h(x) dx \rightarrow 0$$

as $n \rightarrow \infty$. Moreover, by the inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \int_{\mathbb{R}^d} |g_n(x)| |h(x)| dx \\ &\leq \sup_{x \in \mathbb{R}^d} |h(x)| \int_{\mathbb{R}^d} |g_n(x)| dx \\ &\leq \|h\|_{\infty} \|g_n\|_{1,v} \end{aligned} \tag{2.32}$$

and using the same technique in the proof of Theorem 2.6, we say that only possible limit of $(g_n)_{n \in \mathbb{N}}$ in $L_v^1(\mathbb{R}^d)$ is zero. Also, using (2.23) and (2.2), there exists $c > 0$ such that

$$\|g_n\|_{1,v} = \frac{1}{w(t_n)} \|T_{t_n} M_{b_n} g\|_{1,v} = \frac{1}{w(t_n)} \|T_{t_n} g\|_{1,v} \geq c \frac{v(t_n)}{w(t_n)}. \tag{2.33}$$

Since the function $\frac{v(t_n)}{w(t_n)}$ does not tend to zero for $n \rightarrow \infty$ and then there exists $\delta > 0$ such that $\frac{v(t_n)}{w(t_n)} \geq \delta$ as $n \rightarrow \infty$. Then by using (2.33) we have

$$\|g_n\|_{1,v} \geq \frac{v(t_n)}{w(t_n)} c \geq c\delta. \tag{2.34}$$

Hence, $(g_n)_{n \in \mathbb{N}}$ does not converge to zero and also it is impossible to find convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L_v^1(\mathbb{R}^d)$. Consequently, the embedding $I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is never compact.

Suppose that w is constant or bounded function. Since $v \prec w$, the function $\frac{v(x)}{w(x)}$ is constant or bounded and then doesn't tend to zero as $x \rightarrow \infty$. Let us take any fixed function $g \in C_C(\mathbb{R}^d) \cap A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$. Besides we take a sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{T_{t_n} M_{b_n} g}{w(t_n)}$, where $(b_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^d such that $b_{nj} = -t_{nj} \cot \alpha_j$. Thus, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d)$ by (2.30). Let $h \in C_C(\mathbb{R}^d)$ be given. We write

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \frac{1}{w(t_n)} \int_{\mathbb{R}^d} |T_{t_n} M_{b_n} g(x)| |h(x)| dx \\ &\leq \frac{1}{w(t_n)} \|h\|_{\infty} \int_{\mathbb{R}^d} |T_{t_n} g(x)| dx. \end{aligned}$$

Since $g \in C_C(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, $T_{t_n} g \rightarrow 0$ as $n \rightarrow \infty$ and then by using (2.20), (2.21) we obtain

$$\int_{\mathbb{R}^d} g_n(x) h(x) dx \rightarrow 0.$$

Therefore, it is impossible to find convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L_v^1(\mathbb{R}^d)$ by (2.32), (2.33) and (2.34). Thus, the embedding $I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is never compact. \square

Again, we will give an example that the converse of the above theorem is not correct.

Example 2.9. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w, v and ω be weight functions on \mathbb{R}^d such that $v(x) = 1 + \|x\|$, $w(x) = (1 + \|x\|)^2$ for all $x \in \mathbb{R}^d$ and ω is bounded. In that case, the embedding $I :$

$A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is never compact and the function $\frac{v(x)}{w(x)}$ tends to zero as $x \rightarrow \infty$. Indeed, we get

$$1 + \|x\| \leq (1 + \|x\|)^2$$

for all $x \in \mathbb{R}^d$. Since $v \prec w$, then $A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \subset L_v^1(\mathbb{R}^d)$. Also the unit map $I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is continuous. Thus by using Theorem 2.6, it is clear that the embedding $I : A_{\alpha,p}^{w,\omega}(\mathbb{R}^d) \rightarrow L_v^1(\mathbb{R}^d)$ is never compact. On the other hand, it is easy to show that $\frac{v(x)}{w(x)}$ tends to zero as $x \rightarrow \infty$.

Theorem 2.10. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let $1 \leq p < \infty$ and w_1, w_2, ω_1 and ω_2 be weight functions on \mathbb{R}^d . If $w_2 \prec w_1$, $\omega_2 \prec \omega_1$ and functions $\frac{w_2(x)}{w_1(x)}$ or $\frac{\omega_2(x)}{\omega_1(x)}$ do not tend to zero for $x \rightarrow \infty$, then the embedding I from $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ into $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ is not compact.*

Proof. Let $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$. Then it is known that $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d) \subset A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ by Theorem 2.5. Also there exists $c_1, c_2 > 0$ such that $w_2(x) \leq c_1 w_1(x)$ and $\omega_2(x) \leq c_2 \omega_1(x)$ for all $x \in \mathbb{R}^d$. Now, let us define a unit map I from $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ into $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. Then we have

$$\|I(f)\|_{A_{\alpha,p}^{w_2,\omega_2}} = \|f\|_{A_{\alpha,p}^{w_2,\omega_2}} \leq c_1 \|f\|_{1,w_1} + c_2 \|\mathcal{F}_\alpha f\|_{p,\omega_1} \leq c_3 \|f\|_{A_{\alpha,p}^{w_1,\omega_1}}$$

where $c_3 = \max\{c_1, c_2\}$, for all $f \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. So I is continuous.

First of all, let the function $\frac{w_2(x)}{w_1(x)}$ does not tend to zero for $x \rightarrow \infty$. Suppose that $w_1(x) \rightarrow \infty$ as $x \rightarrow \infty$. Again, let us take any fixed function $g \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Also we take sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{T_{t_n} M_{b_n} g}{w_1(t_n)}$, where $(b_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^d such that $b_{nj} = -t_{nj} \cot \alpha_j$. Thus, using by inequality (2.30), it is easy to see that $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Now, assume that there exists convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. Since the inclusion $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d) \subset L_{w_2}^1(\mathbb{R}^d)$, then the subsequence of $(g_n)_{n \in \mathbb{N}}$ is also convergent in $L_{w_2}^1(\mathbb{R}^d)$. But it is well known from the proof of Theorem 2.8 that it is impossible to obtain convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L_{w_2}^1(\mathbb{R}^d)$. That means the embedding I is not compact. Suppose that w_1 is constant or bounded function. Similarly, let us take any fixed function $g \in C_C(\mathbb{R}^d) \cap A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Besides we get sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Then we define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{T_{t_n} M_{b_n} g}{w_1(t_n)}$, where $(b_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^d such that $b_{nj} = -t_{nj} \cot \alpha_j$. Thus, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$ by (2.30). Now, assume that there exists convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. Since the inclusion $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d) \subset L_{w_2}^1(\mathbb{R}^d)$, then the subsequence of $(g_n)_{n \in \mathbb{N}}$ is also convergent in $L_{w_2}^1(\mathbb{R}^d)$. But it is well known from the proof of Theorem 2.8 that it is impossible to obtain convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L_{w_2}^1(\mathbb{R}^d)$. That means the embedding I is not compact.

Now, let the function $\frac{\omega_2(x)}{\omega_1(x)}$ does not tend to zero for $x \rightarrow \infty$. Suppose that $\omega_1(x) \rightarrow \infty$ as $x \rightarrow \infty$. Also, let us take sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Thus we define a sequence $(u_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $u_{nj} = t_{nj} \sin \alpha_j$. Obviously,

$u_n \rightarrow \infty$ as $n \rightarrow \infty$. Now, let us take any fixed function $g \in A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Then we define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = \frac{M_{t_n}g}{\omega_1(u_n)}$. Hence, we get

$$\|g_n\|_{A_{\alpha,p}^{w_1,\omega_1}} = \left\| \frac{M_{t_n}g}{\omega_1(u_n)} \right\|_{A_{\alpha,p}^{w_1,\omega_1}} = \frac{1}{\omega_1(u_n)} \|M_{t_n}g\|_{A_{\alpha,p}^{w_1,\omega_1}}. \tag{2.35}$$

Also we have the inequality

$$\|M_x g\|_{A_{\alpha,p}^{w_1,\omega_1}} \leq \|g\|_{1,w_1} + \omega_1(x_1 \sin \alpha_1, \dots, x_d \sin \alpha_d) \|\mathcal{F}_\alpha g\|_{p,\omega_1} \tag{2.36}$$

for all $x \in \mathbb{R}^d$, by Theorem 5 in [19]. Thus, by using (2.35) and (2.36) we write

$$\|g_n\|_{A_{\alpha,p}^{w_1,\omega_1}} \leq \frac{1}{\omega_1(u_n)} \|g\|_{1,w_1} + \frac{\omega_1(u_n)}{\omega_1(u_n)} \|\mathcal{F}_\alpha g\|_{p,\omega_1} \leq \|g\|_{A_{\alpha,p}^{w_1,\omega_1}}.$$

Therefore, the sequence $(g_n)_{n \in \mathbb{N}}$ is bounded in $A_{\alpha,p}^{w_1,\omega_1}(\mathbb{R}^d)$. Now, we will show that there is no subsequence of $(g_n)_{n \in \mathbb{N}}$ that is convergent in $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. Let $h \in C_C(\mathbb{R}^d)$ be given. We obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^d} g_n(x) h(x) dx \right| &\leq \frac{1}{\omega_1(u_n)} \int_{\mathbb{R}^d} |M_{t_n}g(x)| |h(x)| dx \\ &\leq \frac{1}{\omega_1(u_n)} \|h\|_\infty \|g\|_1. \end{aligned}$$

Since $(u_n)_{n \in \mathbb{N}}$ and $\omega_1(x)$ tend to infinity as $n \rightarrow \infty$ and $x \rightarrow \infty$, respectively, we obtain

$$\int_{\mathbb{R}^d} g_n(x) h(x) dx \rightarrow 0 \tag{2.37}$$

as $n \rightarrow \infty$. Assume that $(g_n)_{n \in \mathbb{N}}$ converges to $f \neq 0$ in $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$. Thus we may write

$$\left| \int_{\mathbb{R}^d} (g_n - f)(x) h(x) dx \right| \leq \int_{\mathbb{R}^d} |(g_n - f)(x)| |h(x)| dx \leq \|h\|_\infty \|g_n - f\|_{A_{\alpha,p}^{w_2,\omega_2}}.$$

Since $\|g_n - f\|_{A_{\alpha,p}^{w_2,\omega_2}} \rightarrow 0$ as $n \rightarrow \infty$, then we have

$$\int_{\mathbb{R}^d} (g_n - f)(x) h(x) dx \rightarrow 0$$

as $n \rightarrow \infty$. Hence by using (2.37), we obtain

$$\int_{\mathbb{R}^d} f(x) h(x) dx = 0$$

for all $h \in C_C(\mathbb{R}^d)$. Then $f = 0$ almost everywhere. This is a contradiction.

Therefore only possible limit of $(g_n)_{n \in \mathbb{N}}$ in $A_{\alpha,p}^{w_2,\omega_2}(\mathbb{R}^d)$ is zero. Let us take $a = (x_1 \sin \alpha_1, \dots, x_d \sin \alpha_d)$. Thus by using Proposition 3 in [19] we write

$$\|\mathcal{F}_\alpha(M_x g)\|_{p,\omega_2} = \|T_a(\mathcal{F}_\alpha g)\|_{p,\omega_2}. \tag{2.38}$$

Besides, it is known by [9] that there exists $c > 0$ such that

$$c\omega_2(x) \leq \|T_x f\|_{p,\omega_2} \tag{2.39}$$

for all $f \in L^p_{\omega_2}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Hence, by using (2.38) and (2.39) we obtain

$$\begin{aligned} \|g_n\|_{A^{w_2, \omega_2}_{\alpha, p}} &\geq \frac{1}{\omega_1(u_n)} \|\mathcal{F}_\alpha(M_{t_n}g)\|_{p, \omega_2} \\ &= \frac{1}{\omega_1(u_n)} \|T_{u_n}(\mathcal{F}_\alpha g)\|_{p, \omega_2} \\ &\geq \frac{\omega_2(u_n)}{\omega_1(u_n)} c. \end{aligned} \tag{2.40}$$

Since the function $\frac{\omega_2(u_n)}{\omega_1(u_n)}$ does not tend to zero for $n \rightarrow \infty$, there exists $\delta > 0$ such that $\frac{\omega_2(u_n)}{\omega_1(u_n)} \geq \delta$ as $n \rightarrow \infty$. Then by using (2.40) we have

$$\|g_n\|_{A^{w_2, \omega_2}_{\alpha, p}} \geq \frac{\omega_2(u_n)}{\omega_1(u_n)} c \geq c\delta.$$

Thus, it is impossible to find convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $A^{w_2, \omega_2}_{\alpha, p}(\mathbb{R}^d)$. Finally, the embedding I is not compact. Assume that ω_1 is constant or bounded function. Let us take any fixed function $g \in C_c(\mathbb{R}^d) \cap A^{w_1, \omega_1}_{\alpha, p}(\mathbb{R}^d)$, a sequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and define a sequence $(g_n)_{n \in \mathbb{N}}$ by $g_n = M_{t_n}g$. Thus, using by inequality (2.18), it is easy to see that $(g_n)_{n \in \mathbb{N}}$ is bounded in $A^{w_1, \omega_1}_{\alpha, p}(\mathbb{R}^d)$. Now, assume that there exists convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $A^{w_2, \omega_2}_{\alpha, p}(\mathbb{R}^d)$. Hence, since the inclusion $A^{w_2, \omega_2}_{\alpha, p}(\mathbb{R}^d) \subset L^1_{w_2}(\mathbb{R}^d)$, then the subsequence of $(g_n)_{n \in \mathbb{N}}$ is also convergent in $L^1_{w_2}(\mathbb{R}^d)$. But it is well known from the proof of Theorem 2.6 that it is impossible to obtain convergent subsequence of $(g_n)_{n \in \mathbb{N}}$ in $L^1_{w_2}(\mathbb{R}^d)$. That means the embedding I is not compact. \square

We conclude the article with the following theorem, which is the main result of the paper.

Theorem 2.11. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\alpha_i \neq k\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$. Let w_1, w_2 and ω_1, ω_2 be weight functions on \mathbb{R}^d and $w_2 \prec w_1, \omega_2 \prec \omega_1$. Then the embedding $I : A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d) \hookrightarrow A^{w_2, \omega_2}_{\alpha, 1}(\mathbb{R}^d)$ is compact if and only if the functions $\frac{w_2(x)}{w_1(x)}$ and $\frac{\omega_2(x)}{\omega_1(x)}$ tend to zero as $x \rightarrow \infty$.*

Proof. Let $w_2 \prec w_1$ and $\omega_2 \prec \omega_1$. Then it is known by Theorem 2.5 that $A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d) \subset A^{w_2, \omega_2}_{\alpha, 1}(\mathbb{R}^d)$. Also it is easy to see that unit map I is bounded $A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d)$ into $A^{w_2, \omega_2}_{\alpha, 1}(\mathbb{R}^d)$.

Now, suppose that the functions $\frac{w_2(x)}{w_1(x)}$ and $\frac{\omega_2(x)}{\omega_1(x)}$ tend to zero as $x \rightarrow \infty$. Let us take a bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d)$. To indicate that the embedding $A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d) \hookrightarrow A^{w_2, \omega_2}_{\alpha, 1}(\mathbb{R}^d)$ is compact, we will prove that (f_n) has a convergent subsequence in $A^{w_2, \omega_2}_{\alpha, 1}(\mathbb{R}^d)$. As the sequence $(f_n)_{n \in \mathbb{N}}$ is bounded in $A^{w_1, \omega_1}_{\alpha, 1}(\mathbb{R}^d)$, there exists $c_1 > 0$ such that

$$\|f_n\|_{A^{w_1, \omega_1}_{\alpha, 1}} \leq c_1 \tag{2.41}$$

for all $n \in \mathbb{N}$. Moreover, boundedness of the unit map I , there exists $c_2 > 0$ such that

$$\|I(f_n)\|_{A^{w_2, \omega_2}_{\alpha, 1}} = \|f_n\|_{A^{w_2, \omega_2}_{\alpha, 1}} \leq c_2 \|f_n\|_{A^{w_1, \omega_1}_{\alpha, 1}}. \tag{2.42}$$

Since, the functions $\frac{w_2(x)}{w_1(x)}$, $\frac{\omega_2(x)}{\omega_1(x)}$ tend to zero as $x \rightarrow \infty$, then there exist $\delta_k, \delta'_k > 0$ such that

$$\frac{w_2(x)}{w_1(x)} < \frac{1}{k}, \quad \text{and} \quad \frac{\omega_2(x)}{\omega_1(x)} < \frac{1}{k} \quad (2.43)$$

where $\|x\| > \delta_k$ and $\|x\| > \delta'_k$, respectively for any $k \in \mathbb{N}$. Thus there are sequences of increasing balls $(B_{\delta_k})_{k \in \mathbb{N}}$ and $(B_{\delta'_k})_{k \in \mathbb{N}}$ which are centered at origin and $\delta_k \rightarrow \infty$, $\delta'_k \rightarrow \infty$ as $k \rightarrow \infty$ such that providing the inequalities (2.43), where $x \in \mathbb{R}^d - B_{\delta_k}$ and $x \in \mathbb{R}^d - B_{\delta'_k}$, respectively for any $k \in \mathbb{N}$. Let $S_k = (B_{\delta_k} \cup B_{\delta'_k})$. Then we can write the inequalities (2.43) such that $x \in \mathbb{R}^d - S_k$ for $k \in \mathbb{N}$.

Suppose that, $(t_n)_{n \in \mathbb{N}}$ be any sequence which is dense in S_1 . Hence by (2.42), we get

$$\begin{aligned} \|\mathcal{F}_\alpha f_n\|_\infty &= \sup_{u \in \mathbb{R}^d} \left| \prod_{j=1}^d \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \int_{\mathbb{R}^d} f_n(t) e^{j=1} \sum_{j=1}^d \left(\frac{i}{2} (u_j^2 + t_j^2) \cot \alpha_j - i u_j t_j \operatorname{cosec} \alpha_j \right) dt \right| \\ &\leq \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right| \int_{\mathbb{R}^d} |f_n(t)| dt \\ &\leq \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right| \|f_n\|_{A_{\alpha,1}^{w_2,\omega_2}} \\ &\leq \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right| c_2 \|f_n\|_{A_{\alpha,1}^{w_1,\omega_1}}. \end{aligned}$$

Also, let $c_0 = \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \alpha_j}{2\pi}} \right| c_1 c_2$. Then by using inequality (2.41), we obtain

$$\|\mathcal{F}_\alpha f_n\|_\infty \leq c_0. \quad (2.44)$$

Therefore there exists a subsequence $(f_{n_1(k)})_{n_1(k) \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha f_{n_1(k)}(t_1))_{n_1(k) \in \mathbb{N}}$ converges. Likewise, there exists a subsequence $(f_{n_2(k)})_{n_2(k) \in \mathbb{N}}$ of $(f_{n_1(k)})_{n_1(k) \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha f_{n_2(k)}(t_2))_{n_2(k) \in \mathbb{N}}$ converges. Continuing this method, there exists a subsequence $(f_{n_l(k)})_{n_l(k) \in \mathbb{N}}$ of $(f_{n_{l-1}(k)})_{n_{l-1}(k) \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha f_{n_l(k)}(t_l))_{n_l(k) \in \mathbb{N}}$ converges. Hence the sequence $(\mathcal{F}_\alpha f_{n_l(k)}(t_r))_{n_l(k) \in \mathbb{N}}$ converges, for all $l \in \mathbb{N}$ such that $r \leq l$. Let $m(k) = n_k(k)$, then the sequence $(\mathcal{F}_\alpha f_{m(k)}(t_l))_{m(k) \in \mathbb{N}}$ converges which is subsequence of $(\mathcal{F}_\alpha f_{n_l(k)}(t_l))_{n_l(k) \in \mathbb{N}}$. Thus there exists a subsequence $(g_m)_{m \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha g_m(t_l))_{m \in \mathbb{N}}$ converges, for all $l \in \mathbb{N}$. Also, since the sequence $(t_n)_{n \in \mathbb{N}}$ is dense in S_1 , then the sequence $(\mathcal{F}_\alpha g_m(x))_{m \in \mathbb{N}}$ converges, for any $x \in S_1$. Using the method reviewed above, we can find a sequence $(u_m)_{m \in \mathbb{N}}$ which is a subsequence of $(g_m)_{m \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha u_m(x))_{m \in \mathbb{N}}$ converges, for all $x \in S_2$. Reiterating this technique, we obtain a sequence $(h_n)_{n \in \mathbb{N}}$ which is a subsequence of $(f_n)_{n \in \mathbb{N}}$ such that the sequence $(\mathcal{F}_\alpha h_n(x))_{n \in \mathbb{N}}$ converges, for all $x \in S_k$. Also, let $\beta = (\beta_1, \beta_2, \dots, \beta_d)$, such that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $\beta_i + \alpha_i = 2\pi$ for each index i with $1 \leq i \leq d$ and $k \in \mathbb{Z}$, and $\prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \beta_j}{2\pi}} \right| = M_1$.

Then by using (2.41) and (2.42) we get

$$\begin{aligned} \|h_n\|_\infty &= \|\mathcal{F}_\beta (\mathcal{F}_\alpha h_n)\|_\infty \leq \prod_{j=1}^d \left| \sqrt{\frac{1-i \cot \beta_j}{2\pi}} \right| \|\mathcal{F}_\alpha h_n\|_1 \\ &\leq M_1 c_2 \|h_n\|_{A_{\alpha,1}^{w_1,\omega_1}} \leq M_1 c_2 c_1. \end{aligned} \quad (2.45)$$

Hence, $(h_n)_{n \in \mathbb{N}}$ is bounded. By applying the steps in the first part of this proof, we have a sequence $(s_n)_{n \in \mathbb{N}}$ that is a subsequence of $(h_n)_{n \in \mathbb{N}}$, where the sequence $(s_n)_{n \in \mathbb{N}}$ converges on S_k , for all $k \in \mathbb{N}$.

Furthermore, by (2.41) and (2.43) we have

$$\begin{aligned}
 & \|s_n - s_m\|_{A_{\alpha,1}^{w_2,\omega_2}} \\
 &= \|s_n - s_m\|_{1,w_2} + \|\mathcal{F}_\alpha(s_n - s_m)\|_{1,\omega_2} \\
 &= \int_{S_k} |s_n(x) - s_m(x)| w_2(x) dx + \int_{\mathbb{R}^d - S_k} |s_n(x) - s_m(x)| w_2(x) dx \\
 &\quad + \int_{S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx + \int_{\mathbb{R}^d - S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx \\
 &\leq \int_{S_k} |s_n(x) - s_m(x)| w_2(x) dx + \int_{S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx \\
 &\quad + \frac{1}{k} (\|s_n\|_{1,w_1} + \|s_m\|_{1,w_1}) + \frac{1}{k} (\|\mathcal{F}_\alpha s_n\|_{1,\omega_1} + \|\mathcal{F}_\alpha s_m\|_{1,\omega_1}) \\
 &\leq \int_{S_k} |s_n(x) - s_m(x)| w_2(x) dx + \int_{S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx + \frac{4c_1}{k}.
 \end{aligned} \tag{2.46}$$

Let $\varepsilon > 0$ be given. Also, we select k large enough where $\frac{4c_1}{k} < \frac{\varepsilon}{3}$. Besides the sequences $(s_n)_{n \in \mathbb{N}}$ and $(\mathcal{F}_\alpha s_n)_{n \in \mathbb{N}}$ converge on $\overline{S_k}$ such that the set $\overline{S_k}$ is closure of S_k . Since the sequences $(s_n)_{n \in \mathbb{N}}$ and $(\mathcal{F}_\alpha s_n)_{n \in \mathbb{N}}$ converge pointwise on $\overline{S_k}$, and according to the inequalities (2.44) and (2.45) and the bounded converge theorem there exists $n_1 \in \mathbb{N}$ such that

$$\int_{S_k} |s_n(x) - s_m(x)| w_2(x) dx < \frac{\varepsilon}{3} \tag{2.47}$$

for all $m, n \geq n_1$ and also there exists $n_2 \in \mathbb{N}$ such that

$$\int_{S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx < \frac{\varepsilon}{3} \tag{2.48}$$

for all $m, n \geq n_2$. Let $N = \max\{n_1, n_2\}$. Thus, combining (2.46), (2.47) and (2.48) we get

$$\begin{aligned}
 \|s_n - s_m\|_{A_{\alpha,1}^{w_2,\omega_2}} &\leq \int_{S_k} |s_n(x) - s_m(x)| w_2(x) dx \\
 &\quad + \int_{S_k} |\mathcal{F}_\alpha s_n(x) - \mathcal{F}_\alpha s_m(x)| \omega_2(x) dx + \frac{4c_1}{k} < \varepsilon
 \end{aligned}$$

for all $m, n \geq N$. This implies that the sequence $(s_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $A_{\alpha,1}^{w_2,\omega_2}(\mathbb{R}^d)$. Thus, the embedding of the space $A_{\alpha,1}^{w_1,\omega_1}(\mathbb{R}^d)$ into $A_{\alpha,1}^{w_2,\omega_2}(\mathbb{R}^d)$ is compact.

Conversely, if the functions $\frac{w_2(x)}{w_1(x)}$ or $\frac{\omega_2(x)}{\omega_1(x)}$ do not tend to zero for $x \rightarrow \infty$, then by Theorem 2.10 the embedding of the space $A_{\alpha,1}^{w_1,\omega_1}(\mathbb{R}^d)$ into $A_{\alpha,1}^{w_2,\omega_2}(\mathbb{R}^d)$ is not compact. This is the desired result. \square

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