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On Homogeneous Randers Metrics

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we study the curvature features of the class of homogeneous Randers metrics. For these metrics, we first find a reduction criterion to be a Berwald metric based on a mild restriction on their Ricci tensors. Then, we prove that every homogeneous Randers metric with relatively isotropic (or weak) Landsberg curvature must be Riemannian. This provides an extension of well-known Deng-Hu theorem that proves the same result for a homogeneous Berwald-Randers metric of non-zero flag curvature.

Keywords: *Homogeneous metric, Ricci tensor, Randers metric.*
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1. Introduction.

Unlike the Riemannian geometry and its natural connection (the Levi-CiVita connection), while in Finsler geometry there are some important connections such as Berwald connection, Chern connection, Cartan connection. All of these connections have their self Riemannian curvatures that are related to other Riemannian curvatures of other Finslerian connections. These connections make different hh-Riemannian curvature and then they provide different Ricci curvature if the definition of Ricci curvature is the same as Riemannian case because in this case and unlike Riemannain case the obtained tensor is not necessarily symmetric. Indeed, in Finsler geometry there is a vast opportunity to define the Ricci tensor. H. Akbar-Zadeh has proposed two Ricci curvature tensors to Finsler geometry: one is defined by $Ric_{jk} := \frac{1}{2}[F^2Ric]_{y^jy^k}$ where Ric is Ricci curvature and trace of R^k_l which is abtained by contracting with y^i and y^j of hh-curvature tensor of Finslerian coonnections i.e. R^m_m and the other can be defined by $\widetilde{Ric}_{ab} := \frac{1}{2}(R_{ab} + R_{ba})$, where R_{ab} is the trace of hh-curvature of Cartan connection defined by $R_{ab} = R^l{}_{alb}$. The substraction of Ric_{ij} and $\widetilde{Ric}_{ij}$ is related to $\mathbf{H}$-curvature which is another important Finslerian quantity. In [2], D. Bao studied the Ricci flow in Finslerian setting by taking into account the first choice of defining the Ricci tensor. In order to provide a natural extension of Einstein metrics from Riemannian geometry to Finsler one, in 1988, Akbar-Zadeh used his first method of defining the Ricci curvature and defined a new curvature so-called scalar curvature [1] and later In 1995, introduced Einstein Finsler manifolds.

In [12], Li-Shen defined a new Ricci curvature tensor and obtain its relation with some non-Riemannian quantities such as χ-curvature and the quantity $\mathbf{H}$. Also, Bao-Robles characterized the Einstein-Randers manifolds [3][16]. This motivates us to reinvestigate the Ricci tensor.

Riemannian homogeneous spaces paly an important role in cosmological models, representation theory and geometric analysis. Similar to the Riemannian setting, a Finsler manifold whose Lie group of isometries acts transitively on it is called homogeneous. In [8], Deng-Hu extended Wallach's theorem from Riemannian manifolds to Finslerian ones and proved that $SU(2)$ is the only Lie group among simply connected compact Lie groups that admits a positively curved left-invariant Finsler structure. . This result provides a characterization of positively curved homogeneous Randers spaces. It is proved that the imposing the conditions of being Einsteinian metric with negative Ricci curvature on a homogeneous Randers space reduces it to a Riemannian

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one. In [11], they proved that a homogeneous Randers metric is Ricci-quadratic if and only if it is a Berwald metric.

Let $R_{ijkl} = R_{ijkl}(x, y)$ denote the Riemannian curvature of the Cartan connection and define the Ricci tensor as $R_{ij} := R_{i\ jr}^{\ r}$. Using it, let us define a new Ricci tensor in Finsler geometry as follows

$$\mathcal{R}_{ij} := R_{ij} - \frac{1}{n+1}\Big[(R_{ijk} + R_{kji} + F^{-1}F_{y^i}R_{kmj}y^m)I^k + I_iR_{kj}y^k\Big], \tag{1.1}$$

where $R_{i\ jr}^{\ r} := g^{rm}R_{imjr}$, C_{kli} and I_i denote the Cartan and mean Cartan torsions of F, respectively [20]. $\mathcal{R}_{ij}$ is not a symmetric tensor, in general. But in a Riemannian space, we get $\mathcal{R}_{ij} = R_{ij}$ which is a symmetric tensor. A Finsler metric F is called $\mathcal{R}$-flat if $\mathcal{R}_{ij} = 0$. We prove the following.

Theorem 1.1. *Every $\mathcal{R}$-flat homogeneous Randers metric is a Berwald metric.*

Let (M, F) be a Finsler manifold. Suppose that $\mathbf{C}_y$, $\mathbf{I}_y$, $\mathbf{L}_y$ and $\mathbf{J}_y$ denote the Cartan, mean Cartan, Landsberg and mean Landsberg curvatures of F, respectively. The Landsberg (resp., mean Landsberg) curvature is the rate of change of the Cartan (resp. mean) torsion along geodesics. By definition, $\mathbf{L}/\mathbf{C}$ and $\mathbf{J}/\mathbf{I}$ are regarded as the relative rate of change of $\mathbf{C}$ and $\mathbf{I}$ along Finslerian geodesics, respectively. Then F is said to be relatively isotropic (resp., mean) Landsberg metric if $\mathbf{L} + cF\mathbf{C} = 0$ (resp., $\mathbf{J} + cF\mathbf{I} = 0$), where $c = c(x)$ is a scalar function on M. In [9], Deng-Hu proved that a homogeneous Randers metric of Berwald type whose flag curvature is everywhere nonzero must be Riemannian. We give the same type of rigidity property for homogeneous Randers metrics which are isotropic mean Landsberg metrics. Indeed, we prove the following.

Theorem 1.2. *Let (M, F) be a homogeneous Randers manifold. If F is an isotropic mean Landsberg metric $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a nonzero and bounded scalar function on M, Then F is Riemannian.*

Theorem 1.2 dose hold for homogeneous Randers metric with vanishing Landsberg curvature.

If F is Riemannian, i.e., $F(y) = \sqrt{\mathbf{g}(y, y)}$ for some Riemannian metric $\mathbf{g}$, then $\mathbf{R}_y := \mathrm{R}(\cdot, y)y$, where $\mathrm{R}(u, v)z$ denotes the Riemannian curvature tensor of $\mathbf{g}$. In this case, $\mathbf{R}_y$ is quadratic in $y \in T_xM$. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be R-quadratic if $\mathbf{R}_y$ is quadratic in $y \in T_xM$ at each point $x \in M$. Every Berwald metric is R-quadratic. In [21], the authors proved that every homogeneous R-qudratic metric is a Landsberg metric. There is a natural extension of Landsberg metrics, namely stretch metrics. In [4], Berwald introduced the new non-Riemannian quantity called by stretch curvature $\mathbf{\Sigma}$ which can be considered as a generalization of Landsberg curvature. The geometric meaning of stretch metrics investigated by Shibata in [19] and Matsumoto in [14]. Recently, the authors prove that every homogeneous (α, β)-metric is a stretch metric if and only if it is a Berwald metric [21]. This throws a light into the well-known Deng-Xu's conjecture that every homogeneous Landsberg metric is a Berwald metric [25]. As its applications, the authors show that the classes of R-quadratic or generalized Landsberg homogeneous (α, β)-metrics are Berwaldian. There is a weaker notion of metrics- weakly stretch metrics. Taking trace with respect to $\mathbf{g}_y$ in first and second variables of $\mathbf{\Sigma}_y$ gives rise the mean stretch curvature $\bar{\mathbf{\Sigma}}_y$. A Finsler metric is said to be weakly stretch metric if $\bar{\mathbf{\Sigma}} = 0$ [24]. Then we have the following

$$\{\text{Berwald metrics}\} \subseteq \{\text{R-quadratic metrics}\} \subseteq \{\text{Landsberg metrics}\}$$
$$\{\text{Generalized Landsberg metrics}\} \subseteq \{\text{Stretch metrics}\} \subseteq \{\text{Weakly stretch metrics}\}.$$

Theorem 1.3. *Let $F = \alpha + \beta$ be a homogeneous Randers metric on a manifold M. Then the following are equivalent:*

(i) F is a R-quadratic metric;

(ii) F is a Landsberg metric;

(iii) F is a generalized Landsberg metric;

(iv) F is a stretch metric;

(v) F is a weakly stretch metric.

In this case, F reduces to a Berwald metric.

2. Preliminary

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y + ru + sv + tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion.

Put

$$\mathbf{C}_y(u, v, w) = C_{ijk}(y)u^iv^jw^k,$$

where $u = u^i\partial_i$, $v = v^j\partial_j$ and $w = w^j\partial_j$. Then

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i\partial y^j\partial y^k}.$$

By definition, F is a Riemannian metric $g_{ij} = g_{ij}(x)$ if and only if $C_{ijk} = 0$.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion [5].

Let (M, F) be an n-dimensional Finsler manifold. For a nonzero vector $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{M}_y(u, v, w) := \mathbf{C}_y(u, v, w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\}.$$

In local coordinates,

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

where $h_{ij} := FF_{y^iy^j}$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$.

Lemma 2.1. (Matsumoto-Hōjō Lemma) *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if the Matsumoto torsion vanishes.*

Given an n-dimensional Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are called spray coefficients and given by following

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

$\mathbf{G}$ is called the spray associated to F. A Finsler metric F is said to be affinely equivalent to another Finsler metric $\bar{F}$ on M if F and $\bar{F}$ induce the same sprays. A curve $c = c(t)$ in TM_0 is called an integral curve of $\mathbf{G}$ if it satisfies $\mathbf{G}(c) = \dot{c}$.

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j\partial y^k\partial y^l}.$$

F is called a Berwald metric if $\mathbf{B}$ vanishes or equivalently G^i are quadratic functions with respect to the direction argument.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u_1, u_2, u_3) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u_1, u_2, u_3), y\big),$$

and locally, $\mathbf{L}_y(u_1, u_2, u_3) := L_{\ ijk}(y)u_1^i u_2^j u_3^k$, where

$$L_{ijk} := -\frac{1}{2}y_l B^l{}_{ijk}.$$

A Finsler metric with vanishing $\mathbf{L}$-curvature is called a Landsberg metric.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

A Finsler metric with vanishing $\mathbf{J}$-curvature is said to be a weakly Landsberg metric.

Let $c(t)$ be a smooth curve and $U(t) = U^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ be a vector field along c. Define the covariant derivative of $U(t)$ along c by

$$D_{\dot c}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}\Big(c(t), \dot c(t)\Big)\Big\}\frac{\partial}{\partial x^i}_{|c(t)}.$$

$U(t)$ is said to be *linearly parallel* if $D_{\dot c}U(t) = 0$. In this case, the Landsberg and mean Landsberg curvatures of F can be defined as follows

$$\begin{aligned}\mathbf{L}_y(u, v, w) : &= \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\Big(U(t), V(t), W(t)\Big)\Big]|_{t=0},\\ \mathbf{J}_y(u) : &= \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\Big(U(t)\Big)\Big]|_{t=0},\end{aligned}$$

where $\sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$.

For a nonzero vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2\mathbf{R}_y$, $\forall\lambda > 0$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\frac{\partial}{\partial x^i}$, where

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature. Let us put

$$R^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\},\quad R^i_{j\ kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}. \tag{2.2}$$

Then

$$R^i_k = R^i_{j\ kl}y^j y^l,\quad R^i{}_{kl} = R^i_{j\ kl}y^j,\quad R^i_{j\ kl} + R^i_{j\ lk} = 0, \tag{2.3}$$

Take a local coordinate system (x^i) in M, the local natural frame $\{\frac{\partial}{\partial x^i}\}$ of T_xM determines a local natural frame $\partial_i|_v$ for π_v^*TM the fibers of the pull-back tangent bundle π^*TM, where $\partial_i|_v = (v, \frac{\partial}{\partial x^i}|_x)$, and $v = y^i\frac{\partial}{\partial x^i}|_x \in TM_0$. The fiber π_v^*TM is isomorphic to $T_{\pi(v)}M$ where $\pi(v) = x$. There is a canonical section ℓ of π^*TM defined by $\ell_v = (v, v)/F(v)$. Suppose that ∇ denotes the Cartan connection on π^*TM. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field for π^*TM with $e_n := \ell$. If $\{\omega^i\}_{i=1}^n$ denotes its dual co-frame field, then put

$$\nabla e_i = \omega_i^{\ j} \otimes e_j,\quad \Omega e_i = 2\Omega_i^{\ j} \otimes e_j,$$

where $\{\Omega_i^{\ j}\}$ and $\{\omega_i^{\ j}\}$ are called respectively, the curvature forms and connection forms of ∇ with respect to $\{e_i\}$. Put $\omega^{n+i} := \omega_n^{\ i} + d(logF)\delta^i_n$. Then $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. Since $\{\Omega_i^{\ j}\}$ are 2-forms on TM_0, they can be expanded as

$$\Omega_i^{\ j} = \frac{1}{2}R_i{}^j{}_{kl}\omega^k \wedge \omega^l + P_i{}^j{}_{kl}\omega^k \wedge \omega^{n+l} + \frac{1}{2}Q_i{}^j{}_{kl}\omega^{n+k} \wedge \omega^{n+l}.$$

Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be the local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$. The objects R, P and Q are called, respectively, the hh-, hv- and vv-curvature tensors of the Cartan connection with the components $R(\bar{e}_k, \bar{e}_l)e_i = R_{i\ kl}^{\ j} e_j$, $P(\bar{e}_k, \dot{e}_l)e_i = P_{i\ kl}^{\ j} e_j$ and $Q(\dot{e}_k, \dot{e}_l)e_i = Q_{i\ kl}^{\ j} e_j$.

Let "|" and "," denotes the horizontal and vertical derivation with respect to the Cartan connection ∇ of F, respectively. It is proved that hv-curvature satisfies $P_{i\ kl}^{\ j}$ if and only if the Finsler metric F is a Landsberg metric. Also, then the Riemannian curvature of ∇ satisfies the following identities

$$
\begin{aligned}
&R_{ijkl} = -R_{jikl}, \qquad R_{ijkl} = -R_{ijlk},\\
&R_{ijkl} + R_{iljk} + R_{iklj} = -\Big\{C_{i\ j}^{\ m} R_{mkl} + C_{i\ k}^{\ m} R_{mjl} + C_{i\ k}^{\ m} R_{mjl}\Big\},\\
&R_{ijkl} = R_{klij} + C_{l\ i}^{\ m} R_{mkj} + C_{l\ j}^{\ m} R_{mki}.
\end{aligned}
$$

In [13], Matsumoto proved the following Bianchi identity of Cartan connection

$$
\begin{aligned}
R_{l\ ij,k}^{\ h} + Q_{l\ kr}^{\ h} R^r_{\ ij} &+ \Big\{R_{l\ ir}^{\ h} C^r_{\ jk} + P_{l\ ir}^{\ h} L^r_{\ jk} + P_{l\ jk|i}^{\ h}\Big\}\\
&+ \Big\{R_{l\ jr}^{\ h} C^r_{\ ik} + P_{l\ jr}^{\ h} L^r_{\ ik} + P_{l\ ik|j}^{\ h}\Big\} = 0.
\end{aligned} \tag{2.4}
$$

By contraction of h and j in (2.4), one can get the following

$$
R_{lk,i} = P_{l\ sr}^{\ s} L^r_{\ ki} - R_{lr} C^r_{\ ki} - P_{l\ si|k}^{\ s} + P_{l\ ki}^{\ r} - R_{l\ kr}^{\ m} C^r_{\ mi} - P_{l\ kr}^{\ m} L^r_{\ si} + Q_{l\ ir}^{\ m} R^r_{\ mk}, \tag{2.5}
$$

which is the relation (2.5) in [20]. Contracting (2.5) by y^l yields

$$
R_{0k,i} = R_{ik} - R_{mr} C^r_{\ ki} y^m - J_{i|k} + L^r_{\ ki} J_r + L^r_{\ ki|r} - R^s_{\ kr} C^r_{\ si} - L^s_{\ kr} L^r_{\ si}. \tag{2.6}
$$

3. Proof of Theorem 1.1.

In this section, we are going to prove Theorem 1.1. First, we remark the following.

Lemma 3.1. ([21]) Let (M, F) be a homogeneous Finsler manifold. Then, every invariant tensor under the isometries of F has bounded norm with respect to it.

Suppose that ϕ is a local isometry of the Finsler metric F, i.e.,

$$
F(x^i, y^i) = F\Big(\phi^i(x), y^j \frac{\partial \phi^i}{\partial x^j}\Big). \tag{3.1}
$$

Setting

$$
\hat{x}^i = \phi^i(x), \quad \hat{y}^i = y^j \frac{\partial \phi^i}{\partial x^j}
$$

we get $F(x^i, y^i) = F(\hat{x}^i, \hat{y}^i)$. We use the following conventions

$$
\phi^i_j := \frac{\partial \phi^i}{\partial x^j}, \quad \phi^i_{jk} := \frac{\partial^2 \phi^i}{\partial x^j \partial x^k}.
$$

Since ϕ is an isometry, the matrix (ϕ^i_j) invertible. Put $(\psi^i_j) := (\phi^i_j)^{-1}$. We have

$$
\frac{\partial \phi^i}{\partial y^j} = 0, \quad \frac{\partial \psi^i}{\partial y^j} = 0.
$$

Put

$$
g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \hat{g}_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial \hat{y}^i \partial \hat{y}^j}.
$$

By definition, we get

$$
\frac{\partial^2 F^2}{\partial y^i \partial y^j} = \frac{\partial^2 F^2}{\partial \hat{y}^r \partial \hat{y}^s} \phi^r_i \phi^s_j.
$$

Thus

$$g_{ij} = \hat{g}_{rs}\phi^r_i\phi^s_j. \tag{3.2}$$

It follows that

$$(g_{ij}) = \Psi^T(\hat{g}_{ij})\Psi, \tag{3.3}$$

where $\Psi := (\phi^r_i)$. Thus

$$(\hat{g}^{ij}) = (\hat{g}_{ij})^{-1} = \Psi(g_{ij})^{-1}\Psi^T.$$

It follows that

$$\hat{g}^{ij} = g^{pq}\phi^i_p\phi^j_q.$$

Moreover, he following holds

$$\frac{\partial^3 F^2}{\partial y^i\partial y^j\partial y^k} = \frac{\partial\phi^r}{\partial x^i}\frac{\partial\phi^s}{\partial x^j}\frac{\partial\phi^t}{\partial x^k}\frac{\partial^3 F^2}{\partial\bar{y}^r\partial\bar{y}^s\partial\bar{y}^t} \tag{3.4}$$

(3.4) implies that

$$C_{ijk} = \tilde{C}_{rst}\phi^r_i\phi^s_j\phi^t_k. \tag{3.5}$$

Multiplying (3.5) with g^{ij} yields

$$I_k = \tilde{I}_t\phi^t_k, \tag{3.6}$$

which shows that every isometry of F preserves the mean Cartan tensor of F.

Now, we deal with the transformation of spray coefficients of F under the isometry ϕ.

$$\hat{G}^i(\hat{x},\hat{y}) = G^i(x,y)\phi^j_i - \frac{1}{2}\phi^j_{rs}y^r y^s, \tag{3.7}$$

where $\hat{G}^i(\hat{x},\hat{y}) := G^i\big(\phi(x),\phi_{*x}(y)\big)$, and consequently, we have

$$\frac{\partial^3\hat{G}^l}{\partial\hat{y}^i\partial\hat{y}^j\partial\hat{y}^k} = \frac{\partial^3 G^r}{\partial y^s\partial y^t\partial y^p}\ \phi^l_r\psi^s_i\psi^t_j\psi^p_k, \tag{3.8}$$

which is equivalent to

$$\hat{B}^l{}_{ijk} = B^r{}_{stp}\phi^l_r\psi^s_i\psi^t_j\psi^p_k. \tag{3.9}$$

Thus, the Berwald curvature of F remains unchanged under any (local) isometry of F. By (3.9), we get

$$\hat{L}_{ijk} = -\frac{1}{2}\hat{y}_m\hat{B}^m_{\ ijk} = -\frac{1}{2}y_q\psi^q_l B^r{}_{stp}\phi^l_r\psi^s_i\psi^t_j\psi^p_k = L_{stp}\psi^s_i\psi^t_j\psi^p_k. \tag{3.10}$$

Multiplying (3.10) with $\hat{g}^{ij}$ implies that

$$\hat{J}_k = -\frac{1}{2}\hat{g}^{ij}\hat{y}_l\hat{B}^l{}_{ijk} = -\frac{1}{2}g^{mn}\phi^i_m\phi^j_n y_q\psi^q_l B^r{}_{stp}\phi^l_r\psi^s_i\psi^t_j\psi^p_k = -\frac{1}{2}g^{st}y_r B^r{}_{stp}\psi^p_k = J_p\psi^p_k. \tag{3.11}$$

Thus the (mean) Landsberg curvature of F remains unchanged under any isometry of F.

In order to prove Theorem 1.1, we need the following key Lemma.

Lemma 3.2. *For an $\mathcal{R}$-flat Randers metric F on an n-dimensional manifold M and any geodesic $\sigma = \sigma(s)$ and any parallel vector field $U = U(s)$ along σ of F, the following functions*

$$\mathbf{I}(s) := \mathbf{I}_{\dot\sigma}(U(s)),\quad \mathbf{J}(s) := \mathbf{J}_{\dot\sigma}(U(s)) \tag{3.12}$$

satisfy

$$\mathbf{I}(s) = \mathbf{I}(0) + s\,\mathbf{J}(0). \tag{3.13}$$

Proof. By assumption, F is $\mathcal{R}$-flat. Then by (1.1), we get

$$R_{ab} = \frac{1}{n+1}\Big[(R_{abp} + R_{pba} + F^{-1}\ell_a R_{pmb}y^m)I^p + I_a R_{pb}y^p\Big], \tag{3.14}$$

where $\ell_i := F_{y^i}$. It follows from (3.14) that

$$R_{ma}y^m = 0, \quad (n+1)R_{as}y^s = 2I^p R_{psa}y^s. \tag{3.15}$$

The Cartan tensor of F is given by the following

$$(n+1)C_{abc} = I_a h_{bc} + I_b h_{ca} + I_c h_{ab}. \tag{3.16}$$

Using the identities $h_{ab|c} = 0$ and $C_{abc|s}y^s = L_{abc}$ infer that

$$(n+1)L_{abc} = J_a h_{bc} + J_b h_{ca} + J_c h_{ab} \tag{3.17}$$

Then (3.17) implies that

$$L^m_{\ ab|m} = \frac{1}{n+1}\Big\{J^m_{\ |m}h_{ab} + J_{b|a} + J_{a|b} - F^{-1}(J_{b|s}y^s\ell_a + J_{a|s}y^s\ell_b)\Big\}, \tag{3.18}$$

By putting (3.16), (3.17) and (3.18) into (2.6), we get

$$\begin{aligned} R_{sa,b}y^s = &\ \frac{1}{n+1}\Big\{J_{b|a} - F^{-1}J_b\ell_a + \frac{n-3}{2}J_aJ_b - \frac{1}{n+1}(R_{ma}y^m - F^{-1}R_{rs}y^ry^s\ell_a)I_b\Big\} \\ + &\ \frac{1}{n+1}\Big\{J_{a|b} - F^{-1}J_a\ell_i + \frac{n-3}{2}J_aJ_b - \frac{1}{n+1}(R_{mb}y^m - F^{-1}R_{rs}y^ry^s\ell_b)I_a\Big\} \\ + &\ \frac{1}{n+1}\Big\{J^r_{\ |r} + (n-1)J^rJ_r - \frac{1}{n+1}R_{sm}y^sI^m\Big\}h_{ba} + \mathcal{R}_{ba}, \end{aligned} \tag{3.19}$$

where $\mathcal{R}_{ab}$ defined by (1.1). Multiplying (3.19) with y^a yields

$$J_{b|s}y^s = (R_{bj} + R_{jb})y^j - \frac{1}{n+1}\Big\{2R_{mrb}I^m + I_iR_{rs}y^s\Big\}y^r - R_{rs,b}y^ry^s. \tag{3.20}$$

By putting (3.15) in (3.20), we get

$$J_{b|l}y^l = I_{b|l|s}y^ly^s = 0. \tag{3.21}$$

By definition of $\mathbf{I}_y$ and $\mathbf{J}_y$, we have

$$\mathbf{J}(s) = \mathbf{I}^{'}(s). \tag{3.22}$$

By (3.21) and (3.22) it follows that $\mathbf{I}^{''}(s) = 0$. Then $\mathbf{I}^{'}(s) = \mathbf{I}^{'}(0)$. By (3.22), we get (3.13). □

Proof of Theorem 1.1: Suppose that $y \in T_xM$ is an arbitrary unit vector and take a tangent vector $u \in T_xM$. Suppose that $\sigma(s)$ is the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$ and let $U(s)$ be the parallel vector field along σ with $U(0) = u$. Define $\mathbf{I}(s)$ and $\mathbf{J}(s)$ as in (3.12). Lemma 3.2 infers

$$\mathbf{I}(s) = \mathbf{I}(0) + s\,\mathbf{J}(0). \tag{3.23}$$

Due to boundedness of the mean Cartan tensor, by letting $t \to \pm\infty$, we get

$$\mathbf{J}_y(u) = \mathbf{J}(0) = 0. \tag{3.24}$$

Hence, F has vanishing $\mathbf{J}$-curvature and by (3.17), we conclude that F has also vanishing $\mathbf{L}$-curvature. On Randers metrics, being Landsbergian and Berwaldian are equivalent. ,This completes the proof. □

4. Proof of Theorems 1.2 and 1.3

In this section, we are going to prove Theorem 1.2. In order to prove it, we remark the following.

Lemma 4.1. *For a homogeneous Randers metric F on a manifold M and $\tau \in C^\infty(M)$, $\mathbf{L} + \tau F\mathbf{C} = 0$ if and only if $\mathbf{J} + \tau F\mathbf{I} = 0$.*

Lemma 4.2. *For a homogeneous Randers metric F on a manifold M and nonzero scalar function $\tau \in C^\infty(M)$, if $\mathbf{J} + \tau F\mathbf{I} = 0$, then F is Riemannian.*

Proof. It is known that that M is geodesically complete. Suppose that $\sigma : \mathbb{R} \longrightarrow M$ is the geodesic with $\sigma(0) = x$ and $\sigma'(0) = y$. Suppose that $U(t)$ is the parallel vector field along $\sigma(t)$ with $U(0) = u$. We define $\mathbf{I}(s) = \mathbf{I}(U(s))$ and $\mathbf{J}(s) = \mathbf{J}(U(s))$. According to $J_i = I_{i|m}y^m$ we have $\mathbf{J}(s) = \mathbf{I}'(s)$. Restricting $\mathbf{J} + \tau F\mathbf{I} = 0$ to the geodesic σ, we get the following ODE

$$\mathbf{I}'(s) + \tau(t)\mathbf{I}(s) = 0,$$

which its general solution is

$$\mathbf{I}(s) = e^{-\int_0^s \tau(t)dt}\mathbf{I}(0), \tag{4.1}$$

where $\tau(s) := \tau(\sigma(s))$. For some positive number $\mathfrak{c}$ we have

$$e^{\mathfrak{c}s} < e^{-\int_0^s \tau(t)dt} < e^{-\mathfrak{c}s}.$$

Suppose that $\mathbf{I}(0)$ is nonzero. Letting $s \to \pm\infty$ yields that $\mathbf{I}(s)$ is unbounded which is a contradiction with boundedness of the mean Cartan tensor of the homogeneous Finsler metric F. Thus, $\mathbf{I}(0)$ vanishes. It follows from (4.1) that $\mathbf{I}(s) = 0$ for all s. Therefore, F is Riemannian by applying Deicke's theorem. □

Now, we are going to prove Theorem 1.3. First, we prove the following.

Lemma 4.3. *Suppose that (M, F) is a weakly stretch Finsler manifold. With the notations of previous lemma, we have*

$$\mathbf{I}(s) = \mathbf{I}(0) + s\,\mathbf{J}(0). \tag{4.2}$$

Proof. By definition, we have

$$\bar{\Sigma}_{ij} = 2(J_{i|j} - J_{j|i}).$$

By assumption F is weakly stretch metric then

$$J_{i|j} = J_{j|i}. \tag{4.3}$$

Contracting (4.3) with y^j, one can get $J_{i|j}y^j = 0$ which gives rise the constancy of the rate of changes of the $\mathbf{J}$-curvature along geodesics of F. From our definition of $\mathbf{J}_y$, we have $\mathbf{J}(s = \mathbf{I}^{'}(s)$. Then, we obtain

$$\mathbf{I}^{''}(s) = \mathbf{J}^{'}(s) = 0. \tag{4.4}$$

Then (4.4) yields (4.2). □

Proof of Theorem 1.3: Let $F = F(x, y)$ be a weakly stretch metric on a manifold M. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c = c(t)$ be the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ and $V(t)$ be the parallel vector field along c with $V(0) = v$. Then by Lemma 4.3, we get

$$\mathbf{I}(s) = \mathbf{I}(0) + s\,\mathbf{J}(0). \tag{4.5}$$

By letting $t \to \pm\infty$ in (4.5) and considering $||\mathbf{I}|| < \infty$, one can get $\mathbf{J}_y(v) = \mathbf{J}(0) = 0$. Hence, F is a weakly Landsberg metric.

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