Ikonion Journal of Mathematics 3(1) (2021) 1-14

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

ISSN: 2687-6531

Adjacent Vertex-Distinguishing Edge-Coloring of Brick-Product

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Keywords
adjacent vertex-distinguishing proper edge - coloring, brick - product, edge-coloring

Abstract — Let G be a finite, simple, undirected and connected graph. $\chi'_{as}(G)$ denotes the minimum number of colors required for a proper edge-coloring of G, in which no two adjacent vertices are incident to edges colored with the same set of colors. In this paper, I am compute sharp bound for adjacent vertex-distinguishing proper edge-coloring of brick-product.

Subject Classification (2020): 05C15, 05C32.

1. Introduction

I am refer the books [4, 11] for graph theoretical notation and terminology. Let G be a finite, simple, undirected and connected graph. Denote by $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. Let $\Delta(G)$ denotes the maximum degree of G. A *proper edge-coloring* σ is a mapping from $E(G)$ to the set of colors such that any two adjacent edges receive distinct colors. For any vertex v of G, let $S_\sigma(v)$ denote the set of the colors of all edges incident to v. A proper edge-coloring σ is said to an *adjacent vertex-distinguishing* (AVD) if $S_\sigma(u) \neq S_\sigma(v)$, for every adjacent vertices u and v. The minimum number of colors required for an adjacent vertex-distinguishing proper edge-coloring of G, denoted by $\chi'_{as}(G)$, is called the *adjacent vertex-distinguishing chromatic index* (*AVD chromatic index*) of G. Thus, $\chi'_{as}(G) \geq \chi'(G)$.

The concept of adjacent vertex-distinguishing edge-coloring has been introduce and studied in [19] Zhang et al. (2002) and pose the following conjecture.

Conjecture 1.1. (Zhang et al. [19]) For any connected graph G ($|V(G)| \geq 6$), there is $\chi'_{as}(G) \leq \Delta(G) + 2$.

If H is a subgraph of G, it is interesting that $\chi'_{as}(H) \leq \chi'_{as}(G)$ is not always true. Let $K_{m,n}$ be the complete bipartite graph, then $\chi'_{as}(K_{2,3}) = 3$ and $K_{2,3} - e$ for any edge, then $\chi'_{as}(K_{2,3} - e) = 4$. Deletion of an edge of a graph may also decrease the coloring number of the graph. Let $n \geq 3$, then $\chi'_{as}(K_{1,n}) = n$ and $\chi'_{as}(K_{1,n} - e) = n - 1$.

The concept of adjacent vertex-distinguishing edge-coloring has been studied in many paper such as [1, 3,

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Article History: Received: 19.09.2020 - Accepted: 05.04.2021 - Published: 02.06.2021

5–10, 12–20].

In [1] Anantharaman (2019) obtained exact values for adjacent vertex-distinguishing edge-coloring of strong product of some graphs. In [3] Axenovich et al. (2016) obtained upper bound for adjacent vertex-distinguishing edge-colorings of graphs. In [5] Balister et al. (2007) obtained upper bound for adjacent vertex-distinguishing edge-coloring some special graphs also consider 3-regular graphs. In [6] Baril et al. (2006) obtained exact values for adjacent vertex-distinguishing edge-coloring of meshes. In [7] Bu et al. (2011) finding adjacent vertex-distinguishing edge-colorings of planar graphs with girth at least six. In [8] Chen et al. (2015) obtained adjacent vertex-distinguishing proper edge-coloring of planar bipartite graphs with $\Delta = 9$, 10, or 11. In [9] Hatami (2005) prove that $\Delta + 300$ is a bound on the adjacent vertex-distinguishing edge chromatic number. In [10] Hocquard et al. (2011) compute adjacent vertex-distinguishing edge-coloring of graphs with maximum degree at least five[1]. In [12] Li et al. (2006) compute adjacent strong edge-coloring of $K(n,m)$. In [13] Lin et al. (2010) compute the adjacent vertex-distinguishing edge-coloring of graphs containing Hamiltonian path and graphs containing dominating path. In [14] Lin-zhong et al. (2003) compute on the adjacent strong edge-coloring of Halin Graphs. In [15] Omai et al. (2017) compute for some result for AVD-edge-coloring on power of path[1]. In [17] Wang et al. (2010) obtained adjacent vertex-distinguishing edge-colorings of graphs with smaller maximum average degree. In [18] Yu et al. (2016) compute adjacent vertex-distinguishing colorings by sum of sparse graphs. In [19] Zhang et al. (2002) obtained some standard result and pose the conjecture for adjacent Strong edge-coloring of graphs. In [20] Zhang et al. (2014) obtained improved upper bound on adjacent vertex-distinguishing chromatic index of a graph.

2. Brick-product

Let $\ell \geq 2$, $m \geq 1$ and $r \geq 0$ be integers such that $m+r$ is even. Let $C_{2\ell}$ be a cycle of length 2ℓ. The (m,r)-*brick-product of* $C_{2\ell}$, denoted by $Br(2\ell,m,r)$, is the graph with adjacency defined in two cases.

• For $m=1$, $r \geq 3$ must be odd and $Br(2\ell,1,r)$ is obtained from the cycle $C_{2\ell} = (v_0, v_1, v_2, \dots, v_{2\ell-1}, v_0)$, by adding chords joining v_{2i} and v_{2i+r} for $i \in \{0,1,\dots,\ell-1\}$ where subscripts are taken modulo 2ℓ.

• For $m \geq 2$, $Br(2\ell,m,r)$ is obtained by first taking the vertex-disjoint union of m copies of $C_{2\ell}$ denoted by

$$C_{2\ell}(i) = (v_{i,0}, v_{i,1}, v_{i,2}, \dots, v_{i,2\ell-1}, v_{i,0}), \quad i \in \{0,1,\dots,m-1\}.$$

Next, for each pair $(i,j) \in \{0,1,\dots,m-2\} \times \{0,1,\dots,2\ell-1\}$ such that i and j have the same parity, an edge is added to join $v_{i,j}$ and $v_{i+1,j}$. Finally, for odd $j \in \{1,3,5,\dots,2\ell-1\}$, an edge is added to join $v_{0,j}$ and $v_{m-1,j+r}$, where the second subscript is modulo 2ℓ ([16]).

By definition, $Br(2\ell,m,r)$ is 3-regular. So $\chi'_{as}(Br(2\ell,m,r)) \geq \Delta+1 = 4$. We show at most brick-product have $\chi'_{as}(Br(2\ell,m,r)) = 4$.

3. $\chi'_{as}(Br(2\ell,m,r))$ for $m \notin \{1,2,5\}$

Theorem 3.1. For $m \notin \{1,2,5\}$, $\chi'_{as}(Br(2\ell,m,r)) = 4$.

Proof.
Let $G = Br(2\ell,m,r)$. I am consider four cases.
Case 1. $m \equiv 0 \pmod 4$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,4,8,\ldots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,5,9,\ldots,m-3\}$

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,6,10,\ldots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{0,4,8,\ldots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{1,5,9,\ldots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{2,6,10,\ldots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{3,7,11,\ldots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,4,8,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,5,9,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,6,10,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Case 2. $m \equiv 1 \pmod 4$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:
for $i \in \{0,3,6\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,4,7\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{2,5,8\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\dots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{10,14,18,\dots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{11,15,19,\dots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{12,16,20,\dots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{0,6\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i = 3$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{1,7\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i = 4$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{2,8\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i = 5$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{9,13,17,\dots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{10,14,18,\dots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{1,3,5,\dots,2\ell-1\}$ and $i \in \{11,15,19,\dots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{0,2,4,\dots,2\ell-2\}$ and $i \in \{12,16,20,\dots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\dots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,6\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{1,7\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,8\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 3$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 5$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{10,14,18,\ldots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{11,15,19,\ldots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{12,16,20,\ldots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 3. $m \equiv 2 \pmod 4$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{2,5\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\ldots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{7,11,15,\ldots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{8,12,16,\ldots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\ldots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i = 0$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i = 1$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i = 2$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i = 3$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i = 4$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i = 5$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{6,10,14,\ldots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{7,11,15,\ldots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{8,12,16,\ldots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{9,13,17,\ldots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i = 0$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 3$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 5$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\dots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{7,11,15,\dots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{8,12,16,\dots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{9,13,17,\dots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 4. $m \equiv 3 \pmod 4$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i = 0$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{4,8,12,\ldots,m-3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{5,9,13,\ldots,m-2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{6,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\ldots,m-1\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ 4 & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i = 0$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i = 1$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i = 2$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{3,7,11,\ldots,m-4\}$, $\sigma(v_{i,j}v_{i+1,j}) = 4$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{4,8,12,\ldots,m-3\}$, $\sigma(v_{i,j}v_{i+1,j}) = 2$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$ and $i \in \{5,9,13,\ldots,m-2\}$, $\sigma(v_{i,j}v_{i+1,j}) = 1$;
for $j \in \{0,2,4,\ldots,2\ell-2\}$ and $i \in \{6,10,14,\ldots,m-5\}$, $\sigma(v_{i,j}v_{i+1,j}) = 3$;
for $j \in \{1,3,5,\ldots,2\ell-1\}$, $\sigma(v_{0,j}v_{m-1,j+r}) = 3$.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i = 0$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 1$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,3\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i = 2$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{3,7,11,\ldots,m-4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{4,8,12,\dots,m-3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,3\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{5,9,13,\dots,m-2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}; \end{cases}$$

for $i \in \{6,10,14,\dots,m-1\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,2,4,\dots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{1,3,5,\dots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,m,r)) = 4$.

4. $\chi'_{as}(Br(2\ell,1,r))$

By definition, $r \in \{3,5,7,\dots\}$. Also, $\ell \geq 3$.

Theorem 4.1. If $\ell \equiv 3 \pmod 6$ and $r \notin \{3,9,15,21,\dots\}$, then $\chi'_{as}(Br(2\ell,1,r)) = 4$.

Proof.
Define $\sigma : E(Br(2\ell,1,r)) \to \{1,2,3,4\}$ as follows:

$$\sigma(v_j v_{j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\dots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\dots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\dots,2\ell-1\}. \end{cases}$$

Remaining edges are colored 4.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are:

$$S_\sigma(v_j) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\dots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\dots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\dots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,1,r)) = 4$.

5. $\chi'_{as}(Br(2\ell,2,r))$

By the definition of $Br(2\ell,2,r)$, r is even.

Theorem 5.1. For $\ell \equiv 0 \pmod 3$, $\chi'_{as}(Br(2\ell,2,r)) = 4$.

Proof.

Let $G = Br(2\ell, 2, r)$. I am consider two cases.

Case 1. $r \notin \{4, 10, 16, \ldots\}$.

Define $\sigma : E(G) \to \{1, 2, 3, 4\}$ as follows:

$$\sigma(v_{0,j}v_{0,j+1}) = \begin{cases} 1 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 2 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 3 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$\sigma(v_{1,j}v_{1,j+1}) = \begin{cases} 3 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 1 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 2 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

$$S_\sigma(v_{0,j}) = \begin{cases} \{1, 3, 4\} & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ \{1, 2, 4\} & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ \{2, 3, 4\} & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$S_\sigma(v_{1,j}) = \begin{cases} \{1, 2, 4\} & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ \{2, 3, 4\} & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ \{1, 3, 4\} & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 2. $r \in \{4, 10, 16, \ldots\}$.

Define $\sigma : E(G) \to \{1, 2, 3, 4\}$ as follows:

$$\sigma(v_{0,j}v_{0,j+1}) = \begin{cases} 1 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 2 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 3 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}; \end{cases}$$

$$\sigma(v_{1,j}v_{1,j+1}) = \begin{cases} 2 & \text{if } j \in \{0, 3, 6, \ldots, 2\ell - 3\}, \\ 3 & \text{if } j \in \{1, 4, 7, \ldots, 2\ell - 2\}, \\ 1 & \text{if } j \in \{2, 5, 8, \ldots, 2\ell - 1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

$$S_\sigma(v_{0,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

$$S_\sigma(v_{1,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{2,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.
Thus, $\chi'_{as}(Br(2\ell,2,r)) = 4$.
This completes the proof.

6. $\chi'_{as}(Br(2\ell,5,r))$

By the definition of $Br(2\ell,5,r)$, r is odd.

Theorem 6.1. For $\ell \equiv 0 \pmod 3$, $\chi'_{as}(Br(2\ell,5,r)) = 4$.

Proof.
Let $G = Br(2\ell,5,r)$. I am consider two cases.
Case 1. $r \notin \{3,9,15,\ldots\}$.
Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:
for $i \in \{0,2,4\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 1 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Edges $\{v_{0,j}v_{1,j}, v_{2,j}v_{3,j} : j \in \{1,3,5,\ldots,2\ell-1\}\} \cup$
$\{v_{1,j}v_{2,j}, v_{3,j}v_{4,j}, v_{0,j}v_{4,j+r} : j \in \{0,2,4,\ldots,2\ell-2\}\}$ are colored 4.
By the construction, σ is a proper edge-coloring.
The induced vertex-color sets are given below:
for $i \in \{0,2,4\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Case 2. $r \in \{3,9,15,\ldots\}$.

Define $\sigma : E(G) \to \{1,2,3,4\}$ as follows:

for $i \in \{0,2\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 1 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 2 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 3 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 3 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 1 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 2 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$\sigma(v_{i,j}v_{i,j+1}) = \begin{cases} 2 & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ 3 & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ 1 & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Remaining edges are colored 4.

By the construction, σ is a proper edge-coloring.

The induced vertex-color sets are given below:

for $i \in \{0,2\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,2,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{2,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i \in \{1,3\}$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{2,3,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{1,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,2,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}; \end{cases}$$

for $i = 4$,

$$S_\sigma(v_{i,j}) = \begin{cases} \{1,2,4\} & \text{if } j \in \{0,3,6,\ldots,2\ell-3\}, \\ \{2,3,4\} & \text{if } j \in \{1,4,7,\ldots,2\ell-2\}, \\ \{1,3,4\} & \text{if } j \in \{2,5,8,\ldots,2\ell-1\}. \end{cases}$$

Observe that σ is an AVD proper edge-coloring of G.

Thus, $\chi'_{as}(Br(2\ell,5,r)) = 4$. We finish this paper with the following problem.

i) For $\ell \equiv 3 \pmod 6$ and $r \in \{3, 9, 15, 21, \ldots\}$, compute $\chi'_{as}(Br(2\ell, 1, r))$.
ii) For $\ell \not\equiv 3 \pmod 6$, compute $\chi'_{as}(Br(2\ell, 1, r))$.
iii) For $\ell \not\equiv 0 \pmod 3$, compute $\chi'_{as}(Br(2\ell, 2, r))$.
iv) For $\ell \not\equiv 0 \pmod 3$, compute $\chi'_{as}(Br(2\ell, 5, r))$.

References

[1] Anantharaman, S., (2019) Adjacent vertex-distinguishing proper edge-coloring of strong product of graphs, Applications and Applied Mathematics: An International Journal (AAM), 14 (2): 1169-1187.

[2] Akgunes, N., Cagan, B., (2018) On the dot product of graphs over monogenic semigroups, Applied Mathematics and Computation, 322: 1-5.

[3] Axenovich, M., Harant, J., Przybylo, J., SotÃ¡k, R., Voigt, M., Weidelich, J., (2016) A note on adjacent vertex distinguishing colorings of graphs, Discrete Applied Mathematics, 205: 1-7.

[4] Balakrishnan, R., Ranganathan, K., (2012) A textbook of graph theory, Second Edition, Springer, New York.

[5] Balister, P.N., Gyori, E., Lehel, J., Schelp, R.H., (2007) Adjacent vertex distinguishing edge-coloring, SIAM Journal on Discrete Mathematics, 21(1): 237-250.

[6] Baril, J.L., Kheddouci, H., Togni, O., (2006) Adjacent vertex distinguishing edge-coloring of meshes, The Australasian Journal of Combinatorics, 35: 89-102.

[7] Bu, Y., Lih, K., Wang, W., (2011) Adjacent vertex distinguishing edge-colorings of planar graphs with girth at least six, Discussiones Mathematicae Graph Theory, 31: 429-439.

[8] Chen, X., Li, Z., (2015) Adjacent vertex-distinguishing proper edge coloring of planar bipartite graphs with $\Delta = 9$, 10, or 11, Information Processing Letters, 115: 263-268.

[9] Hatami, H., (2005) $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, Journal of Combinatorial Theory Series B, 95: 246-256.

[10] Hocquard, H., Montassier, M., (2011) Adjacent vertex-distinguishing edge-coloring of graphs with maximum degree at least five[1], Electronic Notes in Discrete Mathematics, 38: 457-462.

[11] Imrich, W., Klavzar, S., (2000) Product graph: structure and recognition, Wiley Interscience, New York.

[12] Li, J., Zhang, Z., Chen, X., Sun, Y., (2006) A Note on adjacent strong edge coloring of $K(n,m)$, Acta Mathematicae Applicatae Sinica English Series, 22(2): 273-276.

[13] Lin, B., Liu, G., (2010) On the adjacent vertex distinguishing edge coloring of graphs, International Journal of Computer Mathematics, 87(4): 726-732.

[14] Lin-zhong, L., Yin-zhen, L., Zhong-fu, Z., Jian-fang, W., (2003) On the adjacent strong edge coloring of Halin Graphs, Journal of Mathematical Research & Exposition, 23(2): 241-246.

[15] Omai, M.M., de-lmeida, S.M., Sasaki, D., (2017) AVD-edge coloring on power of path[1], Electronic Notes in Discrete Mathematics, 62: 273-278.

[16] Shao, Z., Xu, J., Yeh, R.K., (2016) L(2,1)-labeling for brick product graphs, Journal of Combinatorial Optimization, 31(2): 47-462.

[17] Wang, W., Wang, Y., (2010) Adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree, Journal of Combinatorial Optimization, 19: 471-485.

[18] Yu, X., Qu, C., Wang, G., Wang, Y., (2016) Adjacent vertex distinguishing colorings by sum of sparse graphs, Discrete Mathematics, 339: 62-71.

[19] Zhang, Z., Liu, L., Wang, J., (2002) Adjacent Strong Edge Coloring of Graphs, Applied Mathematics Letters, 15: 623-626.

[20] Zhang, L., Wang, W., Lih, K., (2014) An improved upper bound on adjacent vertex distinguishing chromatic index of a graph, Discrete Applied Mathematics, 162: 348-354.