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Research Article

Linear methods of approximation in weighted Lebesgue spaces with variable exponent

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Abstract

Some estimations in below for the deviations conducted by the Zygmund means and by the Abel-Poisson sums in the weighted Lebesgue spaces with variable exponent are obtained. In the classical Lebesgue spaces these estimations were proved by M. F. Timan. The considered weight functions satisfy the well known Muckenhout condition. For the proofs of main results some estimations obtained in the classical weighted Lebesgue spaces and also an extrapolation theorem proved in the weighted variable exponent Lebesgue spaces are used. Main results are new even in the nonweighted variable exponent Lebesgue spaces.

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1. Introduction

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [1, \infty)$ be a Lebesgue measurable 2π periodic function. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that $\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)}\,dx < \infty$.

We consider the class $\mathcal{P}_0(\mathbb{T})$ of exponents $p(\cdot)$ satisfying the conditions:

$$1 < p_- := ess\inf_{x\in\mathbb{T}} \; p(x) \le ess\sup_{x\in\mathbb{T}} \; p(x) := p^+ < \infty,$$
$$|p(x) - p(y)| \ln(1/|x-y|) \le c, \quad x, y \in \mathbb{T}, \quad 0 < |x-y| \le 1/2.$$

The space $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with the norm $\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \; \rho_{p(\cdot)}(f/\lambda) \le 1\}$.

Let ω be a weight, i.e. an almost everywhere positive, 2π periodic integrable function. For a given ω we define the weighted variable exponent Lebesgue space $L^{p(\cdot)}_{\omega}(\mathbb{T})$ as the set of all Lebesgue measurable 2π periodic functions f such that $f\omega \in L^{p(\cdot)}(\mathbb{T})$. Note that, when $1 \le p^+ < \infty$ the space $L^{p(\cdot)}_{\omega}(\mathbb{T})$ is a Banach space with respect to the norm $\|f\|_{p(\cdot),\omega} := \|f\omega\|_{p(\cdot)}$.

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Variable exponent Lebesgue space was introduced by Orlicz in [19]. Interest in variable exponent Lebesgue spaces has increased since 1990s, because of their use in different application problems in mechanic, especially in fluid dynamic for the modelling of electrorheological fluids and also in the study of image processing and various physical problems (see, for example the monographs [4, 5, 20] and the references cited therein). There are sufficiently investigations, where the fundamental problems of these spaces are investigated in view of potential theory, maximal and singular integral operator theory, especially. Widely presentations of corresponding results can be found in the monographs, cited above. In these spaces there were also investigated some fundamental problems of approximation theory. In particular under some restrictions on variable exponent function $p(\cdot)$ was proved the completeness of polynomials in these spaces and also were constructed different modulus of smoothness, which plays an important role for investigations of quantitative problems of approximation theory (detailed information can be found in the monograph [21]). Later using the results on the boundedness of singular and maximal operators in variable exponent spaces obtained in [7] (see also: [4, 5]), there were proved direct and inverse theorems of approximation theory and also different quantitative estimations relating to the approximation properties of different summation methods in nonweighted and weighted variable exponent Lebesgue spaces [1–3, 8–10, 12–15, 18, 22–24, 27].

Let's give some definitions needed to formulate the main results obtained in this work.

Let $A = \{\lambda_{\nu,r} := \lambda_\nu (r)\}$, $\nu = 0, 1, ..., r$; $r = 0, 1, 2, ...$, be a triangular matrix with the entries $\lambda_{\nu,r}$, satisfying the conditions: $\lambda_0 (r) = 1$, $\lambda_\nu (r) = 0$ for $\nu > r$.

For a given $f \in L^{p(\cdot)}_\omega(\mathbb{T})$, $p(\cdot) \geq 1$, with the Fourier coefficients

$$a_\nu := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos \nu t dt, \text{ and } b_\nu := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin \nu t dt, \quad \nu = 0, 1, 2, ...$$

we generate the series $U_r (f, x, \lambda) := \sum_{\nu=0}^{\infty} \lambda_\nu (r) A_\nu (f, x)$, where $A_0 (f, x) := a_0/2$ and $A_\nu (f, x) := a_\nu \cos \nu x + b_\nu \sin \nu x$.

If

$$\lambda_\nu (r) := \begin{cases} 1 - \left(\frac{\nu}{r+1}\right)^k, & 0 \leq \nu \leq r, \\ 0, & \nu > r \end{cases}$$

for a natural number $k \geq 1$, where $r = 0, 1, 2, ...$, then the series $U_r (f, x, \lambda)$ reduce to *the Zygmund means* $Z^{(k)}_r (f, x)$. In the case of $k = 1$ *the Zygmund means* $Z^{(1)}_r (f, x)$ reduce to the *Fejér means* $F_r (f, x)$.

If $0 \leq r < 1$, then for the sequence $\{\lambda^*_\nu (r)\}$, $\lambda^*_\nu (r) := r^\nu$, $\nu = 0, 1, 2, ...$, the series $U_r (f, x, \lambda^*)$ reduce to the *Abel-Poisson sums* of $f \in L^{p(\cdot)}_\omega (\mathbb{T})$. Hence we have

$$\begin{aligned} Z^{(k)}_r (f, x) &= \sum_{\nu=0}^{r} \left[1 - \left(\frac{\nu}{r+1}\right)^k\right] A_\nu (f, x) , \\ U_r (f, x, \lambda^*) &= \sum_{\nu=0}^{\infty} r^\nu A_\nu (f, x) , \quad 0 \leq r < 1. \end{aligned}$$

Definition 1.1. We say that $\omega \in A_{p(\cdot)}$ if the inequality

$$\sup_I |I|^{-1} \|\omega\chi_I\|_{p(\cdot)} \left\|\omega^{-1}\chi_I\right\|_{p'(\cdot)} < \infty, \quad 1/p (\cdot) + 1/p' (\cdot) = 1$$

holds. Here the supremum is taken over all intervals $I \subset \mathbb{R} := (-\infty, \infty)$ with the characteristic functions χ_I and $|I|$ is the Lebesgue measure of I.

Let

$$E_n (f)_{p(\cdot),\omega} := \inf \left\{\|f - T_n\|_{p(\cdot),\omega} : T_n \in \Pi_n\right\}$$

be the *best approximation number* of f in the space $L_{\omega}^{p(\cdot)}(\mathbb{T})$, where Π_n is the class of trigonometric polynomials of degree not exceeding n.

In this work some estimations in below for the deviations

$$\left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega} \text{ and } \left\| f - U_r(f,\lambda^*) \right\|_{p(\cdot),\omega}$$

in the weighted variable exponent Lebesgue Spaces $L_{\omega}^{p(\cdot)}(\mathbb{T})$ are obtained.

Our main results are following.

Theorem 1.2. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}$. If $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p,k)$ such that for every $r = 0,1,2,\ldots$, the inequality*

$$\frac{1}{(r+1)^k}\left[\sum_{\nu=0}^{r}(\nu+1)^{k\beta-1}E_{\nu}^{\beta}(f)_{p(\cdot),\omega}\right]^{1/\beta} \leq c(p,k)\left\| f - Z_r^{(k)}(f)\right\|_{p(\cdot),\omega}$$

holds, where $k \geq 1$ and $\beta := \max\{2, p_+\}$.

In particular, for the *Fejér means* $F_r(f,x)$ we have

Corollary 1.3. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}$. If $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p)$ such that for every $r = 0,1,2,\ldots$, the inequality*

$$\frac{1}{(r+1)}\left[\sum_{\nu=0}^{r}(\nu+1)^{\beta-1}E_{\nu}^{\beta}(f)_{p(\cdot),\omega}\right]^{1/\beta} \leq c(p)\left\| f - F_r(f)\right\|_{p(\cdot),\omega}$$

holds, where $\beta := \max\{2, p_+\}$.

Theorem 1.4. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}$. If $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, then there exists a positive constant $c(p)$ such that for every $r \in [0,1)$ the inequality*

$$(1-r)\left[\sum_{k=0}^{\infty}r^k(k+1)^{\beta-1}E_{k}^{\beta}(f)_{p(\cdot),\omega}\right]^{1/\beta} \leq c(p)\left\| f - U_r(f,\lambda^*)\right\|_{p(\cdot),\omega}$$

holds, where $\beta := \max\{2, p_+\}$

Note that appropriate estimates from above under more restrictive conditions than the condition $\omega(\cdot) \in A_{p(\cdot)}$, namely when $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$, for some $p_0 \in (1, p_-)$ were obtained in [13].

Combining Theorems 1.2 and 1.4 with the estimations, obtained in [13] we have

Corollary 1.5. *Let $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega^{-p_0} \in A_{(p(\cdot)/p_0)'}$ for some $p_0 \in (1, p_-)$ and let $\gamma := \min\{2, p_-\}$, $\beta := \max\{2, p_+\}$. Then*

i) there exist the constants $c(p,k)$ and $C(p,k)$ such that for every $r = 0,1,2,\ldots$

$$\begin{aligned}
&\frac{1}{(r+1)^k}\left[\sum_{\nu=0}^{r}(\nu+1)^{k\beta-1}E_{\nu}^{\beta}(f)_{p(\cdot),\omega}\right]^{1/\beta} \\
\leq\ & c(p,k)\left\| f - Z_r^{(k)}(f)\right\|_{p(\cdot),\omega} \\
\leq\ & C(p,k)\frac{1}{(r+1)^k}\left[\sum_{\nu=0}^{r}(\nu+1)^{k\gamma-1}E_{\nu}^{\gamma}(f)_{p(\cdot),\omega}\right]^{1/\gamma},
\end{aligned}$$

and

ii) there exist the positive constants $\widetilde{c}(p)$ *and* $\widetilde{C}(p)$ *such that for every* $r \in (0,1)$

$$(1-r)\left[\sum_{k=0}^{\infty} r^k (k+1)^{\beta-1} E_k^{\gamma}(f)_{p(\cdot),\omega}\right]^{1/\beta}$$

$$\leq \widetilde{c}\left\|f - U_r(f,\lambda^*)\right\|_{p(\cdot),\omega} \leq \widetilde{C}(p)(1-r)\left[\sum_{k=0}^{\infty} r^k (k+1)^{\gamma-1} E_k^{\gamma}(f)_{p(\cdot),\omega}\right]^{1/\gamma}.$$

In the case of $p = constant$ these estimates in nonweighted Lebesgue spaces were obtained by M. F. Timan in [26] (see also, [25]). It should be pointed out that Theorems 1.2 and 1.4 are new even in the nonweighted variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{T})$.

2. Auxiliary results

We shall use $c(\cdot)$, $c(\cdot,\cdot)$,..., to denote constants depending in general, only on parameters given in the brackets and non depending of n.

The following extrapolation theorem is a particular case of the more general result proved in [6, Theorem 2.7 and comments in p.1214]:

Theorem 2.1. *Suppose that for some* p_0*,* $1 < p_0 < \infty$*, and every weight* $\omega \in A_{p_0}$ *an operator* $\boldsymbol{T}$ *(which is not linear in general) is bounded in* $L_{\omega}^{p_0}(\mathbb{T})$*, i.e.*

$$\|\boldsymbol{T}(f)\|_{p_0,\omega} \leq c(p_0)\|f\|_{p_0,\omega}$$

for some constant $c(p_0) > 0$*. Then for any pair* $(p(\cdot),\omega)$*, where* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\omega \in A_{p(\cdot)}$ *it is bounded in* $L_{\omega}^{p(\cdot)}(\mathbb{T})$ *and there exists a positive constant* $c(p)$ *such that for every* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$ *the inequality*

$$\|\boldsymbol{T}(f)\|_{p(\cdot),\omega} \leq c(p)\|f\|_{p(\cdot),\omega}$$

holds.

Let $S_n(f)(x) := \sum\limits_{k=0}^{n} A_k(f,x)$, $n = 1,2,...$, where $A_0(f,x) := a_0/2$ and $A_k(f,x) := a_k \cos kx + b_k \sin kx$, be the nth partial sums of Fourier series of f.

Lemma 2.2 ([11])**.** *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$*,* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$*. If* $\omega(\cdot) \in A_{p(\cdot)}$*, then there exists a positive constant* $c(p)$ *such that the inequality*

$$\|S_n(f)\|_{p(\cdot),\omega} \leq c(p)\|f\|_{p(\cdot),\omega}, \quad n = 1,2,...$$

holds.

Lemma 2.3. *Let* $f \in L_{\omega}^{p(\cdot)}(\mathbb{T})$*,* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$*,* $\omega \in A_{p(\cdot)}$*. Then there exist the constants* $c_i(p)$*,* $i = 1,2$*, such that*

$$c_1(p)\|f\|_{p(\cdot),\omega} \leq \left\|\left(\sum_{\mu=0}^{\infty}|\Delta_\mu|^2\right)^{1/2}\right\|_{p(\cdot),\omega} \leq c_2(p)\|f\|_{p(\cdot),\omega},$$

where $\Delta_\mu(x) := \sum\limits_{k=2^{\mu-1}}^{2^\mu-1} A_k(f,x)$ *and* $A_{2^{-1}}(f,x) := 0$*.*

Proof. In classical weighted Lebesgue spaces, namely when $p(\cdot) := p$ is a constant, and $\omega \in A_p$, $1 < p < \infty$, Lemma 2.3 was proved in [17, Theorem 1 and Theorem 2]. If we consider the operator $\boldsymbol{T}: f \to \left(\sum\limits_{\mu=0}^{\infty}|\Delta_\mu|^2\right)^{1/2}$, which is bounded by [17, Theorem 1 and Theorem 2] in $L_{\omega}^{p}(\mathbb{T})$, then by Theorem 2.1 it is bounded also in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega \in A_{p(\cdot)}$. Hence the second inequality of Lemma 2.3 is established. The proof of the first inequality goes similarly. Indeed, if we define the operator $\boldsymbol{T}^{-1}$:$\left(\sum\limits_{\mu=0}^{\infty}\left|\Delta_{\mu}\right|^{2}\right)^{1/2}\rightarrow f$, then by [17, Theorem 1 and Theorem 2] it is bounded in $L_{\omega}^{p}(\mathbb{T})$ and again applying Theorem 2.1 we have that $\boldsymbol{T}^{-1}$ is bounded in $L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_{0}(\mathbb{T})$, $\omega\in A_{p(\cdot)}$. Thus we obtain the first inequality of Lemma 2.3. □

The following lemma in the weighted Lebesgue spaces was proved in [17, Theorem 1 and Theorem 2]. In the spaces $L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_{0}(\mathbb{T})$, $\omega\in A_{p(\cdot)}$ it can be proved like Lemma 2.3 by using Theorem 2.1

Lemma 2.4. *Let $\{f_{n}\}$ $(n=1,2,3,...)$ be a sequence of functions $f_{n}\in L_{\omega}^{p(\cdot)}(\mathbb{T})$ and let$S_{n,k_{n}}(f_{n})$ be the kth partial sum of f_{n} with positive integer $k=k_{n}$. Then for any pair $(p(\cdot),\omega)$, where $p(\cdot)\in\mathcal{P}_{0}(\mathbb{T})$ and $\omega\in A_{p(\cdot)}$ there exists a positive constant $c(p)$ such that*

$$\left\|\left(\sum_{n=1}^{\infty}\left|S_{n,k_{n}}(f_{n})\right|^{2}\right)^{1/2}\right\|_{p(\cdot),\omega}\leq c(p)\left\|\left(\sum_{n=1}^{\infty}\left|f_{n}\right|^{2}\right)^{1/2}\right\|_{p(\cdot),\omega}.$$

Proof. This Lemma in the case of $p(\cdot)=p=constant$ was proved in [17, Theorem 1 and Theorem 2]. Applying the same method used in the proof of Lemma 2.3 it can be proved also in variable exponent cases. □

Lemma 2.5. *Let $\{\lambda_{\mu}\}$ be a sequence of real numbers λ_{μ} such that for all $\mu=1,2,...$ and $m=1,2,...$*

$$\left|\lambda_{\mu}\right|\leq M\ \text{and}\ \sum_{\mu=2^{m-1}}^{2^{m}-1}\left|\lambda_{\mu}-\lambda_{\mu+1}\right|\leq M$$

with some constant M not dependent of μ and m. Let also $f\in L_{\omega}^{p(\cdot)}(\mathbb{T})$, $p(\cdot)\in\mathcal{P}_{0}(\mathbb{T})$, $\omega(\cdot)\in A_{p(\cdot)}$. Then there is a function $F\in L_{\omega}^{p(\cdot)}(\mathbb{T})$ such that the series

$$\lambda_{0}A_{0}(f,x)+\sum_{k=1}^{\infty}\lambda_{k}A_{k}(f,x)$$

is the Fourier series of F and

$$\|F\|_{p(\cdot),\omega}\leq c(p)\|f\|_{p(\cdot),\omega}.$$

Proof. Since $A_{0}(f,x):=a_{0}/2$ and then $\|A_{0}(f,\cdot)\|_{p(\cdot),\omega}\leq c(p)\|f\|_{p(\cdot),\omega}$, without loss of generality we may suppose that $a_{0}=0$. Let for $\mu=1,2,3,...$ and $2^{\mu-1}\leq k\leq 2^{\mu}-1$

$$\Delta_{\mu}(x)\ :=\sum_{k=2^{\mu-1}}^{2^{\mu}-1}A_{k}(f,x),\ \Delta_{\mu,k}(x):=\sum_{j=2^{\mu-1}}^{k}A_{k}(f,x)$$

$$\text{and }\Delta_{\mu}^{*}(x)\ :=\sum_{k=2^{\mu-1}}^{2^{\mu}-1}\lambda_{k}A_{k}(f,x).$$

Then using the inequality [28, Vol.II, p.232]

$$\left|\Delta_{\mu}^{*}(x)\right|^{2}\leq 2M\left(\sum_{s=2^{\mu-1}}^{2^{\mu}-1}\left|\Delta_{\mu,s}(x)\right|^{2}\left|\lambda_{s}-\lambda_{s+1}\right|+\left|\Delta_{\mu}(x)\right|^{2}\left|\lambda_{2^{\mu}}\right|\right)$$

and Lemma 2.4 we have that

$$\rho_{p(\cdot)}\left(\left(\sum_{\mu=1}^{\infty}\left|\Delta_{\mu}^{*}(x)\right|^{2}\right)^{1/2}\omega\right)=\int_{0}^{2\pi}\left(\sum_{\mu=1}^{\infty}\left|\Delta_{\mu}^{*}(x)\right|^{2}\right)^{p(x)/2}\omega^{p(x)}dx$$

$$\leq \int_0^{2\pi} \left(\sum_{\mu=1}^{\infty} 2M \left\{ \sum_{s=2^{\mu-1}}^{2^\mu -1} \left|\Delta_{\mu,s}(x)\right|^2 \left|\lambda_s - \lambda_{s+1}\right| + \left|\Delta_\mu(x)\right|^2 \left|\lambda_{2^{\mu-1}}\right| \right\} \right)^{p(x)/2} \omega^{p(x)} dx$$

$$\leq \int_0^{2\pi} (2M)^{p(x)/2} \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \left\{ \sum_{s=2^{\mu-1}}^{2^\mu -1} \left|\lambda_s - \lambda_{s+1}\right| + \left|\lambda_{2^{\mu-1}}\right| \right\} \right)^{p(x)/2} \omega^{p(x)} dx$$

$$\leq \int_0^{2\pi} (2M)^{p(x)} \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \right)^{p(x)/2} \omega^{p(x)} dx$$

$$\leq c(p) \int_0^{2\pi} \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \right)^{p(x)/2} \omega^{p(x)} dx = c(p)\, \rho_{p(\cdot)} \left(\left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \right)^{1/2} \omega \right),$$

which implies the inequality

$$\left\| \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu^*(x)\right|^2 \right)^{1/2} \right\|_{p(\cdot),\omega} \leq c(p) \left\| \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \right)^{1/2} \right\|_{p(\cdot),\omega}.$$

Now denoting $F(x) := \sum\limits_{k=1}^{\infty} \lambda_k A_k(f,x)$ and applying Lemma 2.3 we get

$$\begin{aligned} \|F\|_{p(\cdot),\omega} &\leq c(p) \left\| \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu^*(x)\right|^2 \right)^{1/2} \right\|_{p(\cdot),\omega} \\ &\leq c(p) \left\| \left(\sum_{\mu=1}^{\infty} \left|\Delta_\mu(x)\right|^2 \right)^{1/2} \right\|_{p(\cdot),\omega} \leq c(p) \|f\|_{p(\cdot),\omega} \end{aligned}$$

□

Lemma 2.6. *If $f \in L_\omega^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\omega(\cdot) \in A_{p(\cdot)}$, then there exists a constant $c(p)$ such that*

$$\|f - S_n(f)\|_{p(\cdot),\omega} \leq c(p) E_n(f)_{p(\cdot),\omega}, \quad n = 1, 2, 3, \ldots .$$

Proof. Let T_n^* $(n =, 1, 2, \ldots)$ be the best approximation trigonometric polynomial of $f \in L_\omega^{p(\cdot)}(\mathbb{T})$. By Lemma 2.2 we have

$$\begin{aligned} \|f - S_n(f)\|_{p(\cdot),\omega} &\leq \|f - T_n^*\|_{p(\cdot),\omega} + \|T_n^* - S_n(f)\|_{p(\cdot),\omega} \\ &\leq \|f - T_n^*\|_{p(\cdot),\omega} + \|S_n(T_n^* - f)\|_{p(\cdot),\omega} \\ &\leq (c_3(p) + 1) \|f - T_n^*\|_{p(\cdot),\omega} = c(p) E_n(f)_{p(\cdot),\omega}. \end{aligned}$$

□

For proofs of Theorems 1.2 and 1.4 we use also the following Lemma, which in the more general space, namely in the Musielak-Orlicz space, was proved in [16, Proposition 4].

Lemma 2.7. *Let $\{f_i\}_{i=1}^n$ be a finite system of nonnegative functions $f_i \in L_\omega^{p(\cdot)}(\mathbb{T})$ where $p(\cdot) \leq q$ ė. on $\mathbb{T}$, for some positive constant q. Then there exists a positive constant $c(p)$ such that*

$$\left(\sum_{i=1}^{n} \|f_i\|_{p(\cdot),\omega}^q \right)^{1/q} \leq c(p) \left\| \left(\sum_{i=1}^{n} f_i^q \right)^{1/q} \right\|_{p(\cdot),\omega}.$$

3. Proofs of main results

Proof of Theorem 1.2. Let $r\in\mathbb{N}$, $\beta:=\max\{2,p_+\}$ and let

$$\sigma_r^{\beta}:=\sum_{\nu=1}^{r}\frac{(\nu+1)^{k\beta-1}}{(r+1)^{k\beta}}E_\nu^{\beta}(f)_{p(\cdot),\omega}.$$

Let also $2^m\le r<2^{m+1}$. Via monotonicity property of $\{E_\nu(f)_{p(\cdot),\omega}\}$ and Lemma 2.3 we have

$$\begin{aligned}\sigma_r^{\beta}&\le\sum_{\nu=0}^{m+1}\sum_{\mu=2^\nu}^{2^{\nu+1}-1}\frac{\mu^{k\beta-1}}{(r+1)^{k\beta}}E_\mu^{\beta}(f)_{p(\cdot),\omega}\\
&\le c_4(p,k)\sum_{\nu=0}^{m+1}\frac{2^{\nu k\beta}}{(r+1)^{k\beta}}E_{2^\nu}^{\beta}(f)_{p(\cdot),\omega}\\
&\le c_4(p,k)\sum_{\nu=0}^{m+1}\frac{2^{\nu k\beta}}{(r+1)^{k\beta}}\left\|f-\sum_{\mu=0}^{2^\nu-1}A_\mu(f)\right\|_{p(\cdot),\omega}^{\beta}\\
&=c_4(p,k)\sum_{\nu=0}^{m+1}\frac{2^{\nu k\beta}}{(r+1)^{k\beta}}\left\|\sum_{\mu=2^\nu}^{\infty}A_\mu(f)\right\|_{p(\cdot),\omega}^{\beta}\\
&=c_4(p,k)\sum_{\nu=0}^{m+1}\frac{2^{\nu k\beta}}{(r+1)^{k\beta}}\left\|\sum_{\mu=\nu}^{\infty}\sum_{l=2^\mu}^{2^{\mu+1}-1}A_l(f)\right\|_{p(\cdot),\omega}^{\beta}\\
&\le c_5(p,k)\sum_{\nu=0}^{m+1}\frac{2^{\nu k\beta}}{(r+1)^{k\beta}}\left\|\left(\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)^{1/2}\right\|_{p(\cdot),\omega}^{\beta}\\
&=c_5(p,k)\sum_{\nu=0}^{m+1}\left\|\left(\frac{2^{2\nu k}}{(r+1)^{2k}}\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)^{1/2}\right\|_{p(\cdot),\omega}^{\beta}.\end{aligned}$$

Then by Lemma 2.7 we get

$$\sigma_r\le c_6(p,k)\left\{\sum_{\nu=0}^{m+1}\left\|\left(\frac{2^{2\nu k}}{(r+1)^{2k}}\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)^{1/2}\right\|_{p(\cdot),\omega}^{\beta}\right\}^{1/\beta}$$

$$\le c_7(p,k)\left\{\left\|\left[\sum_{\nu=0}^{m+1}\left(\frac{2^{2\nu k}}{(r+1)^{2k}}\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)^{\beta/2}\right]^{1/\beta}\right\|_{p(\cdot),\omega}\right\}. \tag{3.1}$$

Hence, if $\beta=2$, then

$$\sigma_r\le c_7(p,k)\left\|\left[\sum_{\nu=0}^{m+1}\left(\frac{2^{2\nu k}}{(r+1)^{2k}}\sum_{\mu=\nu}^{\infty}\Delta_{\mu+1}^2\right)\right]^{1/2}\right\|_{p(\cdot),\omega}, \tag{3.2}$$

and if $\beta=\max\{2,p_+\}=p_+>2$, then $\beta/2>1$ and using the inequality $(a_1)^{\beta/2}+(a_2)^{\beta/2}+\ldots+(a_n)^{\beta/2}\le(a_1+a_2+\ldots+a_n)^{\beta/2}$ in (3.1) we again arrive to the same inequality (3.2). So we conclude that for $\beta=:\max\{2,p_+\}$ the inequality (3.2) holds.

Since $\beta =: \max\{2, p_+\} > 1$, applying the Abel transformation [28, p.1] and the inequality $(a_1 + a_2 + ... + a_n)^{1/2} \leq (a_1)^{1/2} + (a_2)^{1/2} + ... + (a_n)^{1/2}$ and also the Minkowski inequality we have

$$\left\| \left[\sum_{\nu=0}^{m+1} \left(\frac{2^{2\nu k}}{(r+1)^{2k}} \sum_{\mu=\nu}^{\infty} \Delta_{\mu+1}^2 \right) \right]^{1/2} \right\|_{p(\cdot),\omega}$$

$$\leq c_8(p) \left\| \left[\sum_{\nu=0}^{m} \frac{2^{2\nu k}}{(r+1)^{2k}} \Delta_{\nu+1}^2 + \frac{2^{2(m+1)k}}{(r+1)^{2k}} \sum_{\mu=m+1}^{\infty} \Delta_{\mu+1}^2 \right]^{1/2} \right\|_{p(\cdot),\omega}$$

$$\leq c_8(p) \left\| \left[\sum_{\nu=0}^{m} \frac{2^{2\nu k}}{(r+1)^{2k}} \Delta_{\nu+1}^2 \right]^{1/2} \right\|_{p(\cdot),\omega}$$

$$+ c_8(p) \left\| \frac{2^{(m+1)k}}{(r+1)^k} \left(\sum_{\mu=m+1}^{\infty} \Delta_{\mu+1}^2 \right)^{1/2} \right\|_{p(\cdot),\omega}$$

$$=: I_1 + I_2. \tag{3.3}$$

Since $2^m \leq r < 2^{m+1}$ and $(a_1 + a_2 + ... + a_n)^{1/2} \leq (a_1)^{1/2} + (a_2)^{1/2} + ... + (a_n)^{1/2}$, by Lemmas 2.3 and 2.6 we have

$$I_2 \leq c_9(p,k) \left\| \left(\sum_{\mu=m+1}^{\infty} \Delta_{\mu+1}^2 \right)^{1/2} \right\|_{p(\cdot),\omega} \leq c_{10}(p,k) \left\| \sum_{\mu=2^{m+1}}^{\infty} A_\mu(f) \right\|_{p(\cdot),\omega}$$

$$= c_{10}(p,k) \left\| f - \sum_{\mu=0}^{2^{m+1}-1} A_\mu(f) \right\|_{p(\cdot),\omega} \leq c_{11}(p,k) E_r(f)_{p(\cdot),\omega}$$

$$\leq c_{11}(p,k) \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega}. \tag{3.4}$$

Now we consider the system of multipliers $\{h_\mu\}$ for $\nu = 1, 2, ..., 2^{m+1} - 1$

$$h_\mu := \begin{cases} \frac{2^{\nu k}}{\mu^k}, & 2^\nu \leq \mu \leq 2^{\nu+1} - 1,\ \nu = 0, 1, ..., m \\ 0, & \mu \geq 2^{m+1}, \end{cases}$$

which satisfies the conditions of Lemma 2.5. By the inequality $(a_1 + a_2 + ... + a_n)^{1/2} \leq (a_1)^{1/2} + (a_2)^{1/2} + ... + (a_n)^{1/2}$ and by Lemma 2.5 we get

$$I_1 = c_8(p) \left\| \left[\sum_{\nu=0}^{m} \frac{2^{2\nu k}}{(r+1)^{2k}} \Delta_{\nu+1}^2 \right]^{1/2} \right\|_{p(\cdot),\omega}$$

$$\leq c_8(p) \left\| \sum_{\nu=0}^{m} \frac{2^{\nu k}}{(r+1)^k} \Delta_{\nu+1} \right\|_{p(\cdot),\omega}$$

$$= c_8(p) \left\| \sum_{\nu=0}^{m} \sum_{\mu=2^\nu}^{2^{\nu+1}-1} \frac{2^{\nu k}}{\mu^k} \frac{\mu^k}{(r+1)^k} A_\mu(f) \right\|_{p(\cdot),\omega}$$

$$\leq c_9(p) \left\| \sum_{\mu=1}^{r} \frac{\mu^k}{(r+1)^k} A_\mu(f) \right\|_{p(\cdot),\omega}$$

$$= c_9(p) \left\| S_r(f) - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega}.$$

Since by Lemma 2.6

$$\left\| S_r(f) - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega} \le \| S_r(f) - f \|_{p(\cdot),\omega} + \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega}$$

$$\le c(p) E_r(f)_{p(\cdot),\omega} + \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega} \le c_{10}(p) \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega},$$

for I_1 we have the estimation

$$I_1 \le c_{11}(p) \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega}. \tag{3.5}$$

Now, combining the relations (3.1)−(3.5) we obtain that

$$\sigma_r \le c_{12}(p) \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega},$$

that is for $\beta = \max\{2, p_+\}$

$$\frac{1}{(r+1)^k} \left[\sum_{\nu=0}^{r} (\nu+1)^{k\beta-1} E_\nu^\beta(f)_{p(\cdot),\omega} \right]^{1/\beta} \le c(p,k) \left\| f - Z_r^{(k)}(f) \right\|_{p(\cdot),\omega},$$

which proves the desired inequality in Theorem 1.2. □

The proof of Theorem 1.4 goes by similar way.

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